ITB8832 Mathematics for Computer Science
First midterm test: 30 September 2022

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Exercise 1 (3 points)

Use the Well Ordering Principle to prove that:

$$\sum_{k=1}^{n}(2k-1) = n^2 \tag{1}$$

for every integer $n \geq 1$. **Important:** Any solution which does not use the Well Ordering Principle will receive zero points.

Exercise 2 (2 points)

Determine a disjunctive normal form for the following propositional formula:

$$(P \lor Q) \land (P \lor (\overline{Q} \land R)) \land (P \lor (\overline{Q} \land \overline{R})) \tag{2}$$

Any DNF for (2) will be accepted as a solution; it doesn't need to be full.

Exercise 3 (4 points total)

1. (2 points) Let A and B be finite sets such that $|A| = |B|$, and let $f : A \to B$ be a total function. Prove that f is injective if and only if it is surjective. *Hint:* any relation R with domain A is surjective if considered as a relation from A to $R(A)$.

2. (1 point) Give an example of a function $g : \{1,2,3\} \to \{1,2,3\}$ which is injective, but not surjective.

3. (1 point) Give an example of a total relation $R : \{1, 2, 3\} \to \{1, 2, 3\}$ which is surjective, but not injective.

You can solve the three points independently of each other; for example, you can solve points 2 and 3 without having solved point 1.

Exercise 4 (6 points overall)

For each of the following questions, mark the only correct answer:

1. Which one of the following numbers is irrational?
   (a) $\log_4 7$.
   (b) $\log_4 32$.
   (c) $\log_4 16$.
2. Which one of the following sets is well ordered?
   (a) $A ::= \{x \in \mathbb{Z} \mid \exists n \in \mathbb{N} \,.\, x^2 - n = 0\}$.
   (b) $B ::= \{x \in \mathbb{R} \mid \exists n \in \mathbb{N} \,.\, x = \frac{1}{n+1}\}$. [1]
   (c) $C ::= \{x \in \mathbb{Z} \mid x > -400\sqrt{17}\}$.
3. Which one of the following formulas is equivalent to P **implies** (Q **implies** P)?
   (a) (P **implies** Q) **implies** P.
   (b) ((P **implies** Q) **implies** P) **implies** P.
   (c) (P **or** Q) **implies** P.
4. Given as the domain of the discourse the arithmetics of natural numbers and as type for the variable x the set of natural numbers, which one of the following interpretations of $P(x)$ determines a counter-model for the predicate formula $(\exists x \,.\, P(x))$ **implies** $(\forall x \,.\, P(x))$?
   (a) "$x \geq 0$".
   (b) "$x + 1 = 0$".
   (c) "$x = 0$

[1] The original test incorrectly had $x \in \mathbb{Z}$: in that case, $B = \{1\}$ is well ordered.

5. Which one of the following relations is surjective?

   (a) $R : \mathbb{N} \to \mathbb{N}\,, xRy$ **iff** $y = x^3 + 1$.
   (b) $S : \mathbb{Z} \to \mathbb{Z}\,, xSy$ **iff** $y = x^3 + 1$.
   (c) $T : \mathbb{R} \to \mathbb{R}\,, xTy$ **iff** $y = x^3 + 1$.

6. Let A and B be finite sets and let $f : A \to B$ be a surjective function. Under these hypotheses, which one of the following is always true?

   (a) $|f(A)| = |B|$.
   (b) $|A| = |B|$.
   (c) $|f(A)| > |B|$.

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Solutions

Exercise 1

Let

$$C ::= \left\{ n \in \mathbb{N} \mid n \geq 1 \textbf{ and } \sum_{k=1}^{n} (2k-1) \neq n^2 \right\}$$

be the set of counterexamples to (1). By contradiction, assume C is nonempty: by the Well Ordering Principle, C has a smallest element m which, by construction, is a positive integer. It cannot be $m = 1$, because:

$$\sum_{k=1}^{1} (2k-1) = 2 - 1 = 1 = 1^2 \, .$$

Then $m \geq 2$, so $m - 1$ is still a positive integer, and as it is smaller than m, it is not a counterexample to (1). But then:

$$\begin{aligned} \sum_{k=1}^{m} (2k-1) &= \left(\sum_{k=1}^{m-1} (2k-1) \right) + (2m-1) \\ &= (m-1)^2 + 2m - 1 \\ &= m^2 - 2m + 1 + 2m - 1 = m^2 \, . \end{aligned}$$

Then the minimum counterexample m is not a counterexample after all. This is a contradiction, so C is empty and (1) holds for every positive integer n.

Exercise 2

We examine a solution with truth tables, and one with Boolean algebra.

- Truth table:

P	Q	R	$((P \vee Q)$	$\wedge$	$(P \vee$	$(\overline{Q} \wedge R)))$	$\wedge$	$(P \vee$	$(\overline{Q} \wedge \overline{R}))$
T	**T**	**T**	**T**	**T**	**T**	**F**	**T**	**T**	**F**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**	**T**	**F**
T	**F**	**T**	**T**	**T**	**T**	**T**	**T**	**T**	**F**
T	**F**	**F**	**T**	**T**	**T**	**F**	**T**	**T**	**T**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**	**F**	**F**
F	**F**	**T**	**F**	**F**	**T**	**T**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**	**T**	**T**

This gives the full DNF:

$$(P \wedge Q \wedge R) \vee (P \wedge Q \wedge \overline{R}) \vee (P \wedge \overline{Q} \wedge R) \vee (P \wedge \overline{Q} \wedge \overline{R})$$

It can also be observed from the truth table that the formula (1) is true if and only if P is true, so it is equivalent to P, which is already a DNF.

- Boolean algebra:

$$\begin{aligned}
& (P \vee Q) \wedge (P \vee (\overline{Q} \wedge R)) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge (Q \vee (\overline{Q} \vee R))) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge ((Q \vee \overline{Q}) \vee R)) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge (\mathbf{T} \vee R)) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & (P \wedge \mathbf{T}) \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & P \wedge (P \vee (\overline{Q} \wedge \overline{R})) \\
\longleftrightarrow\ & P
\end{aligned}$$

Exercise 3

1. To simplify notation, let $|A| = |B| = n$.

   Many of the proofs from your discussions are based on the following argument, which is correct:

   - As f is a function, it has the [≤ 1 **out**] property, and as it is total, it has the [≥ 1 **out**] property. Then the relational diagram of f has exactly $|A| = n$ arrows.

     If f is injective, then it has the [≤ 1 **in**] property. As there are n arrows and n elements of the codomain, the only possibility is that each $b \in B$ has *exactly* one arrow entering: then f also has the [$= 1$ **in**] property, so it is a bjection, so it is surjective.

     If f is surjective, then it has the [≥ 1 **in**] property. As there are n arrows and n elements of the codomain, the only possibility is that each $b \in B$ has *exactly* one arrow entering: then f also has the [$= 1$ **in**] property, so it is a bjection, so it is injective.

   Of course, me being me, I had devised a much more complicated one:

- First, assume that f is injective. Then f, considered as a function from A to $f(A)$, is a bijection, so $|f(A)| = |A| = n = |B|$. As $f(A) \subseteq B$, the only possibility is that $f(A) = B$, which in turn means that f is surjective. Note that we are using both facts that f is a function, and that it is total.

  Now, assume that f is not injective. Then there must be two elements a_1, a_2 of A such that $f(a_1) = f(a_2)$. Then:

$$\begin{aligned} |f(A)| &= |f(\{a_1, a_2\}) \cup f(A \setminus \{a_1, a_2\})| \\ &\leq |f(\{a_1, a_2\})| + |f(A \setminus \{a_1, a_2\})| \\ &\leq 1 + |A \setminus \{a_1, a_2\}| \\ &= 1 + (n-2) = n - 1 < n = |B| \ , \end{aligned}$$

  so f cannot be surjective. Note that we have used the fact that f is a function (where?) but not that it is total.

2. There are many examples possible. For one, the function $g : \{1,2,3\} \to \{1,2,3\}$ such that $g(1) = 2$ and $g(2)$ and $g(3)$ are undefined, is an injective function which is not surjective: this is possible because g is not total. In fact, the empty relation works as an example in this case.

3. There are many examples possible. For one, the "full relation" $R : \{1,2,3\} \to \{1,2,3\}$ such that xRy whatever $x, y \in \{1,2,3\}$ are, is a total relation which is surjective but not injective; this is possible because R is not a function.

Exercise 4

1. (a) **Yes:** $\log_4 7 = \frac{1}{\log_7 4}$ is the reciprocal of an irrational number, so it is irrational.
   (b) **No:** $32^2 = 4^5$, so $\log_4 32 = 5/2$.
   (c) **No:** $16 = 4^2$, so $\log_4 16 = 2$.

2. (a) **No:** in fact: $A = \mathbb{Z}$.
   (b) **No:** B itself doesn't have a minimum, because if $x = \frac{1}{n+1} \in B$, then $y = \frac{1}{(n+1)+1}$ is an element of B smaller than x.
   (c) **Yes:** every subset of the set of integers which is bounded from below is well ordered.

3. (a) **No:** if P is false and Q is true, then P **implies** (Q **implies** P) is true and P **implies** (Q **implies** P) is false.

   (b) **Yes:** the two formulas are both valid. This one is called *Peirce's law*; Charles Sanders Peirce (1839–1914) was an American philosopher, logician, and scientist.

   (c) **No:** if P is false but Q is true, then P **implies** (Q **implies** P) is true, but (P **or** Q) **implies** P is false.

4. (a) **No:** $\forall x \in \mathbb{N} \,.\, x \geq 0$ is true, and so is
   $(\exists x \in \mathbb{N} \,.\, x \geq 0)$ **implies** $(\forall x \in \mathbb{N} \,.\, x \geq 0)$.

   (b) **No:** $\exists x \in \mathbb{N} \,.\, x + 1 = 0$ is false, so
   $(\exists x \in \mathbb{N} \,.\, x + 1 = 0)$ **implies** $(\forall x \in \mathbb{N} \,.\, x + 1 = 0)$ is true.

   (c) **Yes:** $\exists x \in \mathbb{N} \,.\, x = 0$ is true, but $\forall x \in \mathbb{N} \,.\, x = 0$ is false, so
   $(\exists x \in \mathbb{N} \,.\, x = 0)$ **implies** $(\forall x \in \mathbb{N} \,.\, x = 0)$ is false.

5. (a) **No:** there is no $x \in \mathbb{N}$ such that $xR3$.

   (b) **No:** there is no $x \in \mathbb{Z}$ such that $xS3$.

   (c) **Yes:** given y, let x be the unique real solution of $x^3 - (y - 1) = 0$; then xTy.

6. (a) **Yes:** by definition of surjectivity, $f(A) = \{b \in B \mid \exists a \in A \,.\, f(a) = b\}$ is the entire set B. Actually, f doesn't even need to be a function.

   (b) **No:** it might be $A = \{1, 2\}$, $B = \{1\}$, $f(1) = f(2) = 1$.

   (c) **No:** $f(A)$ is a subset of B, and cannot have more elements than B has.