Mathematics for Computer Science
Self-evaluation exercises for Lecture 2

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Last update: 21 August 2023

Exercise 2.1 (from the classroom test of 03.10.2018)

1. Prove that $\log_{20} 50$ is irrational.
2. Let $a > 1$ and $b > 1$ be integers. Can $\log_a b$ be rational if b is not a power of a?

Exercise 2.2 (from the midterm test of 07.10.2019)

Let a be a real number, different from 1. Use the Well Ordering Principle to prove that, for every nonnegative integer n,

$$1 + a + \ldots + a^n = \frac{1 - a^{n+1}}{1 - a} \,. \tag{1}$$

Important: solutions which do not use the Well Ordering Principle will receive zero points.

Exercise 2.3 (cf. Problem 2.21(a)-(d))

Indicate which of the following sets of numbers have a minimum element and which are well ordered. For those that are not well ordered, give an example of a subset with no minimum element.

(a) The integers $\geq -\sqrt{2}$.

(b) The rational numbers $\geq \sqrt{2}$.

(c) The set of rationals of the form $1/n$ where n is a positive integer.

(d) The set G of rationals of the form m/n where $m, n > 0$ and $n \leq g$, where g is a *googol* 10^{100}.

Exercise 2.4 (cf. Lemma 2.4.6 and Problem 2.20)

We have seen in Lecture 2 that the set:

$$\mathbb{F} = \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\}$$

is well ordered. Prove that the set:

$$\mathbb{N} + \mathbb{F} = \{ n + f \mid n \in \mathbb{N}, f \in \mathbb{F} \}$$

is also well ordered. *Hint:* start with considering:

$$T = \{ n \in \mathbb{N} \mid \text{there exists } f \in \mathbb{F} \text{ such that } n + f \in S \} , \tag{2}$$

where S is a nonempty set of $\mathbb{N} + \mathbb{F}$.

Exercise 2.5 (cf. problem 2.23)

Let S be a subset of the set of real numbers. An *infinite descent* in S is an infinite sequence $\{s_n \mid n \in \mathbb{N}\}$ of elements of S such that:

$$s_n > s_{n+1} \text{ for every } n \in \mathbb{N} . \tag{3}$$

Prove that S is well ordered if and only if it *does not* have an infinite descent.

Exercise 2.6 (from Raymond Smullyan's "The Gödelian Puzzle Book")

You meet a man whom you know to be either a *knight* who only makes true statements, or a *knave* who only makes false statements (but you don't know which of the two). The man makes the following statement:

"Today is not the first day on which I make this statement."

Is he a knight or a knave? *Hint:* choose a "good" subset of the set of natural numbers and use the Well Ordering Principle.

Exercise 2.7 (also from "The Gödelian Puzzle Book")

The Greek philosopher Zeno (489 BC–431 BC) proposed the following argument to "prove" that motion is impossible:

> Consider someone walking in a straight line from a starting point A to an end point B, distant from it a *stadion* (ancient measure unit corresponding to about 177 meters). Before they can reach B, they need to reach the middle point between A and B: call it A_1. Then they must reach the middle point between A_1 and B: call it A_2. Then they must reach the middle point between A and A_2: call it A_3. And so on. After any finite number of steps, the person will not have reached B. Then no finite number of steps will let them reach B, and this proves that motion is impossible.

Given that motion is actually possible, Zeno's argument must contain some mistakes. What is the first of them?

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Solutions

Exercise 2.1

1. By contradiction, assume $\log_{20} 50 = \dfrac{m}{n}$ with m and n integers. As both the base and the argument are larger than 1, the logarithm is positive, so we may suppose m and n positive; also, we can assume that $\gcd(m, n) = 1$.

   By hypothesis, $20^{m/n} = 50$, that is, $20^m = 50^n$. But $20 = 2^2 \cdot 5$ and $50 = 2 \cdot 5^2$, so the equality becomes:

   $$2^{2m} \cdot 5^m = 2^n \cdot 5^{2n} \,.$$

   This is only possible is $n = 2m$ and $m = 2n$: but then, $n = 4n$, which is only possible if $m = n = 0$ against the fact that n is the denominator in a fraction.

2. Yes, it is: it is sufficient that a and b are both powers of the same integer c. For example, if $a = 4 = 2^2$ and $b = 8 = 2^3$, then:

   $$\log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2} \,.$$

Exercise 2.2

Let C be the set of counterexamples to (1):

$$C = \left\{ n \in \mathbb{N} \mid 1 + a + \ldots + a^n \neq \frac{1 - a^{n+1}}{1 - a} \right\} \,.$$

By contradiction, assume that C is nonempty: by the Well Ordering Principle, it has a smallest element c_0. This smallest element must be positive, because for $n = 0$ the sum on the left-hand side of (1) is 1 and the fraction on the right-hand side is $\dfrac{1 - a^{0+1}}{1 - a} = 1$. But if c_0 is positive, then $c_0 - 1$ is nonnegative, and as it is smaller than c_0, it satisfies (1), that is:

$$1 + a + \ldots + a^{c_0 - 1} = \frac{1 - a^{c_0}}{1 - a} \,.$$

But by adding a^{c_0} to both sides of the equality we get:

$$\begin{aligned} 1 + a + \ldots + a^{c_0} &= \frac{1 - a^{c_0}}{1 - a} + a^{c_0} \\ &= \frac{1 - a^{c_0} + (1 - a)a^{c_0}}{1 - a} \\ &= \frac{1 - a^{c_0} + a^{c_0} - a^{c_0+1}}{1 - a} \\ &= \frac{1 - a^{c_0+1}}{1 - a}; \end{aligned}$$

that is, the minimum counterexample is not a counterexample after all. We have reached this contradiction because we had supposed that C is nonempty: therefore, C is empty, and (1) is true for every nonnegative integer n.

Exercise 2.3

(a) An integer m is greater or equal to $-\sqrt{2}$ if and only if it is greater or equal to -1. We know from the lecture that $\{m \in \mathbb{Z} \mid m \geq -1\}$ is well ordered.

(b) This set is not well ordered. To see why, put $x_0 = 2$, $x_1 = 1.5$, $x_2 = 1.42$, $x_3 = 1.415$, $x_4 = 1.4143$, and so on: in general, let x_n be made of the decimal writing of $\sqrt{2}$ up to the nth decimal digit rounded up. Then $x_n > \sqrt{2}$ for every $n \in \mathbb{N}$, but the set $S = \{x_n \mid n \in \mathbb{N}\}$ does not have a minimum, because for every element there is a strictly smaller element.

Alternatively: for every $n \geq 0$ let a_n be the truncation to the nth decimal digit of the decimal expansion of $\sqrt{2}$, so that $a_0 = 1$, $a_1 = 1.4$, $a_2 = 1.41$, $a_3 = 1.414$, and so on. Let $x_n = 3 - a_n$: then $x_n > \sqrt{2}$ for every $n \geq 0$, because

$$3 - a_n > 3 - \sqrt{2} > 3 - \frac{3}{2} = \frac{3}{2} > \sqrt{2}\,.$$

But the set $\{x_n \mid n \geq 0\}$ does not have a minimum, because if $m < n$ then $x_m > x_n$.

(c) This set is not well ordered: no point $x = 1/n$ can be the minimum, because $1/(n+1) < 1/n$ if n is a positive integer.

(d) The set G is well ordered! To see this, let:

$$a = 1 \cdot 2 \cdots (g - 1) \cdot g = g!$$

be the *factorial* of g, that is, the product of all positive integers from 1 to g included. Then, since $n \leq g$ when $x = m/n \in G$, for every such x the number ax is a positive integer; also, if $x \leq y$, then $ax \leq ay$. So, however given a nonempty subset S of G, the set $T = \{ax \mid x \in S\}$ is a nonempty subset of positive integers: if m is the minimum of T, then m/a is the minimum of S.

Exercise 2.4

Let S be a nonempty subset of $\mathbb{N} + \mathbb{F}$. Define T according to (2). As S is nonempty, T is nonempty: let n_0 be the smallest element of T. Now let:

$$U = \{f \in \mathbb{F} \mid n_0 + f \in S\}.$$

By construction, U is nonempty: let f_0 be its smallest element. We will prove that $n_0 + f_0$ is the smallest element of S.

To do this, let $s \in S$. By definition, there exist $n \in \mathbb{N}$ and $f \in \mathbb{F}$ such that $s = n + f$. By definition of n_0, it must be $n_0 \leq n$. We have two cases:

1. $n_0 < n$. Then $n_0 + f_0 < n_0 + 1 \leq n \leq n + f = s$.

2. $n_0 = n$. Then $f_0 \leq f$ by definition of f_0, so again $n_0 + f_0 \leq n_0 + f = n + f = s$.

Exercise 2.5 (cf. problem 2.23)

If a sequence such as in (3) exists, then the set of its terms does not have a minimum: however given an element, there will be another element which is strictly smaller. In this case, S has a subset which is not well ordered, so it is not well ordered.

If S is not well ordered, take a nonempty subset T of S which has no minimum. Choose $s_0 \in T$: as s_0 is not the minimum of T, there exists $s_1 \in T$ which is strictly smaller than s_0. Similarly, as s_1 is not the minimum of T, there exists $s_2 \in T$ which is strictly smaller than s_1. Iterating the procedure, we obtain a sequence of elements of S such as in (3). More in detail:

1. We choose the starting element $s_0 \in T$ as we want.

2. For every $n \in \mathbb{N}$, after we have chosen $s_n \in T$, we choose $s_{n+1} \in T$ so that it is smaller than s_n: this is possible, because T has no minimum, so in particular s_n is not the minimum of T.

The sequence $\{s_n \mid n \in \mathbb{N}\}$ which we obtained is an infinite descent in T, thus in S too.

Note that our proof made the silent assumption that the set S is nonempty. This is not problematic, because the definition of well-ordered set is that every *nonempty* subset has a minimum. As the empty set does not have any nonempty subset, it doesn't have nonempty subsets without a minimum either, so it is well ordered; it has no infinite descents either.

Exercise 2.6

Even if the statement is self-referential, we know that it has been made by either a knight or a knave, so it must have a truth value.

Count the days since the birth of the man, starting with day 0. Since there is a day (namely, today) when he made that statement, by the Well Ordering Principle there must have been a *first* day when he made it. But on that first day, the statement was false! Since knights only make true statements, the man is a knave.

Exercise 2.7

Raymond Smullyan explains it better than I ever could:

> This is about the only argument I know in which the first false step is the conclusion! Everything up there is correct—it is indeed true that to get from A to B one must go through infinitely many steps of the sort that Zeno describes, but so what? Zeno never proved that you cannot go through infinitely many points in a finite length of time, and it cannot be proved, because it is false!