Mathematics for Computer Science
Exercise session 2, 7 September 2022

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Problems for Section 1.8

Problem 1.17.

Prove that $\log_4 6$ is irrational.

Problem 1.19.

Prove by contradiction that $\sqrt{3} + \sqrt{2}$ is irrational.
Hint: $(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$.

Problems from Section 2.2

Problem 2.2 (with some small changes).

The *Fibonacci numbers* $F(0), F(1), F(2), \ldots$ are defined as follows:

$$F(n) = \begin{cases} 0 & \text{if } n = 0\,, \\ 1 & \text{if } n = 1\,, \\ F(n-1) + F(n-2) & \text{if } n > 1\,, \end{cases}$$

Exactly which sentence(s) in the following bogus proof contain logical errors? Explain.

Theorem (Bogus theorem). *Every Fibonacci number is even.*

Bogus proof. Let all the variables n, m, k mentioned below be nonnegative integer valued.

1. Let $EF(n)$ mean that $F(n)$ is even.

2. Let C be the set of counterexamples to the assertion that $EF(n)$ holds for all $n \in \mathbb{N}$, namely,

$$C ::= \{n \in \mathbb{N} \mid \textbf{not}(EF(n))\}\,.$$

3. Assume C is nonempty. By WOP, it has a minimum m.

4. Then $m > 0$, since $F(0) = 0$ is an even number.

5. Since m is a minimum counterexample, $F(k)$ is even for all $k < m$.

6. In particular, $F(m-1)$ and $F(m-2)$ are both even.

7. But $F(m) = F(m-1) + F(m-2)$, and the right-hand side is even.

8. That is, $EF(m)$ is true, and m is not a true counterexample.

9. Then C is empty, and $F(n)$ is even for all $n \in \mathbb{N}$.

Problem 2.4.

Use the *Well Ordering Principle* to prove that

$$\sum_{k=0}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \tag{1}$$

for all nonnegative integers n.

Problems for Section 2.4

Problem 2.5

Use the Well Ordering Principle to prove that there is no solution over the positive integers to the equation:

$$4a^3 + 2b^3 = c^3\,.$$

Problems for Section 3.1

Problem 3.2.

Your class has a textbook and a final exam. Let P, Q, and R be the following propositions:

- P ::= "You get an A on the final exam."
- Q ::= "You do every exercise in the book."
- R ::= "You get an A in the class."

Translate following assertions into propositional formulas using P, Q, R, and the propositional connectives **and** , **not**(), **implies** .

(a) You get an A in the class, but you do not do every exercise in the book.

(b) You get an A on the final exam, you do every exercise in the book, and you get an A in the class.

(c) To get an A in the class, it is necessary for you to get an A on the final.

(d) You get an A on the final, but you don't do every exercise in this book; nevertheless, you get an A in this class.

Problem 3.5.

Sloppy Sam is trying to prove a certain proposition P. He defines two related propositions Q and R, and then proceeds to prove three implications:

$$P \textbf{ implies } Q\,, \quad Q \textbf{ implies } R\,, \quad R \textbf{ implies } P\,.$$

He then reasons as follows:

> If Q is true, then since I proved Q **implies** R, I can conclude that R is true. Now, since I proved R **implies** P, I can conclude that P is true. Similarly, if R is true, then P is true and so Q is true. Likewise, if P is true, then so are Q and R. So any way you look at it, all three of P, Q and R are true.

(a) Exhibit truth tables for

$$(P \textbf{ implies } Q) \textbf{ and } (Q \textbf{ implies } R) \textbf{ and } (R \textbf{ implies } P) \tag{2}$$

and for

$$P \textbf{ and } Q \textbf{ and } R\,. \tag{3}$$

Use these tables to find a truth assignment for P, Q, R so that (2) is **T** and (3) is **F**.

(b) You show these truth tables to Sloppy Sam and he says "OK, I'm wrong that P, Q and R all have to be true, but I still don't see the mistake in my reasoning. Can you help me understand my mistake?" How would you explain to Sammy where the flaw lies in his reasoning?

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Solutions

Problem 1.17.

By contradiction, assume $\log_4 6 = m/n$ for suitable positive in- tegers m, n. As we saw during the lecture, we may suppose also that m and n are positive and relatively prime. Then $6^n = 4^{n \log_4 6} = 4^m$, which is impossible, because 6^n is divisible by 3, and 4^m is not.

Important note: As 4 is not prime, we *cannot* conclude that, since 6 is not an integer power of 4, then $\log_4 6$ is irrational. For example, 8 is not an integer power of 4, but $\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2}$ is rational.

Problem 1.19.

We follow the hint and perform the multiplication:

$$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2}) = 3 - 2 = 1\,.$$

This means that $\sqrt{3} - \sqrt{2}$ is the multiplicative inverse of $\sqrt{3} + \sqrt{2}$. By contradiction, assume $\sqrt{3} + \sqrt{2} = m/n$ is rational. Then $\sqrt{3} - \sqrt{2} = n/m$ is rational too, and so is their difference $2\sqrt{2}$. But then, so is $\sqrt{2}$: contradiction.

If, instead of the difference $2\sqrt{2}$, we consider the sum $2\sqrt{3}$, we reach a similar contradiction. Indeed, an argument similar to our proof of the irrationality of $\sqrt{2}$ leads us to the conclusion that $\sqrt{3}$ is irrational.

Problem 2.2 (with some small changes).

The problem is with point 6. Until now, we only know that m is positive: it could well be 1. (It is so indeed, but that's not the point.) But if $m = 1$, then $m - 2 = -1$ is not a natural number; and we have only defined the Fibonacci numbers as a function on the naturals, not on all integers! For what we know, $F(m-2)$ might not exist.[1]

Problem 2.4.

First, a note on notation. Let a be an integer, and for each integer $k \geq a$ let x_k be a complex number. Then for $n \geq a$ integer the *sum, for k from a to*

[1] As a curiosity: it is possible to define the Fibonacci numbers on negative integers, and it turns out that it must be $F(-1) = 1$. More in general, if n is a positive integer, then $F(-n) = (-1)^{n-1} F(n)$.

n, *of* x_k is defined as follows:

$$\sum_{k=a}^{a} x_k = x_a\,;\quad \sum_{k=a}^{n} x_k = \left(\sum_{k=a}^{n-1} x_k\right) + x_n \quad \text{for every } n > a\,.$$

This is an example of a *recursive definition*, where the current value is constructed from the previous ones. We will see more of these in later chapters.

Let C be the set of counterexamples to (1), namely,

$$C ::= \left\{ n \in \mathbb{N} \mid \sum_{k=0}^{n} k^2 \neq \frac{n(n+1)(2n+1)}{6} \right\}.$$

If C is nonempty, then it has a minimum element m: such m must be positive, because for $n = 0$ both sides of (1) are zero. Since m is the minumum of C, $m - 1$, which is still a natural number as m is positive, *does* satisfy (1): we then have

$$\sum_{k=0}^{m-1} k^2 = \frac{(m-1)m(2(m-1)+1)}{6}\,.$$

But then,

$$\begin{aligned}
\sum_{k=0}^{m} k^2 &= \sum_{k=0}^{m-1} k^2 + m^2 \\
&= \frac{(m-1)m(2(m-1)+1)}{6} + m^2 \\
&= \frac{(m^2-m)(2m-1)+6m^2}{6} \\
&= \frac{2m^3-3m^2+m+6m^2}{6} \\
&= \frac{2m^3+3m^2+m}{6} \\
&= \frac{m(2m^2+3m+1)}{6} \\
&= \frac{m(m+1)(2m+1)}{6}\,:
\end{aligned}$$

that is, m *does* satisfy (1) after all. The contradiction stems from our hypothesis that C be nonempty: hence, C is empty, and (1) holds for every nonnegative integer m.

Problem 2.5

Let c_0 be the smallest positive integer such that positive integers a_0 and b_0 exist such that $4a_0^3 + 2b_0^3 = c_0^3$. We observe that $c_0 > 1$, because the left-hand side must be even: indeed, c_0 itself must be even, so it must be $c_0 = 2c_1$ for some positive integer c_1. We then have:

$$4a_0^3 + 2b_0^3 = 8c_1^3\,,$$

which, dividing by 2, yields:

$$2a_0^3 + b_0^3 = 4c_1^3\,.$$

Now, the right-hand side is even, so both summands on the left-hand side must be even: this means that b_0 must be even, so we write $b_0 = 2b_1$ for a suitable positive integer h. Again, we get, first, $2a_0^3 + 8b_1^3 = 4c_1^3$, then, dividing by 2,

$$a_0^3 + 4b_1^3 = 2c_1^3\,.$$

This time, with the same logic, $a_0 = 2a_1$ for a suitable positive integer a_1: substituting and replacing, we find...guess what?,

$$4a_1^3 + 2b_1^3 = c_1^3\,,$$

which is a solution over the positive integers with $c_1 < c_0$. (Note that we need to have proved that $c_0 > 0$; otherwise, $c_1 = c_0/2$ could have been zero as well.) We have thus discovered that the smallest counterexample c_0 was not the smallest: then there was no c_0 in the first place, and the equation does not have a solution on the positive integers.

Problem 3.2.

(a) In this case, R is verified, Q is not, and P is irrelevant: the assertion translates as R **and not**(Q).

(b) Here P, Q, and R are all verified, so this assertion translates as P **and** Q **and** R. Recall that **and** is associative, so (P **and** Q) **and** R is equivalent to P **and** (Q **and** R).

(c) Here we have a clear implication, and a causal one too! What the assertion says, is that if you get an A in the class, it means that you had gotten an A in the final: the translation in mathematical language is then R **implies** P.

(d) In this case, P is true, Q is false, and R is true: the assertion translates as P **and not**(Q) **and** R.

At the end of the exercise, observe how, in mathematical language, "but" and "nevertheless" mean the same as "and". The differences in tone of the three words are lost in translation.

Problem 3.5.

(a) We first construct the truth table for P **and** Q **and** R, as it is almost immediate:

P	Q	R	P **and** Q **and** R
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**F**	**T**	**F**
T	**F**	**F**	**F**
F	**T**	**T**	**F**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

For the formula

$$S ::= (P \textbf{ implies } Q) \textbf{ and } (Q \textbf{ implies } R) \textbf{ and } (R \textbf{ implies } P)$$

we proceed in two steps: first, we construct the truth values for each of the implications; then, we compute those for their conjunction.

P	Q	R	P **implies** Q	Q **implies** R	R **implies** P	S
T	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**F**	**T**	**F**
T	**F**	**T**	**F**	**T**	**T**	**F**
T	**F**	**F**	**F**	**T**	**T**	**F**
F	**T**	**T**	**T**	**T**	**F**	**F**
F	**T**	**F**	**T**	**F**	**T**	**F**
F	**F**	**T**	**T**	**T**	**F**	**F**
F	**F**	**F**	**T**	**T**	**T**	**T**

We then see that, if P, Q and R are all **F**, then (2) is **T** and (3) is **F**.

(b) Sam is silently assuming that *some* of P, Q and R are true. But why should it be so? All he has proved is that they are *equivalent:* either they all all true, or all false. To check which is the case, he must find a proof or disproof of any of the three, but at least one of them, which does *not* depend on the others, but only on other things which he *knows*, not just assumes, to be true.