Mathematics for Computer Science
Self-evaluation exercises for Chapter 3

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Exercise 3.1 (from the classroom test of 3 October 2018)

Find a disjunctive normal form for the following formula:

$$(P \textbf{ or } Q) \textbf{ implies } \textbf{not}(R \textbf{ and } P)$$

Use either a truth table, or logical equivalences.

Exercise 3.2 (cf. Problem 1.16(d))

A finite set of propositional formulas $X = \{P_1, \ldots, P_n\}$ is *consistent* if there exists an assignment of truth values to *all* the variables which appear in *any* formulas in which *all* propositions are true. For example:

- The set $\{P \textbf{ and } \textbf{not}(Q), Q \textbf{ or } R\}$ is consistent, because setting $P = \mathbf{T}$, $Q = \mathbf{F}$, and $R = \mathbf{T}$ makes both P **and not**(Q) and Q **or** R true.
- The set $\{A \textbf{ and } \textbf{not}(A)\}$ is not consistent, because A **and not**(A) is unsatisfiable.

Construct a formula S such that S is valid if and only if X is *not* consistent.

Exercise 3.3 (cf. Problem 3.18(c))

We have seen during the classroom exercises that every propositional formula can be rewritten as an equivalent formula where only the connectives **or** and **not**() appear. Consider now the operator **nand** defined by:

$$A \textbf{ nand } B ::= \textbf{not}(A \textbf{ and } B)\,.$$

Prove that every propositional formula can be rewritten as an equivalent formula where only the connective **nand** appears.

Exercise 3.4 (cf. Problem 3.28)

Express each of the following statements using quantifiers, logical connectives, and/or the following predicates:

- $P(x)$ ::= 'x is a monkey"
- $Q(x)$ ::= 'x is a 6.042 TA"
- $R(x)$ ::= 'x comes from the 23rd century"
- $S(x)$ ::= 'x likes to eat pizza"

where x ranges over all living things.

(a) No monkey likes to eat pizza.

(b) Nobody from the 23rd century dislikes eating pizza.

(c) All 6.042 TAs are monkeys.

(d) No 6.042 TA comes from the 23rd century.

(e) Does part (d) follows from parts (a), (b) and (c)? If so, give a proof. If not, give a counterexample.

(f) Translate into English: $\forall x \,.\, (R(x) \textbf{ or } S(x) \textbf{ implies } Q(x))$

(g) Translate into English:

$$\exists x \,.\, (R(x) \textbf{ and } \textbf{not}(Q)(x)) \textbf{ implies } \forall x \,.\, (P(x) \textbf{ implies } S(x))$$

Exercise 3.5 (from the classroom test of 3 October 2018)

Find a counter-model for the following predicate formula:

$$(\exists x \,.\, \forall y \,.\, (P(x) \textbf{ implies } Q(y))) \textbf{ implies } (\forall x \,.\, (P(x) \textbf{ implies } \exists y \,.\, Q(y))) \;.$$

Exercise 3.6 (cf. first midterm test of 2021)

Let F be a propositional formula depending on the propositional variables $P_1, P_2, \ldots, P_n$. Let now $G(x)$ be the predicate formula obtained by starting from F and replacing, for every i from 1 to n, every occurrence of the propositional variable P_i with a predicate $Q_i(x)$, where the variable x is the same for all predicates. For example:

- If $F ::= P_1$ **and** $(P_2$ **or** $P_3)$, then $G(x) ::= Q_1(x)$ **and** $(Q_2(x)$ **or** $Q_3(x))$.
- If $F ::= P_1$ **implies** $(P_2$ **implies** $P_1)$, then $G(x) ::= Q_1(x)$ **implies** $(Q_2(x)$ **implies** $Q_1(x))$.

Prove that if F is valid, then $\forall x \,.\, G(x)$ is also valid. (A predicate formula is valid if it doesn't have any counter-models.) *Hint:* proof by contraposition.

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Solutions

Exercise 3.1

For a truth table:

P	Q	R	$(P$ **or** $Q)$	**implies**	**not**$(R$ **and** $P)$
T	**T**	**T**	**T**	**F**	**F**
T	**T**	**F**	**T**	**T**	**T**
T	**F**	**T**	**T**	**F**	**F**
T	**F**	**F**	**T**	**T**	**T**
F	**T**	**T**	**T**	**T**	**T**
F	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**	**T**

Choosing the lines where the formula is true, we reach the following disjunctive normal form:

$$\begin{aligned}(P \textbf{ and } Q \textbf{ and } \overline{R}) \quad &\textbf{or} \quad (P \textbf{ and } \overline{Q} \textbf{ and } \overline{R}) \\ &\textbf{or} \quad (\overline{P} \textbf{ and } Q \textbf{ and } R) \\ &\textbf{or} \quad (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \\ &\textbf{or} \quad (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } R) \\ &\textbf{or} \quad (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R})\end{aligned}$$

For logical equivalences:

1. First, we rewrite the implication:

   $$((P \textbf{ or } Q) \textbf{ implies not}(R \textbf{ and } P)) \textbf{ iff } (\textbf{not}(P \textbf{ or } Q) \textbf{ or not}(R \textbf{ and } P))$$

2. Next, we apply de Morgan's laws to only have negation on single variables:

   $$\begin{aligned}\textbf{not}(P \textbf{ or } Q) \quad &\textbf{iff} \quad \overline{P} \textbf{ and } \overline{Q} \\ \textbf{not}(R \textbf{ and } P) \quad &\textbf{iff} \quad \overline{R} \textbf{ or } \overline{P}\end{aligned}$$

   and by applying associativity we get the following formula, equivalent to the original one:

   $$(\overline{P} \textbf{ and } \overline{Q}) \textbf{ or } \overline{R} \textbf{ or } \overline{P}$$

3. The formula above is a disjunction of conjunctions, so we can apply distributivity and the equivalence A **iff** A **and** $(B$ **or** **not**$(B))$ to rewrite each term of the disjunction as a conjunction so that P, Q and R, or their negations, appear exactly once:

$$\begin{aligned} \overline{P} \textbf{ and } \overline{Q} \quad &\textbf{iff} \quad \overline{P} \textbf{ and } \overline{Q} \textbf{ and } (R \textbf{ or } \overline{R}) \\ &\textbf{iff} \quad (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } R) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R}) \end{aligned}$$

$$\begin{aligned} \overline{R} \quad &\textbf{iff} \quad (P \textbf{ or } \overline{P}) \textbf{ and } \overline{R} \\ &\textbf{iff} \quad (P \textbf{ and } \overline{R}) \textbf{ or } (\overline{P} \textbf{ and } \overline{R}) \\ &\textbf{iff} \quad (P \textbf{ and } Q \textbf{ and } \overline{R}) \textbf{ or } (P \textbf{ and } \overline{Q} \textbf{ or } \overline{R}) \\ &\qquad \textbf{or } (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R}) \end{aligned}$$

$$\begin{aligned} \overline{P} \quad &\textbf{iff} \quad \overline{P} \textbf{ or } (Q \textbf{ or } \overline{Q}) \\ &\textbf{iff} \quad (\overline{P} \textbf{ and } Q) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q}) \\ &\textbf{iff} \quad (\overline{P} \textbf{ and } Q \textbf{ and } R) \textbf{ or } (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \\ &\qquad \textbf{or } (\overline{P} \textbf{ and } Q \textbf{ and } \overline{R}) \textbf{ or } (\overline{P} \textbf{ and } \overline{Q} \textbf{ and } \overline{R}) \end{aligned}$$

4. By substituting equivalent formulas and applying commutativity and absorption, we reach precisely the disjunctive normal form we have found earlier.

Exercise 3.2

We first consider a "dual" form of the problem by considering a formula T which is *satisfiable* (instead of valid) if and only if X is consistent. Such formula is clearly the conjunction of the finitely many formulas that appear in X:

$$T ::= P_1 \textbf{ and } P_2 \textbf{ and } \dots \textbf{ and } P_n \,.$$

Such formula is also *un*satisfiable if and only if X is *not* consistent. But we know that a formula is unsatisfiable if and only if its negation is valid. Then the formula S that we are looking for is simply the negation of T:

$$\begin{aligned} S ::= \textbf{not}(T) \quad &= \quad \textbf{not}(P_1 \textbf{ and } P_2 \textbf{ and } \dots \textbf{ and } P_n) \\ &\longleftrightarrow \quad \textbf{not}(P_1) \textbf{ or } \textbf{not}(P_2) \textbf{ or } \dots \textbf{ or } \textbf{not}(P_n) \,. \end{aligned}$$

Exercise 3.3

It is sufficient to prove that **or** and **not**() can be rewritten in terms of **nand**. The latter is easier: by absorption, A is equivalent to A **and** A, so **not**(A) is equivalent to **not**(A **and** A), which is simply A **nand** A. For **or**, we use Boolean algebra:

$$\begin{aligned} A \textbf{ or } B \quad & \textbf{iff} \quad \textbf{not}(\textbf{not}(A) \textbf{ and } \textbf{not}(B)) \\ & \textbf{iff} \quad \textbf{not}((A \textbf{ nand } A) \textbf{ and } (B \textbf{ nand } B)) \\ & \textbf{iff} \quad (A \textbf{ nand } A) \textbf{ nand } (B \textbf{ nand } B)\,. \end{aligned}$$

Exercise 3.4

(a) $\forall x\,.\,(P(x)$ **implies not**$(S(x)))$.

(b) $\forall x\,.\,(R(x)$ **implies** $S(x))$.

(c) $\forall x\,.\,(Q(x)$ **implies** $P(x))$.

(d) $\forall x\,.\,(R(x)$ **implies not**$(Q(x)))$.

(e) Yes, it does. Suppose parts (a), (b) and (c) are all true. By contradiction, assume that (d) is false. Then there exists an x_0 which is a 6.042 TA and comes from the 23rd century. On the one hand, as x_0 comes from the 23rd century, by (b), they like eating pizza. On the other hand, as x_0 is a 6.042 TA, by (c), they are a monkey. But then, x_0 is a monkey who likes eating pizza, which contradicts (a).

(f) Anyone who either comes from the 23rd century or likes to eat pizza is a 6.042TA.

(g) If there is someone who comes from the 23rd century but is not a 6.042TA, then every monkey likes to eat pizza.

Exercise 3.5

We want the main implication to be false, so the antecedent must be true and the consequent false. Now, if $P(x)$ is false for some x, then for *that* x and for *every* y the formula $P(x)$ **implies** $Q(y)$ is true; on the other hand, if $P(x)$ is true for some x but $Q(y)$ is false for every y, then for *that* x the formula $P(x)$ **implies** $\exists y\,.\,Q(y)$ is false.

Let then x and y take values in the set $\mathbb{N}$ of nonnegative integers; let $P(x) ::= x = 0$ and $Q(y) ::= x < 0$. Then the antecedent of the main implication becomes:

$$\exists x \in \mathbb{N} \,.\, \forall y \in \mathbb{N} \,.\, (x = 0 \textbf{ implies } y < 0)\,,$$

which is true, because we can set $x = 1$; but the consequent becomes

$$\forall x \in \mathbb{N} \,.\, (x = 0 \textbf{ implies } \exists y \in \mathbb{N} \,.\, y < 0)\,,$$

which is false, because for $x = 0$ the implication has a true antecedent and a false consequent.

Alternatively[1], as we only need two values for x and one for y, we could choose a domain where $x \in X = \{x_{\mathbf{T}}, x_{\mathbf{F}}\}$, $y \in Y = \{y_{\mathbf{F}}\}$, $P(x_{\mathbf{T}}) = \mathbf{T}$, $P(x_{\mathbf{F}}) = \mathbf{F}$, and $Q(y_{\mathbf{F}}) = \mathbf{F}$. Indeed, we could just choose $x_{\mathbf{T}} = \mathbf{T}$, $x_{\mathbf{F}} = y_{\mathbf{F}} = \mathbf{F}$, $P(x) ::= x$, and $Q(y) ::= y$. Then $P(x_{\mathbf{F}})$ **implies** $\forall y \,.\, Q(y)$ is interpreted as **F implies** $\forall y \,.\, Q(y)$, which is true; but with this choice of the type of y, $Q(y)$ can only be false, so $P(x_{\mathbf{T}})$ **implies** $\exists y \,.\, Q(y)$ is interpreted as **T implies** $\exists y \,.\, \mathbf{F}$, which is false.

Exercise 3.6

We prove the contrapositive: if $\forall x \,.\, G(x)$ *does* have a counter-model, then F is *not* valid.

Consider a domain D, a type X for the variable x, and an interpretation of the predicates $Q_1(x), \ldots, Q_n(x)$ that makes $\forall x \,.\, G(x)$ false. By definition, there exists $x_0 \in X$ such that $G(x_0)$ is false. Define the truth values of the variables P_i of F as being the same as those of the corresponding propositions $Q_i(x_0)$ in the counter-model we have defined: that is, if $Q_i(x_0)$ is true in the counter-model, then $P_i = \mathbf{T}$, and if $Q_i(x_0)$ is false in the counter-model, then $P_i = \mathbf{F}$. By construction, this assignment of truth values makes F false, so F is not valid.

[1] This variant was suggested by a student.