Objectivity and constitutive modelling

Fülöp[2], T., Papenfuss, C. and Ván[12], P.,

[1]RMKI, Dep. Theor. Phys., Budapest, Hungary
[2]Montavid Research Group, Budapest, Hungary

- Objectivity – contra Noll
- Kinematics – rate definition
- Frame independent Liu procedure
- Second and third grade solids (waves)

Problems with the traditional formulation

Rigid rotating frames:

$$\hat{t} = t, \qquad \hat{\mathbf{x}} = \mathbf{h}(t) + \mathbf{Q}(t)\mathbf{x}$$ Noll (1958)

Four-transformations, four-Jacobian:

$$\hat{c}^a = \hat{J}^a{}_b c^b, \qquad \text{where} \qquad \hat{J}^a{}_b = \frac{\partial \hat{x}^a}{\partial x^b} = \begin{pmatrix} 1 & 0 \\ \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x} & \mathbf{Q} \end{pmatrix}$$

$$\begin{pmatrix} \hat{c}^0 \\ \hat{\mathbf{c}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x} & \mathbf{Q} \end{pmatrix} \begin{pmatrix} c^0 \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} c^0 \\ (\dot{\mathbf{h}} + \dot{\mathbf{Q}}\mathbf{x})c^0 + \mathbf{Q}\mathbf{c} \end{pmatrix}$$

Consequences:

$\Rightarrow$ four-velocity is an objective vector,
$\Rightarrow$ time cannot be avoided,
$\Rightarrow$ transformation rules are not convenient
$\Rightarrow$ Why rigid rotation?

$\Rightarrow$ arbitrary reference frames
$\Rightarrow$ reference frame INDEPENDENT formulation!

The aspect of spacetime:

Special relativity: no absolute time, no absolute space,
$\exists$ absolute spacetime; three-vectors $\rightarrow$ four-vectors,
x(t) $\rightarrow$ world line (curve in the 4-dimensional spacetime).

The nonrelativistic case (Galileo, Weyl): $\exists$ absolute time,
no absolute space, spacetime needed ($t' = t$, $x' = x - vt$)

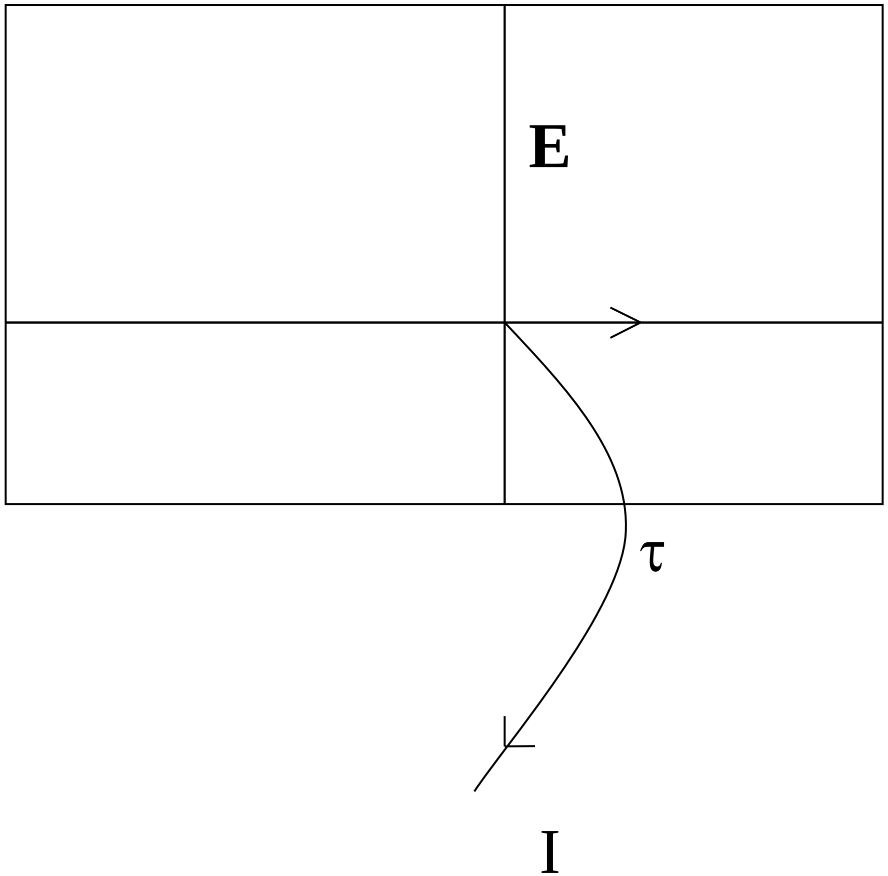
4-vectors: $\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$, world lines: $\mathbf{x}(t) \rightarrow \begin{pmatrix} t \\ \mathbf{x}(t) \end{pmatrix}$

(4-coordinates w.r.t. an inertial coordinate system K)

The spacelike 4-vectors $\begin{pmatrix} 0 \\ \mathbf{x} \end{pmatrix}$ are absolute: form a 3-dimensional

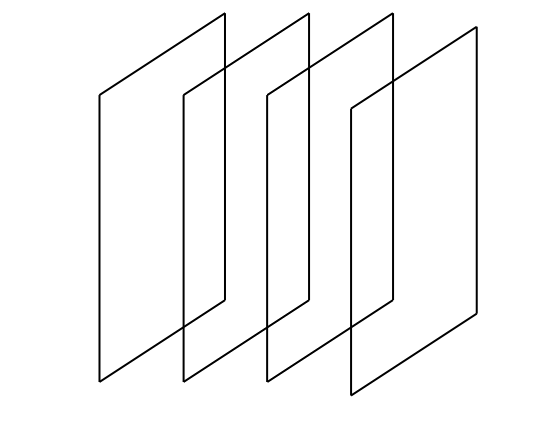
Euclidean vector space.

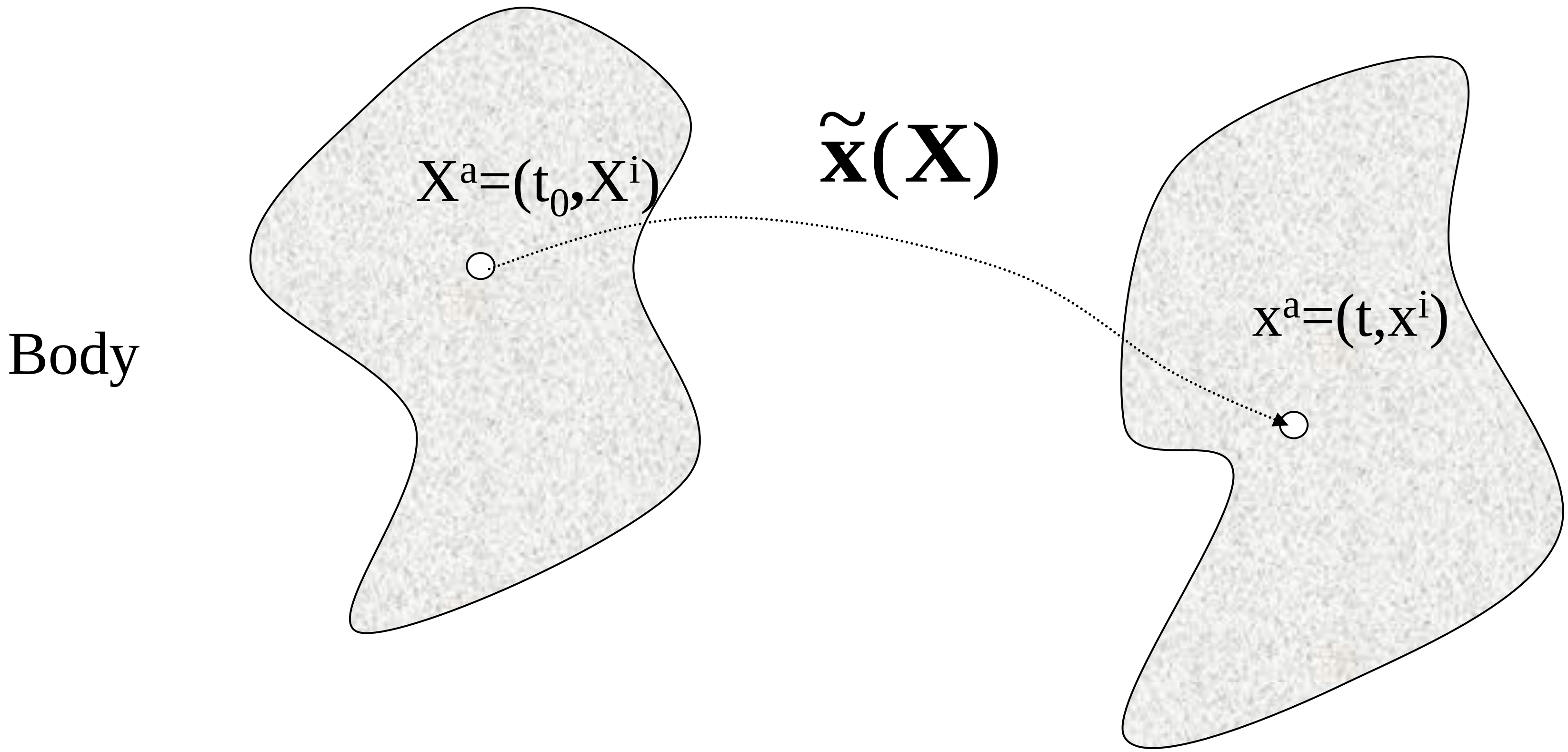
What is non-relativistic space-time?



M=R^3�R

M=E�I





$$\tilde{Y}^a{}_b = \tilde{\partial}_b \tilde{x}^a \prec \begin{pmatrix} t \\ \tilde{x}^i(t, X_i) \end{pmatrix} (\tilde{\partial}_t, \tilde{\partial}_j) = \begin{pmatrix} 1 & 0_j \\ \tilde{\partial}_t \tilde{x}^i & \tilde{\partial}_j \tilde{x}^i \end{pmatrix} = \begin{pmatrix} 1 & 0_j \\ \tilde{v}^i & \tilde{F}^i{}_j \end{pmatrix}$$

$$a, b \in \{0,1,2,3\}, \qquad i, j \in \{1,2,3\}$$

deformation gradient

Objective derivative of a vector

$$\tilde{\partial}_a \hat{c}^b \prec \begin{pmatrix} c^0 \\ \left(\tilde{F}^{-1}\right)^j{}_k c^k \end{pmatrix} (\tilde{\partial}_t, \tilde{\partial}_i) = \begin{pmatrix} \dfrac{d}{dt} c^0 & \tilde{\partial}_i c^0 \\ \dfrac{d}{dt}\left(\left(\tilde{F}^{-1}\right)^j{}_k \bar{c}^k\right) & \tilde{\partial}_i\left(\left(\tilde{F}^{-1}\right)^j{}_k \bar{c}^k\right) \end{pmatrix} =$$

$$\begin{pmatrix} \dot{c}^0 & \tilde{\nabla} c^0 \\ \mathbf{F}^{-1}\left(\dot{\bar{\mathbf{c}}} - \nabla\mathbf{v}\cdot\bar{\mathbf{c}}\right) & \mathbf{F}^{-1}\left(\tilde{\nabla}\bar{\mathbf{c}} - \nabla\mathbf{F}\cdot\bar{\mathbf{c}}\right) \end{pmatrix}$$

$$\bar{\mathbf{c}} = \mathbf{c} - c^0\mathbf{v}$$

Upper convected

Tensorial property (order, co-) determines the form of the derivative.

Kinematics

Spacetime-allowed quantities:

- the velocity field v [the 4-velocity field $\begin{pmatrix} 1 \\ \mathbf{v} \end{pmatrix}$],
- its integral curves (world lines of material points),
- $\partial_i v^j, \frac{1}{2}(\partial_i v^j + \partial_j v^i), \frac{1}{2}(\partial_i v^j - \partial_j v^i)$

The problem of reference time: is the continuum completely relaxed, undisturbed, totally free at t_0, at every material point?

In general, not → incorrect for reference purpose.

The physical role of a strain tensor: to quantitatively characterize not a *change* but a *state*:

not to measure a change since a t_0 but to express the difference
of the current local geometric status from the status in a fully relaxed, ideally
undisturbed state.

What one can have in general for a strain tensor:

a rate equation + an initial condition

(which is the typical scheme for field quantities, actually)

Spacetime requirements → the change rate of strain is to be determined by $\partial_i v^j$

Example: instead of $\mathbf{E}^{Cauchy} = \left(\nabla_R \mathbf{u}\right)^S$

from now on:

$$\boxed{\dot{\mathbf{E}}^{Cauchy} = \left(\nabla_R \mathbf{v}\right)^S + \text{initial condition}}$$

(e.g. knowing the initial stress and a linear elastic const. rel.)

Remark: the *natural strain measure* rate equation is

$$\dot{\mathbf{A}} = \left(\nabla_R \mathbf{v}\right)\mathbf{A} + \mathbf{A}\left(\nabla_R \mathbf{v}\right)^T \quad \text{generalized left Cauchy-Green}$$

Frame independent constitutive theory:

$\Rightarrow$ variables:

spacetime quantities (v!)

body related rate definition of the deformation

$\Rightarrow$ balances:

frame independent spacetime relations

$\Rightarrow$ pull back to the body (reference time!)

Entropy flux is constitutive!

Question: fluxes?

Constitutive theory of a second grade elastic solid

Balances:

$$\lambda: \quad \rho_0 \dot{e} + \tilde{\partial}_i w^i = 0,$$

$$\lambda_i: \quad \rho_0 \dot{v}^i - \tilde{\partial}_j T^{ij} = 0^i,$$

$$\Lambda^j{}_i: \quad \dot{F}^i{}_j - \tilde{\partial}_j v^i = 0^i{}_j.$$

$\tilde{\partial}$ material space derivative

$T^i{}_j$ first Piola-Kirchhoff stress

Constitutive space: $e, \tilde{\partial}_i e, v^i, \tilde{\partial}_j v^i, F^i{}_j, \tilde{\partial}_k F^i{}_j$

Constitutive functions: $w^i, T^{ij}, \quad s, J^i$

Remark:

$$\tilde{Y}^a{}_b = \tilde{\partial}_b \tilde{x}^a = \begin{pmatrix} 1 & 0_j \\ \tilde{v}^i & \tilde{F}^i{}_j \end{pmatrix}, \quad \tilde{\partial}_k \tilde{Y}^a{}_b = \tilde{\partial}_{kc} \tilde{x}^a = \begin{pmatrix} 0_k & 0_{kj} \\ \tilde{\partial}_k \tilde{v}^i & \tilde{\partial}_k \tilde{F}^i{}_j \end{pmatrix}$$

Second Law:

$$\rho_0 \dot{s} + \tilde{\partial}_i J^i - \lambda(\rho_0 \dot{e} + \tilde{\partial}_i w^i) - \lambda_i(\rho_0 \dot{v}^i - \tilde{\partial}_j T^{ij}) - \Lambda^j{}_i(\dot{F}^i{}_j - \tilde{\partial}_i v^j) \geq 0,$$

Consequences:

Function restrictions:

$$w^i(e, v^i, F^i{}_j), T^{ij}(e, v^i, F^i{}_j), \quad s(e, v^i, F^i{}_j),$$

$$J^i = \partial_e s w^i - \partial_{v^j} s T^{ji} + K(e, v^i, F^i{}_j)$$

- Internal energy $\quad s(e - \frac{v^2}{2}, F^i{}_j)$

Dissipation inequality:

$$\mathbf{q} \cdot \nabla \frac{1}{T} + \frac{1}{T}\left(\mathbf{T} - \rho_0 \partial_{F^i{}_j} f\right) : \dot{\mathbf{F}} \geq 0$$

Constitutive theory of a third grade elastic solid

Balances:

$$\lambda: \quad \rho_0 \dot{e} + \tilde{\partial}_i w^i = 0,$$

$$\lambda_i: \quad \rho_0 \dot{v}^i - \tilde{\partial}_j T^{ij} = 0^i,$$

$$\hat{\lambda}_i{}^j: \quad \rho_0 \tilde{\partial}_j \dot{v}^i - \tilde{\partial}_{jk} T^{ik} = 0^i{}_j,$$

$$\Lambda^j{}_i: \quad \dot{F}^i{}_j - \tilde{\partial}_j v^i = 0^i{}_j,$$

$$\hat{\Lambda}^{jk}{}_i: \quad \tilde{\partial}_k \dot{F}^i{}_j - \tilde{\partial}_{kj} v^i = 0^i{}_{jk}$$

Constitutive functions:

$$w^i, T^{ij}, \quad s, J^i$$

Constitutive space: $e, \tilde{\partial}_i e, v^i, \tilde{\partial}_j v^i, \tilde{\partial}_{jk} v^i, F^i{}_j, \tilde{\partial}_k F^i{}_j, \tilde{\partial}_{kl} F^i{}_j$

Second Law:

$$\rho_0 \dot{s} + \tilde{\partial}_i J^i - \lambda(\rho_0 \dot{e} + \tilde{\partial}_i w^i) - \lambda_i(\rho_0 \dot{v}^i - \tilde{\partial}_j T^{ij}) - \Lambda^j{}_i(\dot{F}^i{}_j - \tilde{\partial}_i v^j)$$

$$- \hat{\lambda}_i{}^j\left(\rho_0 \tilde{\partial}_j \dot{v}^i - \tilde{\partial}_{jk} T^{ik}\right) - \hat{\Lambda}^{jk}{}_i\left(\tilde{\partial}_k \dot{F}^i{}_j - \tilde{\partial}_{kj} v^i\right) \geq 0,$$

Consequences:

Function restrictions:

$$T^{ij}(e, v^i, \tilde{\partial}_j v^i, F^i{}_j, \tilde{\partial}_k F^i{}_j), \quad s(e, v^i, \tilde{\partial}_j v^i, F^i{}_j, \tilde{\partial}_k F^i{}_j),$$

$$J^i = \partial_e s w^i - \partial_{v^j} s T^{ji} + ... + K(e, v^i, \tilde{\partial}_j v^i, F^i{}_j, \tilde{\partial}_k F^i{}_j)$$

- Internal energy $\quad s(e - \frac{v^2}{2} - \frac{\alpha}{2}\mathrm{Tr}(\dot{\mathbf{F}} \cdot \dot{\mathbf{F}}) - ..., F^i{}_j)$

Dissipation inequality:

Double stress

$$\mathbf{q} \cdot \nabla \frac{1}{T} + \frac{1}{T}\left(\mathbf{T} - \alpha \nabla\nabla \cdot \mathbf{T} - \rho_0 \partial_{F^i{}_j} f + \rho_0 \tilde{\partial}_k \left(\partial_{\tilde{\partial}_k F^i{}_j} f\right)\right) : \dot{\mathbf{F}} \geq 0$$

Waves- 1d

Quadratic energy – without isotropy

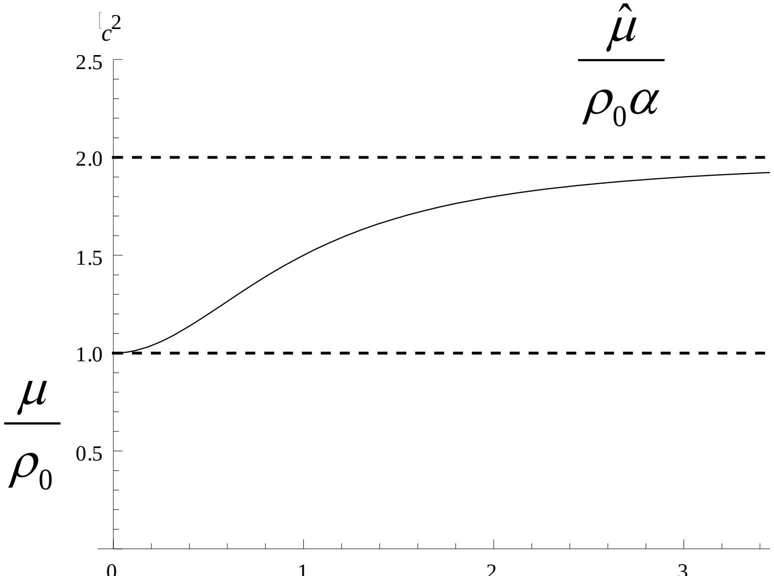
$$f(F^i{}_j, \tilde{\partial}_k F^i{}_j) = \frac{\mu}{2} F^i{}_j F^j{}_i + \frac{\hat{\mu}}{2} \tilde{\partial}_k F^i{}_j \tilde{\partial}^k F^j{}_i + \ldots$$

$$\left(\mathbf{T} - \alpha \nabla\nabla \cdot \mathbf{T} - \rho_0 \partial_{F^i{}_j} f + \rho_0 \tilde{\partial}_k \left(\partial_{\tilde{\partial}_k F^i{}_j} f\right)\right) : \dot{\mathbf{F}} \geq 0$$

$$\boxed{\begin{aligned} &T - \mu F - \alpha \hat{\partial}_{xx} T + \hat{\mu} \hat{\partial}_{xx} F - \eta \hat{\partial}_t F = 0, \\ &\rho_0 \hat{\partial}_{tt} F - \hat{\partial}_{xx} T = 0. \end{aligned}}$$

- Stable
- Dispersion relation

$$c^2 = \left(\frac{\varpi}{k}\right)^2 = \frac{k^2}{\rho_0} \frac{\mu + \hat{\mu} k^2}{1 + \alpha k^2}$$



Summary

- Noll is wrong.

- Non-relativistic spacetime cannot be avoided.

- Deformation and strain – wordline based rate definition.

- Gradient elasticity theories are valid.

Thank you for your attention!

Thermodinamics - Mechanics