Mathematics for Computer Science
Self-evaluation exercises for Lecture 1

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Exercise 1.1 (cf. Problem 1.10(b) in the textbook)

Let w, x, y, z be nonnegative integers such that:

$$x^2 + y^2 + z^2 = w^2 \,. \tag{1}$$

Let P be the proposition "w is even" and let Q be the proposition "x, y, and z are all even". Prove that

$$P \textbf{ iff } Q \,,$$

that is, whatever is our choice of w, x, y, z such that (1) is satisfied, the proposition P is true if and only if the proposition Q is true.

Hint: What is the remainder of the division of m^2 by 4 if m is even? What is it if m is odd?

Exercise 1.2

We have seen in classroom a proof of the implication:

If $1 = -1$, then $2 = 1$.

Modify the argument to obtain a proof of the following implication:

If $2 + 2 = 5$, then I am the Pope.

(There are proofs available in the literature and on the Web, but it is good to try by oneself first.)

Exercise 1.3 (from "What Is the Name of This Book?" by Raymond Smullyan)

You meet two men, of whom you know that each one is either a *knight* who only makes true statements, or a *knave* who only makes false statements; however, you don't know whether they are knights or knaves.

You ask them: "Are you knights or knaves?" One of the two remains silent, but the other says: "We are both knaves."

What are they?

Exercise 1.4 (cf. Problem 1.19)

An integer m is a *divisor* of an integer n if there exists an integer k such that $m \cdot k = n$. Note that, with this definition, every integer is a divisor of 0.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_d x^d$ be a polynomial of degree $d \geq 1$ with integer coefficients. The *rational root theorem* says that, if for two relatively prime integers m, n the value m/n is a root of $p(x)$ (that is, $p(m/n) = 0$) then m is a divisor of the *constant term* a_0 and n is a divisor of the *leading coefficient* a_d.

1. Prove the rational root theorem.

2. Use the rational root theorem to prove that, if the integer k is not the rth power of some other integer, then the rth root of k is irrational. *Hint:* Prove the contrapositive.

Note: you *do not* need to have solved point 1 before you solve point 2, but you *must* use it.

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Solutions

Exercise 1.1

Let's follow the hint. If $m = 2n$ is even, then $m^2 = 4n^2$, and its remainder in the division by 4 is 0. If $m = 2n + 1$ is odd, then:

$$m^2 = (2n + 1)^2 = 4n^2 + 4n + 1 \,,$$

and the remainder in the division of m^2 by 4 is 1.

One possible selection of cases is made according to the number of summands on the left-hand side of (1). It can be done as follows:

1. If x, y, and y are all even, then on the one hand, Q is true; and on the other hand, as w^2 must be even as a sum of even summands, w must be even in the first place, so P is true. Summarizing, if x, y, and z are all even, then P and Q are both true.

2. If exactly one of x, y, and z is odd, then on the one hand, Q is false; and on the other hand, as the remainder of the division of the left-hand side by 4 is 1, w^2 must be odd, and w must be odd in the first place, so P is false too. Summarizing, if exactly one of x, y, and z is odd, then P and Q are both false.

3. If exactly two of x, y, and z are odd, then things get interesting. Indeed, in this case, the remainder of the division of the left-hand side by 4 is the sum of one 0 and two 1s, so it is 2; but then, the remainder of the division of w^2 by 4 must also be 2, and this is impossible, because such remainder can only be either 0 or 1. This means that, *if (1) is satisfied, then it is impossible that exactly two of x, y, and z are odd:* as we are assumeing that (1) is satisfied, this case simply never happens. There is nothing wrong with this: when we do a proof by cases, it is important that every instance of the problem belongs to at least one case, but not that every case includes some instance of the problem.

4. For reasons similar to the previous case, under the premise that (1) is satisfied, it is impossible that all of x, y, and z are odd.

Exercise 1.2

The following proof is due to the logician Bertrand Russell:

Suppose that $2 + 2 = 5$. By subtracting 2 to each side we obtain $2 = 3$. By swapping the sides we obtain $3 = 2$. By subtracting 1 to each side we obtain $2 = 1$.

Now, I and the Pope are two. Since $2 = 1$, I and the Pope are one: that is, I am the Pope.

Exercise 1.3

The statement "we are both knaves" has been made by someone who either only makes true statements or only makes false statements, so it has a definite truth value. It also cannot be true: otherwise, the man who made it must have been a knave, and knaves only make false statements. Then it is false, so the man who spoke must be a knave; as this man is a knave and they are not both knaves, the other man is a knight.

Exercise 1.4

1. By hypothesis,

$$a_0 + a_1 \cdot \frac{m}{n} + a_2 \cdot \frac{m^2}{n^2} + \ldots + a_d \cdot \frac{m^d}{n^d} = 0\,.$$

By multiplying both sides by n^d (which is not 0, because it is the denominator of a fraction) we obtain the equivalent equality:

$$a_0 n^d + a_1 m n^{d-1} + \ldots + a_d m^d = 0\,. \tag{2}$$

By moving $a_0 n^d$ on the right-hand side in (2), we obtain:

$$a_1 m n^{d-1} + \ldots + a_d m^d = -a_0 n^d\,.$$

The left-hand side is clearly divisible by m, and so must be the right-hand side. But by hypothesis, m and n do not have prime factors in common, so m must be a divisor of a_0.

The proof that n is a factor of a_d is similar, and left as an exercise.

2. We prove the contrapositive: if $\sqrt[r]{k}$ is rational, then k is the rth power of some integer. Write $\sqrt[r]{k} = m/n$; we can assume that the fraction m/n is irreducible. Then m/n is a rational root of the polynomial $p(x) = x^r - k$: by the rational root theorem, n must be a divisor of 1, which is only possible if $n = 1$ or $n = -1$. But in this case, m/n is either m or $-m$, so it is an integer: then either $k = m^r$ or $k = (-m)^r$.