Some notes on Besicovitch and Weyl distances over higher-dimensional configurations

Silvio Capobianco[1]

Institute of Cybernetics at Tallinn University of Technology
Akadeemia tee 21, 12618 Tallinn, Estonia

Abstract

The Besicovitch and Weyl topologies on the space of configurations take a point of view completely different from the usual product topology; as such, the properties of the former are much different than that of the latter. The one-dimensional case has already been the subject of thorough studies; we carry it on on greater dimension.

Keywords: Pseudo-distance, Besicovitch, Weyl, cellular automaton

1 Introduction

The Besicovitch and Weyl pseudo-distances were introduced in the context of cellular automata as a way to overcome several unwanted properties of the ordinary product topology, not last the fact that any distance inducing it cannot be translation invariant. In the case of one-dimensional configurations—*i.e.*, bi-infinite words—Blanchard, Formenti and Kůrka [2] define two pseudo-distances on the space $\mathcal{C} = \{0,1\}^{\mathbb{Z}}$. The basic idea is to take a "window" of the form $U_n = [-n, \dots, n]$, and evaluate the *density* of the set of points under the window where two configurations take different values. From this basic idea, two quantities arise:

(i) For the Besicovitch pseudo-distance, the window is *kept in place*, progressively enlarged, and the upper limit d_B of the density computed.

(ii) For the Weyl pseudo-distance, the window is *moved all around* between enlargements, and the upper limit d_W of the *maximum density* computed.

One then considers the *quotient space* $\mathcal{C}_B$ (resp., $\mathcal{C}_W$), where two configurations $c, c' \in \mathcal{C}$ are identified iff $d_B(c, c') = 0$ (resp., $d_W(c, c') = 0$). Both spaces, at least in the one-dimensional case, behave very differently from the usual product space: for instance, they are both pathwise connected while $\mathcal{C}$ is totally disconnected. The

[1] Email: silvio@cs.ioc.ee

most interesting feature, however, is that cellular automata (ca) induce transformations on $\mathcal{C}_B$ and $\mathcal{C}_W$, whose properties can provide information on those of the original ca.

In this paper, which is ideally a continuation of both [2] and our previous work [4], we display some preliminary findings in our search for extensions of the results of [2] in the broader context of *finitely generated groups*, which includes all of the usual d-dimensional grids commonly used by ca theorists and practitioners. Besicovitch and Weyl pseudo-distances (namely, d_B and d_W) are defined by linking them to some *exhaustive sequence* of finite sets growing up to cover the whole group. In this setting, it is already known from [4] that d_B is translation invariant only under certain conditions, which are fulfilled when the group is $\mathbb{Z}^d$ and the exhaustive sequence is chosen as either the von Neumann or Moore disks. We focus on properties *of the spaces*, such as density of subsets and compactness; properties that belong to *single configurations*, such as those linked to occurrences of patterns, are not the subject of the present paper.

First, on $\mathbb{Z}^d$ the notion of convergence (and consequently, the space itself) does not depend on the choice of disks. Also, several properties known for the Besicovitch and Weyl space over $\mathbb{Z}$ remain true when moving to $\mathbb{Z}^d$. Finally, ca induce continuous transformations of the Besicovitch and Weyl space, and equicontinuity is preserved in the passage—with an improvement.

2 Background

Let G be a group. We write $H \leq G$ if H is a subgroup of G. The classes of the equivalence relation on G defined by $x \rho y$ iff $xy^{-1} \in H$ are called the **right cosets** of H. If U is a **set of representatives** of the right cosets of H (one representative per coset) then $(h, u) \mapsto hu$ is a bijection between $H \times U$ and G. The number $[G : H]$ of right cosets of $H \leq G$ is called the **index** of H in G.

Let $f, g : \mathbb{N} \to [0, +\infty)$. We write $f(n) \preccurlyeq g(n)$ if there exist $n_0 \in \mathbb{N}$ and $C, \beta > 0$ such that $f(n) \leq C \cdot g(\beta n)$ for all $n \geq n_0$; we write $f(n) \approx g(n)$ if $f(n) \preccurlyeq g(n)$ and $g(n) \preccurlyeq f(n)$. Observe that, if either f or g is a polynomial, the choice $\beta = 1$ is always allowed.

Call 1_G the identity of the group G. Product and inverse are extended to subsets of G element-wise. If $E \subseteq G$ is finite and nonempty, the **closure** and **boundary** of $X \subseteq G$ w.r.t. E are the sets $X^{+E} = \{g \in G : gE \cap X \neq \emptyset\} = XE^{-1}$ and $\partial_E X = X^{+E} \setminus X$, respectively; in general, $X \not\subseteq X^{+E}$ unless $1_G \in E$. $S \subseteq G$ is a **set of generators** if the graph $(G, \mathcal{E}_S)$, where $\mathcal{E}_S = \{(x, xz) : x \in G, z \in S \cup S^{-1}\}$, is connected. A group is **finitely generated** (briefly, f.g.) if it has a finite set of generators (briefly, f.s.o.g.). The **distance** between g and h w.r.t. S is their distance in the graph $(G, \mathcal{E}_S)$; the **length** of $g \in G$ w.r.t. S is its distance from 1_G. The **disk** of center g and radius r w.r.t. S will be indicated by $D_{r,S}(g)$; we will omit g if equal to 1_G, and S if irrelevant or clear from the context. Observe that $D_r(g) = gD_r$, and that $(D_{n,S})^{+D_{R,S}} = D_{n+R,S}$. For the rest of the paper, we will only consider f.g. infinite groups.

The **growth function** of G w.r.t. S is $\gamma_S(n) = |D_{n,S}|$. It is well-known [5] that $\gamma_S(n) \approx \gamma_{S'}(n)$ for any two f.s.o.g. S, S'. G is of **sub-exponential growth** if $\gamma_S(n) \preccurlyeq \lambda^n$ for all $\lambda > 1$; G is of **polynomial growth** if $\gamma_S(n) \approx n^k$ for some $k \in \mathbb{N}$. Observe that, if $G = \mathbb{Z}^d$, then $\gamma_S(n) \approx n^d$.

A sequence $\{X_n\}$ of *finite* subsets of G is **exhaustive** if $X_n \subseteq X_{n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} X_n = G$. $\{D_n\}$ is an exhaustive sequence. If $\{X_n\}$ is exhaustive and $U \subseteq G$, the **lower** and **upper density** of U w.r.t. $\{X_n\}$ are, respectively, the lower limit $\text{dens inf}_{\{X_n\}}U$ and the upper limit $\text{dens sup}_{\{X_n\}}U$ of the quantity $|U \cap X_n|/|X_n|$. An exhaustive sequence is **amenable** [5,6,8] if the limit of $|\partial_E X_n|/|X_n|$ is zero for every finite $E \subseteq G$; a group is amenable if it has an amenable sequence. If G is of sub-exponential growth, then $\{D_n\}$ contains an amenable subsequence, and is itself amenable if G is of polynomial growth (cf. [5]).

If $2 \leq |Q| < \infty$ and G is a f.g. group, the space $\mathcal{C} = Q^G$ of **configurations** of G over Q, endowed with the product topology, is homeomorphic to the *Cantor set*. For $c \in \mathcal{C}$, $g \in G$, $c^g \in \mathcal{C}$ is defined by $c^g(h) = c(gh)$ for all $h \in G$; transformations of $\mathcal{C}$ of the form $c \mapsto c^g$ for a fixed $g \in G$ are called **translations**. A **cellular automaton** (briefly, CA) over G is a triple $\mathcal{A} = \langle Q, \mathcal{N}, f\rangle$, where the **set of states** Q is finite and has at least two elements, the **neighborhood index** $\mathcal{N} \subseteq G$ is finite and nonempty, and the **local evolution function** f maps $Q^{\mathcal{N}}$ into Q. The map $F_{\mathcal{A}} : Q^G \to Q^G$ defined by

$$(F_{\mathcal{A}}(c))(g) = f\left(c^g|_{\mathcal{N}}\right) \tag{1}$$

is the **global evolution function** of $\mathcal{A}$. Observe that $F_{\mathcal{A}}$ is continuous in the product topology and commutes with translations. $\mathcal{A}$ is injective, surjective, and so on, if $F_{\mathcal{A}}$ is.

A **pseudo-distance** on a set X is a map $d : X \times X \to [0, +\infty)$ satisfying all of the axioms for a distance, except $d(x, y) > 0$ for every $x \neq y$. If d is a pseudo-distance on X, then $x_1 \sim x_2$ iff $d(x_1, x_2) = 0$ is an equivalence relation, and $d(\kappa_1, \kappa_2) = d(x_1, x_2)$ with $x_i \in \kappa_i$ is a distance on $X/\sim$.

Let $U, W \subseteq G$ be nonempty. A (U, W)**-net** is a set $N \subseteq G$ such that the sets xU, $x \in N$, are pairwise disjoint, and $NW = G$. Any subgroup is a (U, U)-net for any set U of representatives of its right cosets. For every nonempty $U \subseteq G$ there exists a (U, UU^{-1})-net. Moreover, if N is a (U, W)-net and $\phi(x) \in xU$ for every $x \in N$, then $\phi(N)$ is a $(\{1_G\}, U^{-1}W)$-net.

3 The Besicovitch and Weyl distances on f.g. groups

Given $c_1, c_2 \in Q^G$ and finite $U \subseteq G$ we define the *Hamming (pseudo-)distance* between c_1 and c_2 w.r.t. U as $H_U(c_1, c_2) = |\{x \in U \mid c_1(x) \neq c_2(x)\}|$. If $U = D_{n,S}$ we may write $H_{n,S}(c_1, c_2)$ instead of $H_{D_{n,S}}(c_1, c_2)$.

If $\{X_n\}$ is an exhaustive sequence for the group G, then

$$d_{B,\{X_n\}}(c_1, c_2) = \limsup_{n\in\mathbb{N}} \frac{H_{X_n}(c_1, c_2)}{|X_n|} \tag{2}$$

and

$$d_{W,\{X_n\}}(c_1, c_2) = \limsup_{n\in\mathbb{N}} \left(\frac{1}{|X_n|} \sup_{g\in G} H_{X_n}(c_1^g, c_2^g) \right) \tag{3}$$

are pseudo-distances on $\mathcal{C}$, and are distances if and only if G is finite. Otherwise, they are not continuous in the product topology, as shown by $c_k(g) = c(g)$ if and only if $g \in X_k$.

Definition 3.1 The quantity (2) is called the *Besicovitch distance* of c_1 and c_2 w.r.t. $\{X_n\}$. The quotient space $\mathcal{C}_{B,\{X_n\}} = \mathcal{C}/\sim_{B,\{X_n\}}$, where $c_1 \sim_{B,\{X_n\}} c_2$ iff $d_{B,\{X_n\}}(c_1, c_2) = 0$, is called the *Besicovitch space* induced by $\{X_n\}$. The *Weyl distance* and *Weyl space* are similarly defined according to (3).

If $X_n = D_{n,S}$ for some f.s.o.g. S, we write $d_{B,S}$ and $d_{W,S}$ instead of $d_{B,\{D_{n,S}\}}$ and $d_{W,\{D_{n,S}\}}$. S shall be skipped if irrelevant or clear from the context.

The definition of d_W seems to stray from the idea of "moving the window around"; in fact, it looks more like "keeping the window still, and moving the *configurations* around". The two viewpoints however, yield identical *observations* when it comes to the suprema: in fact, for any $g \in G$ and finite $U \subseteq G$, $H_U(c_1^g, c_2^g) = H_{gU}(c_1, c_2)$.

It is already known that several properties of d_B (and possibly d_W) depend on the properties of $\{X_n\}$: for instance, in [4] we show a case where d_B is not translation-invariant. The next result is thus rather surprising.

Theorem 3.2 $(\mathcal{C}_{B,\{X_n\}}, d_{B,\{X_n\}})$ *is a complete metric space.*

That is, if a sequence $\{c_k\}$ of configurations satisfies *Cauchy condition*

$$\forall \varepsilon > 0 \, \exists k_\varepsilon \mid d_{B,\{X_n\}}(c_k, c_{k+p}) < \varepsilon \, \forall k > k_\varepsilon \, \forall p \in \mathbb{N} \,, \tag{4}$$

then there exists a configuration c such that $\lim_{k\to\infty} d_{B,\{X_n\}}(c_k, c) = 0$.

Proof The following proof is modeled on that of [2, Proposition 2]: given a sequence $\{c_k\}$ satisfying (4), we pick up a "nice" subsequence and find a limit for it. It is then easy to check that the whole sequence converges to the same limit.

Let $R \geq 2$. Choose $\{k_m\}$ so that $d_B(c_{k_m}, c_{k_{m+1}}) < R^{-m-1}$ for all m. Let $\{\lambda_m\}$ satisfy the following properties:

(i) $|X_{\lambda_{m+1}}| \geq R \cdot |X_{\lambda_m}|$ for every $m \in \mathbb{N}$.

(ii) $\sup_{n \geq \lambda_m} \dfrac{H_{X_n}(c_{k_m}, c_{k_{m+1}})}{|X_n|} \leq R^{-m}$ for every $m \in \mathbb{N}$.

Such $\{\lambda_m\}$ exists because $\{X_n\}$ is exhaustive and $d_B(c_{k_m}, c_{k_{m+1}}) \leq R^{-m-1}$. Note that property (ii) implies

$$H_{X_n}(c_{k_m}, c_{k_{m+p}}) \leq |X_n| \cdot R^{-m} \cdot \frac{1 - R^{-p}}{1 - R^{-1}} \tag{5}$$

for all $p \geq 1$, $n \geq \lambda_{m+p}$.

Call $\Delta_m = X_{\lambda_{m+1}} \setminus X_{\lambda_m}$. Put

$$c(x) = \begin{cases} c_{k_m}(x) & \text{if } \; x \in \Delta_m \,, \\ \text{arbitrary} & \text{if } \; x \in X_{\lambda_0} \,. \end{cases} \tag{6}$$

We must prove that, if m is large enough, then $d_{B,S}(c_{k_m}, c)$ is arbitrarily small.

Let $n > \lambda_m$. Choose $M \geq m$ s.t. $n \in \{\lambda_M + 1, \dots, \lambda_{M+1}\}$. Then

$$\begin{aligned}
H_{X_n}(c_{k_m}, c) &= H_{X_{\lambda_m}}(c_{k_m}, c) + \sum_{i=m+1}^{M-1} H_{\Delta_i}(c_{k_m}, c) + H_{X_n \setminus X_{\lambda_M}}(c_{k_m}, c) \\
&= H_{X_{\lambda_m}}(c_{k_m}, c) + \sum_{i=m+1}^{M-1} H_{\Delta_i}(c_{k_m}, c_{k_i}) + H_{X_n \setminus X_{\lambda_M}}(c_{k_m}, c_{k_M}) \\
&\leq |X_{\lambda_m}| + \frac{1}{1 - R^{-1}} \sum_{i=m+1}^{M-1} |X_{\lambda_{i+1}}| R^{-m} + \frac{1}{1 - R^{-1}} |X_n| R^{-m} .
\end{aligned}$$

But because of property (i),

$$\sum_{i=m+1}^{M-1} |X_{\lambda_{i+1}}| \leq |X_{\lambda_M}| \cdot \sum_{j=1}^{M-m-1} R^{-j} \leq \frac{1}{1 - R^{-1}} |X_{\lambda_M}| .$$

Consequently, and since $R \geq 2$,

$$\frac{H_{X_n}(c_{k_m}, c)}{|X_n|} \leq \frac{|X_{\lambda_m}|}{|X_n|} + \frac{1}{(1 - R^{-1})^2} \cdot \frac{|X_{\lambda_M}|}{|X_n|} \cdot R^{-m} + \frac{1}{1 - R^{-1}} R^{-m}$$

for all $n \geq \lambda_m$, so that $d_{B,S}(c_{k_m}, c) \leq 6R^{-m}$ because of (5). □

Theorem 3.2 is surprising in that it is true whatever $\{X_n\}$ is. Completeness of $\mathcal{C}_B$ is especially remarkable, because this space is usually not compact. [2] We remark (cf. [2]) that $\mathcal{C}_W$ is not complete even when $G = \mathbb{Z}$ and $X_n = [-n, \dots, n]$.

Lemma 3.3 ([4, Lemma 3.10]) *Let $\{X_n\}$ be amenable and N be a (U, W)-net with $|U|, |W| < \infty$. Then* $\text{dens inf}_{\{X_n\}} N \geq 1/|W|$ *and* $\text{dens sup}_{\{X_n\}} N \leq 1/|U|$.

Theorem 3.4 *Let $\{X_n\}$ be an amenable sequence for G. Then $(\mathcal{C}_{B,\{X_n\}}, d_{B,\{X_n\}})$ is a perfect metric space.*

Proof Let $c \in \mathcal{C}$, $\varepsilon > 0$. Let $E \subseteq G$ be finite and $\varepsilon \cdot |E| > 1$. Let N be a (E, EE^{-1})-net. Let $c_\varepsilon \in \mathcal{C}$ satisfy $c_\varepsilon(g) = c(g)$ iff $g \notin N$. Then $d_{B,\{X_n\}}(c, c_\varepsilon) = \text{dens sup}_{\{X_n\}} N \in \left[1/|EE^{-1}|, 1/|E|\right] \subseteq (0, \varepsilon)$. □

In general, the classes of $\sim_B$ and $\sim_W$ depend on the choice of $\{X_n\}$. However, if the group "does not grow too fast" and the X_n's are disks, then all the f.s.o.g. for G determine the same *notion of convergence* for d_B and d_W.

Theorem 3.5 *Let G be a group of polynomial growth. Let S, S' be f.s.o.g. for G. There exists $C = C_{S,S'} > 0$ such that*

$$d_{B,S}(c_1, c_2) \leq C \cdot d_{B,S'}(c_1, c_2) \ \forall c_1, c_2 \in \mathcal{C} . \tag{7}$$

In particular:

[2] For metric spaces, compactness implies completeness.

(i) *If* $\lim_{k\to\infty} d_{B,S}(c_k, c) = 0$ *for some* S*, then* $\lim_{k\to\infty} d_{B,S}(c_k, c) = 0$ *for every* S*.*

(ii) *If* $d_{B,S}(c_1, c_2) = 0$ *for some* S*, then* $d_{B,S}(c_1, c_2) = 0$ *for all* S*.*

The above remain true if $d_{B,S}$ *is replaced with* $d_{W,S}$*.*

Proof Let d be such that $\gamma(n) \equiv n^d$. Choose $\alpha_S, \alpha_{S'}, n_0 > 0$ so that $\gamma_S(n) \geq \alpha_S \cdot n^d$ and $\gamma_{S'}(n) \leq \alpha_{S'} \cdot n^d$ for every $n > n_0$. Choose $\beta > 0$ so that $D_{1,S} \subseteq D_{\beta,S'}$. Put $C = C_{S,S'} = \alpha_{S'}\beta^d/\alpha_S$. It is straightforward to check that, whatever c_1 and c_2 are,

$$\frac{H_{n,S}(c_1, c_2)}{\gamma_S(n)} \leq C \cdot \frac{H_{\beta n,S'}(c_1, c_2)}{\gamma_{S'}(\beta n)} \tag{8}$$

for every $n > n_0$. Then $d_{B,S}(c_1, c_2) \leq C \cdot \limsup_n \dfrac{H_{\beta n,S'}(c_1, c_2)}{\gamma_S(\beta n)} \leq C \cdot d_{B,S'}(c_1, c_2)$. But (8) holds for any c_1 and c_2, so that

$$\max_{g\in G} \frac{H_{n,S}(c_1^g, c_2^g)}{\gamma_{S'}(n)} \leq C \cdot \max_{g\in G} \frac{H_{\beta n,S'}(c_1^g, c_2^g)}{\gamma_{S'}(\beta n)}$$

as well. Then $d_{W,S}(c_1, c_2) \leq C \cdot d_{W,S'}(c_1, c_2)$. □

A noteworthy property of the Besicovitch distance on $Q^{\mathbb{Z}}$, other than invariance by translations, is that it is positive between distinct periodic configurations. To extend such result to more general groups—whose geometry might sometimes defy intuition—we need a definition of periodicity that does not rely on the "shape of a period".

Definition 3.6 Let $c \in Q^G$. The *stabilizer* of c is the subgroup $\mathrm{St}(c) = \{g \in G \mid c^g = c\}$. c is *periodic* if $[G : \mathrm{St}(c)] < \infty$.

For instance, $c(x) = x \mod 2$ is periodic *because* it remains unchanged precisely when translated by an even number of steps, *i.e.*, its stabilizer is $2\mathbb{Z}$, which has index 2 in $\mathbb{Z}$. By a standard argument in group theory, if c_1 and c_2 are periodic and $H = \mathrm{St}(c_1) \cap \mathrm{St}(c_2)$, then $[G : H] \leq [G : \mathrm{St}(c_1)] \cdot [G : \mathrm{St}(c_2)]$.

Theorem 3.7 *Let* $\{X_n\}$ *be an amenable sequence for* G*. Let* c_1 *and* c_2 *be distinct periodic configurations. Then*

$$d_{B,\{X_n\}}(c_1, c_2) \geq \frac{1}{[G : \mathrm{St}(c_1) \cap \mathrm{St}(c_2)]^2} > 0\,. \tag{9}$$

Proof Let U be a set of representatives of the right cosets of $H = \mathrm{St}(c_1) \cap \mathrm{St}(c_2)$ in G: then $|U| < \infty$ and $c_1(u) \neq c_2(u)$ for some $u \in U$. Let $\phi(x) = xu$ for every $x \in H$: then $\phi(H)$ is a $(\{1_G\}, U^{-1}U)$-net and $c_1(g) \neq c_2(g)$ for all $g \in \phi(H)$. Then $d_{B,\{X_n\}}(c_1, c_2) \geq \mathrm{dens\,sup}_{\{X_n\}}\phi(H) \geq 1/|U^{-1}U| \geq 1/|U|^2$ because of Lemma 3.3.□

4 Besicovitch and Weyl spaces on $\mathbb{Z}^d$

In this section, unless differently stated, we will always suppose $G = \mathbb{Z}^d$ and $X_n = D_{n,\mathcal{M}_d}$, where $\mathcal{M}_d$ is the *d-dimensional Moore neighborhood*

$$\mathcal{M}_d = \left\{ z \in \mathbb{Z}^d \mid |z_i| \leq 1 \, \forall i \in \{1, \ldots, d\} \right\} . \tag{10}$$

Because of Theorem 3.5, the results hold for $(\mathcal{C}_{B,S}, d_{B,S})$ and $(\mathcal{C}_{W,S}, d_{W,S})$ whatever the f.s.o.g. S is. In this context, d_B is a straightforward extension of the Besicovitch distance as defined in [2]. In [3] we prove that this is also true for the Weyl distance.

We have seen that periodic configurations are separated by both d_B and d_W. We now see another difference between the Cantor space and the Besicovitch and Weyl spaces: there are configurations that cannot be "approximated with arbitrarily high precision" by periodic configurations.

Theorem 4.1 *The set of (classes of) periodic configurations is not dense in either* $(\mathcal{C}_B, d_B)$ *or* $(\mathcal{C}_W, d_W)$.

Proof Consider the sequence $X_n = \{-n, \ldots, n-1\}^d$. It is straightforward to check that $d_{B,\{X_n\}}(c_1, c_2) = d_{B,\mathcal{M}_d}(c_1, c_2)$ whatever c_1 and c_2 are.

Let $a, b \in Q$ with $a \neq b$. Consider $c \in Q^{\mathbb{Z}^d}$ defined by

$$c(x) = \begin{cases} a \text{ if } \ x_1 < 0 \, , \\ b \text{ if } \ x_1 \geq 0 \, . \end{cases} \tag{11}$$

Let c' be a periodic configuration. Since $\mathrm{St}(c')$ is a subgroup of finite index in the f.g. group $\mathbb{Z}^d$, it has itself a finite set Σ of generators. But any $\sigma \in \Sigma$ can be re-written as a linear combination of the e_i's, where $(e_i)_j = \delta^i_j$; consequently, $\mathrm{St}(c')$ has a sub-group generated by multiples of the e_i's. It is thus *not* restrictive to suppose that a period of c' is represented by a d-hypercube of the form $\{0, \ldots, L-1\}^d$.

Now, for any $x = (x_1, \ldots, x_d) \in X_{mL}$ with $x_1 \geq 0$, let $x' = (x_1 - mL, \ldots, x_d)$. Then either $c'(x) \neq c(x)$, or $c'(x') \neq c(x')$, or both. Thus, $\dfrac{H_{X_{mL}}(c, c')}{(2mL)^d} \geq \dfrac{1}{2}$ for any m, so that $d_{W,\mathcal{M}_d}(c, c') \geq d_{B,\mathcal{M}_d}(c, c') \geq 1/2$. □

For $d = 1$, [1, Proposition 9] indicates dense subspaces of $\mathcal{C}_B$: that of *Toeplitz* configurations and, consequently, that of *quasi-periodic* configurations.[3] None of these is dense in $\mathcal{C}_W$ (cf. [1, Proposition 14]).

We conclude the section with a simple topological result. We recall that a space X is *infinite-dimensional* if for every $n \in \mathbb{N}$ there exists a continuous embedding of $[0,1]^n$ into X. It is proved in [2] that $\mathcal{C}_{B,\mathcal{M}_1}$ and $\mathcal{C}_{W,\mathcal{M}_1}$ are infinite-dimensional when the set of states is $\{0,1\}$. Let $\pi : \{0,1\} \to Q$ be injective and define $f : \{0,1\}^{\mathbb{Z}} \to Q^{\mathbb{Z}^d}$ by $f(c)(z_1, \ldots, z_d) = \pi(c(z_1))$. Then $d_{B,\mathcal{M}_d}(f(c_1), f(c_2)) = d_{B,\mathcal{M}_1}(c_1, c_2)$ and $d_{W,\mathcal{M}_d}(f(c_1), f(c_2)) = d_{W,\mathcal{M}_1}(c_1, c_2)$, so f induces maps $\phi_n : [0,1]^n \to \mathcal{C}_{B,\mathcal{M}_d}$ and $\psi_n : [0,1]^n \to \mathcal{C}_{W,\mathcal{M}_d}$ which are both injective and continuous. We thus have

[3] A one-dimensional configuration is Toeplitz if each pattern is repeated periodically; it is quasi-periodic if each pattern occurs with bounded gaps.

Theorem 4.2 *$\mathcal{C}_B$ and $\mathcal{C}_W$ are infinite-dimensional. Consequently, they have the power of continuum.*

5 Cellular Automata on Besicovitch and Weyl spaces

Now that we have modified the space of configurations, we would like to be able to run cellular automata on them. A sufficient condition for a CA $\mathcal{A}$ to do this, is that $F_{\mathcal{A}}$ is *Lipschitz continuous* w.r.t. d_B (resp., d_W). Recall that $f : X \to X$ is Lipschitz continuous w.r.t. d if there exists $L > 0$ such that

$$d(f(x_1), f(x_2)) \leq L \cdot d(x_1, x_2) \; \forall x_1, x_2 \in X \,. \tag{12}$$

In [4, Theorem 3.7] we prove that any CA is Lipschitz continuous w.r.t. $d_{B,\{X_n\}}$, provided $\{X_n\}$ is either amenable or a sequence of disks. Since CA commute with translations, the argument used to prove [4, Theorem 3.7] can be adapted to work for the Weyl distance.

Theorem 5.1 *Let G be a f.g. group and let $\mathcal{A} = \langle Q, \mathcal{N}, f\rangle$ be a* CA *over G.*

(i) *If $\{X_n\}$ is amenable, then $F_{\mathcal{A}}$ satisfies (12) w.r.t. $d_{W,\{X_n\}}$ with $L = |\mathcal{N} \cup \{1_G\}|$.*

(ii) *If $\{X_n\} = \{D_{n,S}\}$ for some f.s.o.g. S, and $\mathcal{N} \subseteq D_{r,S}$, then $F_{\mathcal{A}}$ satisfies (12) w.r.t. $d_{W,\{X_n\}}$ with $L = (\gamma_S(r))^2$.*

From Theorem 5.1 and [4, Theorem 3.7] follows that, for any $c \in Q^{\mathbb{Z}^d}$ and any d-dimensional CA $\mathcal{A}$ with global rule $F_{\mathcal{A}}$, two Lipschitz continuous transformations $F_B : \mathcal{C}_B \to \mathcal{C}_B$ and $F_W : \mathcal{C}_W \to \mathcal{C}_W$ are well-defined, respectively, as $F_B([c]_B) = [F_{\mathcal{A}}(c)]_B$ and $F_W([c]_W) = [F_{\mathcal{A}}(c)]_W$. This remains true over more complex groups, provided that the sequence used to construct the distance is "good enough".

From Theorem 5.1 and [4, Theorem 3.7] follows another fact. Recall that, given a function $f : X \to X$, the *k-th iterate of f* is defined as $f^{(0)}(x) = x$ and $f^{(k+1)}(x) = f(f^{(k)}(x))$ for every $x \in X$.

Definition 5.2 Let d be a pseudo-distance on X and $f : X \to X$ be a function. f is *uniformly equicontinuous* (briefly, u.e.) on X w.r.t. d if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $d(x_1, x_2) < \delta$, then $d(f^{(k)}(x_1), f^{(k)}(x_2)) < \varepsilon$ for every $k \in \mathbb{N}$.

According to Definition 5.2, a map is u.e. when all its iterates are uniformly continuous with the same ε-δ relation. This is much more than requiring that all the iterates be continuous at x with the same ε-δ relation—*i.e.*, just being *equicontinuous at x*—for every $x \in X$. However, for compact systems such as CA under the product topology, the two notions coincide (cf. [1, Proposition 3]).

Let $\mathcal{A}$ be an equicontinuous CA. Then, whatever the f.s.o.g. S is, there exists ρ_0 such that, if $c_1|_{D_{\rho_0}} = c_2|_{D_{\rho_0}}$, then $F_{\mathcal{A}}^{(k)}(c_1)(1_G) = F_{\mathcal{A}}^{(k)}(c_2)(1_G)$ for every $k \in \mathbb{N}$. Consequently, any iterate $F_{\mathcal{A}}^{(k)}$ is the global evolution function of some CA of the form $\langle Q, D_{\rho_0}, f_k\rangle$.

We now apply Theorem 5.1. If either $\{X_n\}$ is amenable or $\{X_n\} = \{D_{n,S}\}$, then an upper bound for $d_{W,\{X_n\}}(F_{\mathcal{A}}^{(k)}(c_1), F_{\mathcal{A}}^{(k)}(c_2))$ is $(\gamma_S(\rho_0))^2 \cdot d_{W,\{X_n\}}(c_1, c_2)$. These are true whatever the iteration k is. Similarly, [4, Theorem 3.7] tells us that the same considerations hold for $d_{B,\{X_n\}}$. We have thus proved

Theorem 5.3 *Let $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ be a* CA *on G. Suppose $F_{\mathcal{A}}$ is equicontinuous at all points (equivalently, u.e.) in the product topology. Also suppose that the hypotheses in either point (i) or (ii) of Theorem 5.1 are satisfied. Then all the iterates of F_B and F_W are Lipschitz continuous with the same constant L. In particular, F_B and F_W are uniformly equicontinuous.*

6 Conclusion

The topic of translation-invariant pseudo-distances for CA spaces is relatively new but very appealing. This is just a short miscellany of preliminary results, and several conjectures on the properties of higher-dimensional CA in these settings are yet to be verified or refuted. In particular, we could not (yet) either prove or disprove that $\mathcal{C}_{B,\mathcal{M}_d}$ is pathwise connected for $d > 1$. We hope that our small contributions may provide ground for further results, and possibly draw more attention on these fascinating subjects.

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