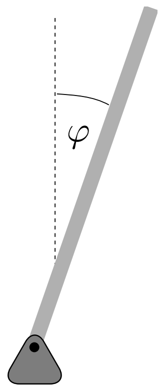
The 1st Gulf Physics Olympiad — Theoretical Competition

Riyadh, Saudi Arabia — Monday, March 21st 2016

- The examination lasts for 5 hours. There are 3 problems worth in total 30 points. **Please note that the point values of the three theoretical problems are not equal.**
- **You must not open the envelope with the problems before the signal of the beginning of competition.**
- **You are not allowed to leave your working place without permission.** If you need any assistance (broken calculator, need to visit a restroom, etc), please raise your hand until an organizer arrives.
- Use only the front side of the sheets of paper.
- For each problem, there are **dedicated Solution Sheets** (see header for the number and pictogram). Write your solutions onto the appropriate Solution Sheets. For each Problem, the Solution Sheets are numbered; use the sheets according to the enumeration. **Always mark which Problem Part and Question you are dealing with.** Copy the final answers into the appropriate boxes of the **Answer Sheets**. There are also **Draft** papers; use these for writing things which you don't want to be graded. If you have written something that you don't want to be graded onto the Solution Sheets (such as initial and incorrect solutions), cross these out.
- If you need more paper for a certain problem, please raise your hand and tell an organizer the problem number; you are given two Solution sheets (you can do this more than once).
- **You should use as little text as possible:** try to explain your solution mainly with equations, numbers, symbols and diagrams. Though in some places textual explanation may be unavoidable.
- **After the signal signifying the end of examination you must stop writing immediately.** Put all the papers into the envelope at your desk. **You are not allowed to take any sheet of paper out of the room.** If you have finished solving before the final sound signal, please raise your hand.

Problem T1. Stabilizing unstable states *(11 points)*

Part A. Stabilization via feedback *(3.5 points)*



Let us study, how an initially unstable equilibrium position can be stabilized. First we consider a reversed pendulum: a thin long rod of homogeneous mass distribution and length l is fixed at its lowest point to a hinge so that it can freely rotate around the hinge. We describe the position of the rod via the angle φ between the rod and a vertical line. We shall assume that $\varphi \ll 1$ (φ is much smaller than 1). The free fall acceleration $g = 9.8\,\mathrm{m/s^2}$.

i. (1.5 pts) Express the angular acceleration of the rod $\ddot{\varphi}$ in terms of φ, and the parameters l and g. Show that the inclination angle φ as a function of time t is expressed as $\varphi(t) = A\mathrm{e}^{t/\tau} + B\mathrm{e}^{-t/\tau}$, where A and B are constants which depend on the initial position and initial angular speed of the rod, and τ is a characteristic time. Express τ in terms of l and g. (You may use dimensional analysis, but then you'll lose 0.5 pts.) *Hint:* for a rod of length l and mass m, the moment of inertia with respect to its endpoint is $\frac{1}{3}ml^2$.

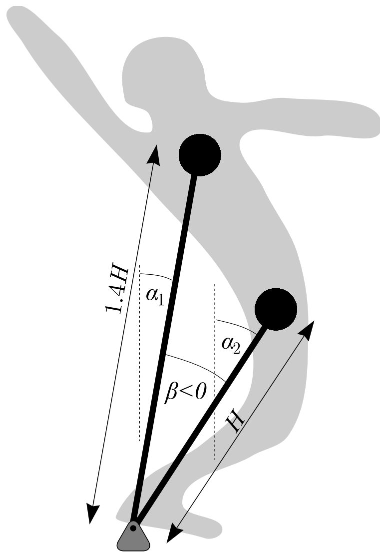
ii. (0.5 pts) Now, a boy tries to keep a long thin rod standing vertically on his palm. For instance, as soon as the rod starts falling leftwards, he moves his palm to an even greater distance leftwards so that the rod's centre of gravity would be positioned rightwards from the rod's support point. Then, the torque of the gravity force would rotate the rod rightwards, decreasing the previously observed leftwards angular speed. Estimate, for which rod lengths the boy can keep the rod vertically if his reaction time is estimated as $\tau_r = 0.2\,\mathrm{s}$. (The reaction time is the time lag between the command sent by brain to hands, and the corresponding motion of the hands.)

iii. (0.5 pts) Humans and birds keep their standing position similarly and move the support centre (the point at the bottom of their feet where the total normal force is applied), e.g. by adjusting the angle between a leg and the foot, so as to oppose the falling motion of the upper part of their body. A small bird of length $l_b = 6\,\mathrm{cm}$ can stand on its feet; estimate the upper bound for its reaction time.

iv. (1 pt) Equilibrium on a bike is also kept by displacing the support centre which lies on the line connecting the wheel-ground contact points; that line can be conveniently displaced by turning the handlebar while driving forth. Estimate the minimal driving speed v_m of a bicyclist by which the equilibrium can be maintained in such a way. Assume that for the bicyclist, the characteristic falling time is the same as for a rod of length $L = 2\,\mathrm{m}$; the distance between the centres of the wheels $d = 1\,\mathrm{m}$.

Part B. Tightrope walker *(3.5 points)*

A tightrope walker cannot move the support point in the direction perpendicular to the rope. His equilibrium is kept by displacing the centre of gravity, instead. Let us make a simple model of a man balancing on a rope.



Lower half of the body is modelled by a point mass m at height H, and the upper half of the body — by an equal point mass m at the height $1.4H$. The mutual position of these point masses can be changed by bowing right or left; for the sake of simplicity, let us assume that the distance of the point masses from the rope will remain unchanged, i.e. these behave as if being fixed to the endpoints of two thin rods of lengths H and $1.4H$ respectively, see figure. Let the rods form angles α_1 and α_2 with the vertical line (positive angles correspond to clock-wise rotation), so that the angle between the rods is $\beta = \alpha_1 - \alpha_2$. A tightrope walker can control the value of the angle β by bowing.

i. (1 pt) Let us assume that initially, the tightrope walker was standing in an almost perfect equilibrium ($\alpha_1 = \alpha_2 = 0$). Due to instability of this equilibrium, he **starts slowly falling clock-wise**, which he notices at $t = t_0$ when $\alpha_1 = \alpha_2 = \alpha_0 > 0$. He bows rapidly to stop falling: assume that the angle β takes almost **instantaneously** a new value β_0. Express the new values of the angles α_1 and α_2 in terms of β and α_0.

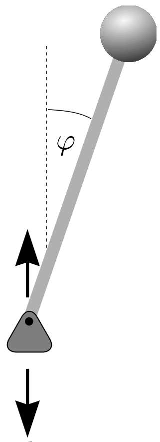
ii. (0.5 pts) So, the tightrope walker is now bowing and keeps this body shape ($\beta = \beta_0$) for the time period T_b, upon which he straightens himself almost instantaneously and makes thereby $\beta = 0$. His aim is to resume the motionless standing position with $\alpha_1 = \alpha_2 = 0$. Should he have bowed clock-wise ($\beta_0 > 0$) or counter-clock-wise? Motivate your answer.

iii. (1 pt) From now on, we assume that $\alpha_0 \ll \beta_0$. Immediately after he has straightened himself, neither his angular speed $\dot{\alpha}_1 = \dot{\alpha}_2$ nor angle α_1 are zero: zero values will be achived much later. Which value (expressed in terms of H and g) should the ratio $\dot{\alpha}_1/\alpha_1$ take at that moment?

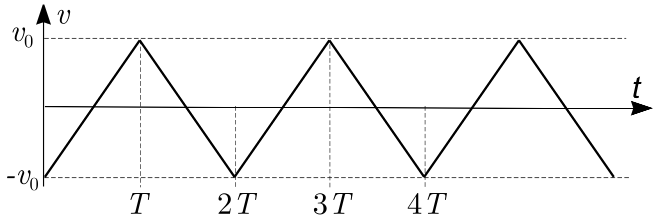
iv. (1 pt) Express the required duration T_b in terms of α_0, β_0, H, and g assuming that $\alpha_0 \ll \beta_0$.

Part C. Kapitza's pendulum *(4 points)*

In 1908 Andrew Stephenson found that the upper position of a pendulum can be stable, if its suspension point oscillates with a high frequency. The explanation of this phenomenon was provided in 1951 by Russian physicist Pyotr Kapitza. In what follows we'll find the stability criterion of such a pendulum. Apart from being just a nice toy, the Kapitza's pendulum demonstrates the method of separating fast and slow processes which plays an important role in physics. High frequency oscillations can drive a slow motion in various systems, e.g. high frequency electric fields act on charges with an effective average force known as the ponderomotive force.



We consider a pendulum of length l, similar to that of Part-A-Question-i, but now the rod is massless, with a point mass at its end, and the suspension point oscillates vertically (see the figure). Let the velocity v of the suspension point depend on time t as shown in the graph below ($v > 0$ corresponds to upward motion); the oscillations' half-period $T \ll l/v_0$. We also assume that $v_0/T \gg g$ so that for questions i–ii you may ignore the free fall acceleration. In order to simplify calculations, you'll need to study this process in the frame of reference of the suspension point (keep in mind: reference frame's acceleration $\vec{a}$ gives rise to an inertial force $-M\vec{a}$ acting on a body of mass M).



i. (1.5 pts) Suppose that at $t = T/2$, the pendulum was motionless and inclined by a small angle φ_0. Sketch the graph of the inclination angle φ as a function of time, and determine the angular displacement of the pendulum $\Delta\varphi$ for the moment $t = T$, i.e. $\Delta\varphi = \varphi(T) - \varphi(T/2)$. You may assume in your calculations that $\Delta\varphi \ll \varphi_0$ (this is valid because $T \ll l/v_0$).

ii. (1.5 pts) Since we still neglect gravity, only inertial force exerts a torque on the pendulum. Determine the average value of this torque (with respect to the suspension point, averaged over the full period $2T$).

iii. (1 pt) Now, let us take into account that there is also the gravity field of the Earth. Determine, which inequality must be satisfied for g, T, l and v_0 in order to ensure the stability of the vertical position of such a pendulum (some of these parameters may not be needed for your inequality).

Problem T2. Gravitational waves *(10 points)*

Part A. Dipole radiation *(2.4 points)*

Static electric and gravity fields are described by identical set of equations — as long as we are far from black holes. However, if we add terms describing time variations of the fields, the equations become different. Therefore, expressions for electromagnetic waves cannot be directly carried over to gravitational waves. Still, for expressions given below, the difference will be only in the value of numerical prefactors.

Charges moving with acceleration lose kinetic energy by radiating electromagnetic waves; this radiation is known as **the dipole radiation**. The total radiation power is expressed as

$$P_{ed} = \frac{\ddot{\vec{d}}^{2}}{6\pi\varepsilon_0 c^3}, \tag{1}$$

where $\ddot{\vec{d}}$ is the second time derivative of the dipole moment, c is the speed of light, and ε_0 — vacuum permittivity. Dipole moment for a system of charges q_i is defined as $\vec{d} = \sum_i \vec{r}_i q_i$, where $\vec{r}_i$ is vector pointing from the origin to the position of i-th charge. For harmonically oscillating dipoles, the radiated wave frequency equals to the frequency of oscillations.

i. (1.4 pts) Consider an electron of charge $-e$ and mass m, circulating around an atomic nucleus of charge $+Ze$ at distance r; neglect quantum mechanical effects. Express the total radiated power, and the wavelength λ of the radiated waves in terms of e, Z, m, r, and physical constants.

ii. (1 pt) Let us try to carry over Eq. (1) to gravitational waves; then, the total radiation power P_{gd} would be proportional to $\ddot{\vec{d}}_g^{\,2}$, where $\vec{d}_g$ is the gravitational dipole moment, and two dots denote the second time-derivative. Analogously to the electrical dipole, gravitational dipole moment for a system of point masses m_i is defined as $\vec{d}_g = \sum_i \vec{r}_i m_i$. Show that always $P_{gd} = 0$.

Part B. Quadrupole radiation *(7.6 points)*

Let us consider a binary star consisting of two stars of equal mass M which rotate around a circular orbit of radius R with angular speed ω.

i. (1 pt) Express ω in terms of M, R, and constants.

ii. (0.8 pts) While there is no gravitational dipole radiation, there is a quadrupole one. In analogy with the dipole radiation, it should be proportional to squared time-derivatives of the quadrupole moment. For this problem, it is enough to know that for our binary star, the gravitational quadrupole moment components are of the order of MR^2. So, we expect the total radiation power to have a form $P_{qg} = AM^2R^4$, where the factor A may depend on ω and physical constants (here ω is an independent parameter, though for a binary star it depends on M and R). Find expression for P_{qg} using dimensional analysis.

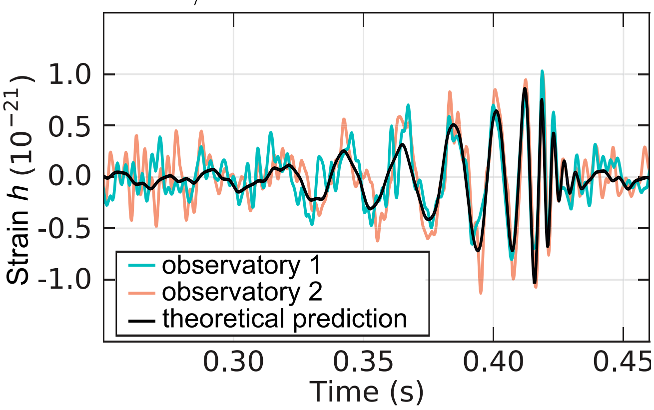
iii. (0.8 pts) The effect of gravitational waves is measured by strain $h = \Delta l/l$; here l is a distance between two points in space, and Δl is the change of that distance due to the wave. As usual for waves, the energy flux density S (radiation energy per unit time and unit area) is proportional to the squared wave amplitude: $S = Kh_0^2$ (h_0 denotes the wave amplitude). Based on dimensional arguments, express the factor K in terms of constants and the angular frequency of the wave ω.

iv. (1 pt) The dipole radiation is distributed over propagation directions anisotropically, but let us ignore this: for the sake of simplicity, assume isotropic radiation. Express the amplitude h_0 of gravitational waves at distance L in terms of M, R, and physical constants.

The energy of the binary star decreases in time due to the emission of gravitational waves. So, the distance R between the two stars decreases. This process will continue until the stars collide and merge (R becomes of the order of the radius of a star). In LIGO experiment (reported on 11th February 2016), gravitational waves emitted right before a merger of two black holes were observed. For the radius of a black hole, we'll use the Schwarzschild radius R_s which is defined as such a critical distance from a point mass M that light cannot escape due to gravitational pull from distances $r < R_s$. To derive properly an expression for R_s, theory of general relativity is needed.

v. (1 pt) Express R_s in terms of M and physical constants. Use the following fact: if we neglect general relativity and use special relativity together with Newtonian gravitation law, we obtain a result which is exactly half of the correct one.

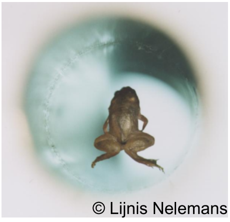
vi. (1.5 pts) In LIGO experiment, using a 4-km-long laser interferometer, the strain h (see question iii) was measured as a function of time; the result is given in the graph below. Using this graph and assuming that the masses of the two black holes were equal, estimate the mass of each of them numerically. Gravitational constant $G = 6.67 \times 10^{-11}\,\mathrm{m^3s^{-2}kg^{-1}}$; $c = 3.00 \times 10^8\,\mathrm{m/s}$.



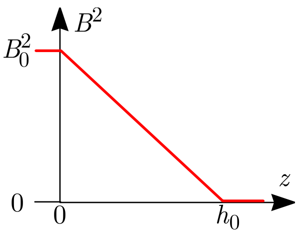
vii. (1.5 pts) Using the same data as for question vi, estimate the distance to these black holes.

Problem T3. Magnetars *(9 points)*

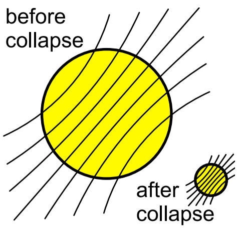
Magnetic fields are everywhere around us. Some typical magnetic B-field values: Earth's magnetic field: $25 - 60\,\mu$T; at Sunspots: 0.3 T; strong permanent magnets: around 1 T; continuously maintained magnetic fields in laboratory: up to 45 T; neutron stars and magnetars: up to 10^{11} T. In what follows we study few aspects of strong magnetic fields.



Magnetic field energy density $w = B^2 \frac{1}{2\mu\mu_0}$, where $\mu_0 \approx 1.3 \times 10^{-6}\,\mathrm{N/A^2}$ is the vacuum permeability, and μ — the relative permeability of the medium. System tries to move towards a lower energy state and so ferromagnetic materials with $\mu \gg 1$ are pulled towards regions with strong magnetic fields, and diamagnetic materials with $\mu < 1$ are pushed out. For diamagnetic materials, the magnetic *suspectibility* $\chi = \mu - 1$ is small, $|\chi| \ll 1$, and so the effect is small unless the field is strong. Water is a diamagnetic with $\chi = -9 \times 10^{-6}$ and animals are mostly made of water. So, a frog can levitate in a magnetic field if the field is strong enough, see the photo.



i. (1.5 pts) Let the frog height h_f be not more than $h_0 = 10\,\mathrm{mm}$, and let us assume simplifyingly that the squared magnetic field depends linearly on height z, see figure. Find how strong magnetic field B_0 (in Teslas) is needed to keep this frog in levitation. Assume that the frog is made entirely of water (density $\rho = 1000\,\mathrm{kg/m^3}$); free fall acceleration $g = 9.8\,\mathrm{m/s^2}$. *Hint:* for $|\chi| \ll 1$, we can write $w \approx B^2 \frac{1-\chi}{2\mu_0}$; hence, the energy density associated with the presence of water is $\Delta w = B^2 \frac{1-\chi}{2\mu_0} - B^2 \frac{1}{2\mu_0} = -B^2 \frac{\chi}{2\mu_0}$.

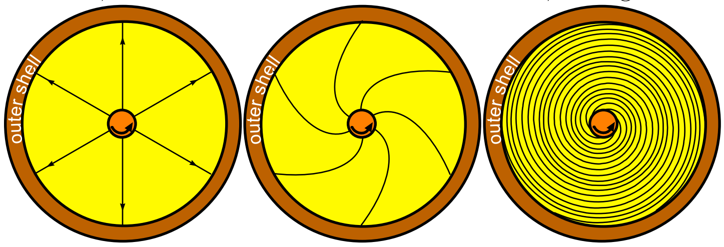


Stars are made of a plasma which is a good electrical conductor. Because of that, magnetic field lines behave as if being "frozen" into the moving plasma (this follows from the Faraday's induction law and Kirchoff's voltage law: due to the absence of electrical resistance, the voltage drop along a closed fictitious contour inside the plasma must be zero, hence the magnetic flux cannot change). If a star were to collapse into a neutron star, this effect would lead to an instantaneous increase of the magnetic field, see the sketch of the magnetic field lines before and after the collapse (recall that magnetic field strength is proportional to the density of field lines).

ii. (1 pt) Assuming that the polar magnetic field of a star is $B_s = 100\,\mu$T and its average density $\rho_s = 1400\,\mathrm{kg/m^3}$, what would be its polar magnetic field strength B_c after its collapse into a neutron star due to the compression of magnetic field lines as depicted above? The neutron star density $\rho_n = 5 \times 10^{17}\,\mathrm{kg/m^3}$.

iii. (1 pt) In reality, magnetic fields of neutron stars are generated differently. Let us consider a very simplified model. Interior part of the star has collapsed to a neutron star's size and density, but the exterior parts remains of the same size. Assume that before the collapse, the star was rotating as a solid body with angular speed ω_s. Express the new angular speed of the interior part of the star ω_n in terms of ω_s, ρ_s and ρ_n.

iv. (1.5 pts) Rotation speeds of the inner- and outer parts are different, hence the field lines will be stretched, see figure.



For the sake of simplicity: (a) we use 2-dimensional geometry, i.e. consider stars as being cylindrical; (b) while the initial field was a dipole field, we assume that it was cylindrically symmetric as shown in figure; (c) endpoints of field lines are attached to the inner cylinder (the neutron star) and to the outer cylindrical shell (the remnant of the original star). Let the initial magnetic field at the outer shell be B_0. Express the magnetic field B as a function of time t in the region where field lines are being stretched for $t \gg 1/\omega_n$ in terms of B_0 and ω_n.

v. (1 pt) So, the energy is converted during the star collapse as follows: gravitational energy is converted into kinetic one (let us neglect thermal energy), which is later on converted into the magnetic one. Based on this scenario, estimate the maximal strength of the magnetic field $B_{\max}$ for a neutron star of mass $M_n = 4 \times 10^{30}\,\mathrm{kg}$ and radius $R_n = 13\,\mathrm{km}$. Recall that $G = 6.67 \times 10^{-11}\,\mathrm{m^3 s^{-2} kg^{-1}}$.

vi. (1 pt) Very strong magnetic fields affect chemical properties of matter by changing the shape of electron orbits. This happens when the Lorenz force acting on an orbital electron becomes stronger than the Coulomb force due to the atomic nucleus. Estimate the strength of the magnetic field B_H needed to distort the electron orbit of an hydrogen atom which has radius $R_H = 5 \times 10^{-11}\,\mathrm{m}$. Note that $\frac{1}{4\pi\varepsilon_0} = 9 \times 10^9\,\mathrm{m}/F$, $e = 1.6 \times 10^{-19}\,\mathrm{C}$, and electron mass $m_e = 9.1 \times 10^{-31}\,\mathrm{kg}$.

vii. (2 pts) In very strong magnetic fields, atomic electron clouds take cylindrical shape. Estimate the length-to-diameter ratio $\kappa = l/d$ of such electron clouds for hydrogen atoms near a neutron star, in magnetic field $B_n = 10^8\,\mathrm{T}$. Note that the Planck's constant $h = 6.6 \times 10^{-34}\,\mathrm{J \cdot s}$. *Hint:* the radius of the cyclotron orbit for an electron in quantum-mechanical ground state can be estimated using uncertainty principle.