# *Garden-of-Eden-like theorems for amenable groups*

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In the light of recent results by Bartholdi, we consider several properties that, for classical cellular automata, are known to be equivalent to surjectivity. We show that the equivalence still holds for amenable groups, and give counter-examples for non-amenable ones.

**Keywords:** cellular automata, amenability, group theory, topological dynamics, symbolic dynamics, ergodic theory, random theory

## 1   Introduction

Retrieving global properties of cellular automata (CA) has been a main topic of research since the field was established. Indeed, the Garden-of-Eden theorem by Moore [Moo62] and its converse by Myhill [Myh62], which link surjectivity of the global map of 2D CA to *pre-injectivity* (a property that may be described as the impossibility of erasing finitely many errors in finite time) also have the distinction of being the first rigorous results of cellular automata theory. Several more properties were later proved to be equivalent to surjectivity in $d$-dimensional CA, such as *balancedness* of the local map [MK76] and the sending of algorithmically random configurations into algorithmically random configurations [CHJW01].

With the subsequent efforts to extend the definition of CA to more general situations than the usual Euclidean lattices, an unexpected phenomenon appeared: the Garden-of-Eden property actually depends on properties of the involved groups! In particular, counterexamples to both Moore's and Myhill's theorem are well known for CA on the free group on two generators (cf. [CSMS99]). However, from a reading of the original proofs, a key fact emerges, which is crucial for the proofs themselves: in $\mathbb{Z}^d$, the size of a hypercube is a $d$-th power of the side, but the number of sites on its outer surface is a polynomial of degree $d - 1$. In other words, it seems that, to get Moore's or Myhill's theorem for CA on a group $G$, we need that in $G$ *the sphere grows more slowly than the ball*. What is actually sufficient is a slightly weaker property called *amenability*, which can be stated as the existence of a translation-invariant finitely additive

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probability measure on $G$. Bartholdi's theorem [Bar10] states then that the amenable groups are precisely those where surjective CA are pre-injective, and preserve the *product measure* on configurations.

In this paper, which illustrates work in progress, we extend the range of Bartholdi's theorem by characterizing amenable groups as those where surjective CA have additional properties. We start by considering *balancedness* [MK76], which is the combinatorial variant of measure preservation. We then include the *nonwandering* property, an important feature of dynamical systems. Finally, and for groups that have a decidable *word problem*, we prove that amenable groups are those where, in line with [CHJW01], CA preserve *descriptional complexity*.

To sum up, we get the following statement.

**Theorem 1** *Let $G$ be a finitely generated group. The following are equivalent.*

1. *$G$ is amenable.*

2. *Every surjective CA on $G$ is pre-injective.*

3. *Every surjective CA on $G$ preserves the uniform product measure.*

4. *Every surjective CA on $G$ is balanced.*

5. *Every surjective CA on $G$ is nonwandering.*

*If, in addition, $G$ has decidable word problem, then the above are equivalent to the following:*

- *Every surjective CA sends random configurations into random configurations.*

## 2  Preliminaries

### 2.1  Groups

Let $G$ be a group. We call $1_G$, or simply 1, its identity element. Given a set $X$, the family $\sigma = \{\sigma_g\}_{g \in G}$ of transformations of $X^G$, called *translations*, defined by

$$\sigma_g(c)(z) = c^g(z) = c(gz) \ \ \forall g \in G \tag{1}$$

is a *right action* of $G$ on $X^G$, that is, $\sigma_{gh} = \sigma_h \circ \sigma_g$ for every $g, h \in G$. This is consistent with defining the product $\phi\psi$ of functions as the composition $\psi \circ \phi$. Other authors (cf. [CSC10]) define $\sigma_g(c)(x)$ as $c(g^{-1}x)$, so that $\sigma$ becomes a *left* action. However, most of the definitions and properties we deal with do not depend on the "side" of the multiplication: we will therefore stick to (1).

A set of *generators* for $G$ is a subset $S \subseteq G$ such that for each $g \in G$ there is a word $w = w_1 \ldots w_n$ on $S \cup S^{-1}$ such that $g = w_1 \cdots w_n$. The minimum length of such a word is called *length* of $g$ w.r.t. $S$, and indicated by $\|g\|_S$, or simply $\|g\|$. $G$ is *finitely generated* (briefly, f.g.) if $S$ can be chosen finite. A group $G$ is *free* on a set $S$ if it is isomorphic to the group of reduced words on $S \cup S^{-1}$. For $r \geq 0$, $g \in G$ the *disk of radius $r$ centered in $g$* is $D_r(g) = \{h \in G \mid \|g^{-1}h\| \leq r\}$. The points of $D_r(g)$ can be "reached" from the "origin" $1_G$ by first "walking" up to $g$, then making up to $r$ steps: this is consistent with the definition of translations by (1), where to determine $c^g(z)$ we first move from 1 to $g$, then from $g$ to $gz$. We write $D_r$ for $D_r(1)$. We also put $U^{-r} = \{z \in G \mid D_r(z) \subseteq U\}$ and $\partial_{-r}U = U \setminus U^{-r}$. For our purposes, we will only consider f.g. groups.

A group $G$ is *residually finite* (briefly, r.f.) if for every $g \neq 1$ there exists a homomorphism $\phi : G \to H$ such that $H$ is finite and $\phi(g) \neq 1$. Equivalently, $G$ is r.f. if the intersection of all its subgroups of finite index is trivial. It follows from the definitions that, if $G$ is r.f. and $U \subseteq G$ is finite, then there exists $H \leq G$ s.t. $[G : H] \leq \infty$ and $U \cap H \subseteq \{1_G\}$.

**Lemma 2 ([Fio00, Lemma 2.3.2])** *Let $G$ be a residually finite (not necessarily f.g.) group and let $F$ be a finite subset of $G$ not containing $1_G$. Then there exists a subgroup $H_F$ of finite index in $G$, which does not intersect $F$, and such that the $H_F u$, $u \in F$, are pairwise disjoint.*

The *word problem* (briefly, w.p.) for a group $G$ with a set of generators $S$ is the set of words on $S \cup S^{-1}$ that represent the identity element of $G$. Although this set may depend on the choice of the presentation, its decidability does not; and although the problem is not decidable even for finitely generated groups, it is for the Euclidean groups $\mathbb{Z}^d$, the free groups, and more.

The *stabilizer* of $c$ is the subgroup $\mathrm{st}(c) = \{g \in G \mid c^g = c\}$: be aware, that $\mathrm{st}(c)$ might not be a normal subgroup. $c$ is *periodic* if $[G : \mathrm{st}(c)] < \infty$. If $[G : H] < \infty$ and $H \leq \mathrm{st}(c)$ we say that $c$ is *$H$-periodic*. The family of periodic configurations in $Q^G$ is indicated by $\mathrm{Per}(G, Q)$.

A group $G$ is *amenable* if it satisfies one of the following equivalent conditions:

1. There exists a finitely additive probability measure $\mu$ on $G$ with $\forall A \subseteq G, \forall g \in G, \mu(gA) = \mu(A)$.

2. For every finite $U \subseteq G$ and $\varepsilon > 0$ there exists a finite $K \subseteq G$ such that

$$|UK \setminus K| < \varepsilon|K| \tag{2}$$

Similar definitions want $\mu$ *right*-invariant and (2) replaced by $|KU \setminus K| < \varepsilon|K|$, or $\mu$ both left- and right-invariant and difference in (2) replaced by symmetric difference: in fact, all these definitions are equivalent. Also, if every f.g. subgroup of a given group is amenable, then the group is itself amenable.

A *bounded-propagation* $2 : 1$ *compressing map* over a group $G$ is a map $\phi : G \to G$ such that, for some finite *propagation set* $S \subseteq G$, $\phi(g)^{-1}g \in S$ for every $g \in G$, and $|\phi^{-1}(g)| = 2$ for every $g \in G$. In particular, such a map must be surjective, and $|S| \geq 2$. By [CSC10, Theorem 4.9.2], a group has a bounded-propagation $2 : 1$ compressing map if and only if it is *not* amenable. For instance, in the case of the free group over generators $a, b$, one can define: $\phi(x) = y$ if $x \notin \{a^n | n \in \mathbb{N}\}$ is written in an irreducible way as $yc$ for some $c \in \{a, b\}$; $\phi(x) = x$ otherwise. Here $S = \{1, a, b\}$ and any point $y$ has two preimages: $y$ and $yb$ if $y$ is written in an irreducible way as $wa^{-1}$ or $a^n$; $y$ and $ya$ if $y$ is written in an irreducible way as $wb^{-1}$; $ya$ and $yb$ otherwise.

## 2.2 Cellular automata

A *cellular automaton* (briefly, CA) on a group $G$ is a triple $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ where the *alphabet* $Q$ is a finite set, the *neighborhood index* $\mathcal{N} \subseteq G$ is finite and nonempty, and $f : Q^{\mathcal{N}} \to Q$ is a *local function*. This, in turn, induces a *global function* on any *configuration* $c : G \to Q$, defined by

$$F_{\mathcal{A}}(c)(g) = f\left(c^g|_{\mathcal{N}}\right) = f\left(c|_{g\mathcal{N}}\right) . \tag{3}$$

Through (3) we also consider, for every finite $E \subseteq G$, a function between patterns $f : Q^{E\mathcal{N}} \to Q^E$ defined by $f(p)_j = f(p|_{j\mathcal{N}})$. *Hedlund's theorem* [CSC10, Theorem 1.8.1] states that global functions of CA are exactly those functions from $Q^G$ to itself that commute with translations and are continuous

in the product topology. We recall that a base for this topology is given by the *cylinders* of the form $C(E, p) = \{c \in Q^G \mid c|_E = p\}$, with $E$ a finite shape of $G$ and $p : E \to Q$ a *pattern*: observe that, for countable groups, this base is countable. Also, the cylinders of the form $C(q, z) = \{c \mid c(z) = q\}$ form a (countable) subbase. If $p = c|_E$ we may write $C(p)$ instead of $C(c, E)$.

An *occurrence* of a pattern $p : E \to Q$ in $c \in Q^G$ is an element $g \in G$ such that $c^g|_E = p$; the pattern $p_g : gE \to Q$ defined by $p_g(gz) = p(z)$ is then a *copy* of $p$. For compactness reasons, a CA $\mathcal{A}$ has no *Garden-of-Eden configurations* (i.e., $c \in Q^G \setminus F_{\mathcal{A}}(Q^G)$) if and only if it has no *orphan patterns*, i.e., if every pattern has an occurrence in some $F_{\mathcal{A}}(c)$. Two configurations are *asymptotic* if they differ on at most finitely many points; a CA is *pre-injective* if distinct asymptotic configurations have distinct images. *Moore's Garden-of-Eden theorem* [Moo62] states that surjective CA on $\mathbb{Z}^d$ are pre-injective; *Myhill's theorem* [Myh62] states the converse implication.

A cellular automaton $\mathcal{A}$ over $Q^G$ is *nonwandering* if for any open set $U \subset Q^G$ there exists $t \geq 1$ such that $F_{\mathcal{A}}^t(U) \cap U \neq \emptyset$; it is *transitive* if for any two open sets $U$ and $V$ there exists some $t \geq 1$ such that $F_{\mathcal{A}}^t(U) \cap V \neq \emptyset$. (In particular, a transitive CA is nonwandering). A state $q_0 \in Q$ is *spreading* for $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ if for any $u \in Q^{\mathcal{N}}$ such that $u_i = q_0$ for some $i \in \mathcal{N}$ we have $f(u) = q_0$.

**Remark 3** *A nonwandering non-trivial* CA *has no spreading state.*

By non-trivial, we mean that $|\mathcal{N}| > 1$ and $|Q| > 1$. Indeed, take a cylinder $U = C(\mathcal{N} \cup \{1_G\}, c)$ where $c_i = q_0 \neq c_{1_G}$ for some $i \in \mathcal{N} \setminus \{1_G\}$: then $F^t(U) \cap U = \emptyset$ for any $t \geq 1$.

Let $\mathcal{N} \subseteq G \leq \Gamma$ and $f : Q^{\mathcal{N}} \to Q$. The triple $\langle Q, \mathcal{N}, f \rangle$ describes both a CA $\mathcal{A}$ over $G$ and a CA $\mathcal{A}'$ on $\Gamma$. We then say that $\mathcal{A}'$ is the CA *induced by $\mathcal{A}$ on $\Gamma$*, or that $\mathcal{A}$ is the *restriction of $\mathcal{A}'$ to $G$*.

## 2.3  Measures and randomness

Let $\Sigma$ be a $\sigma$-algebra on $Q^G$. If $\mu : \Sigma \to [0, 1]$ is a measure on $Q^G$, a measurable function $F : Q^G \to Q^G$ determines a new measure $F\mu : \Sigma \to [0, 1]$ defined as $F\mu(U) = \mu(F^{-1}(U))$. We say that $F$ *preserves* $\mu$ if $F\mu = \mu$. If $Q$ is finite, $G$ is countable, and $\Sigma$ is the *Borel $\sigma$-algebra* generated by the open sets, by standard facts in measure theory, a measure $\mu$ is completely determined by its value on the cylinders. In particular, the measure defined by $\mu_{\Pi}(C(E, p)) = |Q|^{-|E|}$ is called the *uniform product measure*, because it is a product of independent uniform measures on the alphabet. *Bartholdi's theorem* [Bar10] states that the amenable groups are precisely those where surjective CA preserve $\mu_{\Pi}$ and are pre-injective.

Let $\mu$ be some probability measure over $Q^G$. We say that a continuous function $F : Q^G \to Q^G$ is $\mu$-*recurrent* if for any measurable set $A \subset Q^G$ of measure $\mu(A) > 0$, there exists some time step $t \geq 1$ such that $\mu(A \cap F^t(A)) > 0$. If $\mu$ has full support, then this implies that $F$ is nonwandering. Moreover, the *Poincaré recurrence theorem* states that any $F$ that preserves $\mu$ is $\mu$-recurrent.

We say that $\mu$ is $F$-*ergodic* (or $F$ is $\mu$-*ergodic*) if $F$ preserves $\mu$ and every $F$-invariant set $U$ (*i.e.*, $F^{-1}(U) = U$) has $\mu(U) \in \{0, 1\}$. In that case, the *Birkhoff ergodic theorem* gives that $\mu$-*almost every point is $\mu$-typical for $F$*, that is,

$$\mu\left(\left\{x \in Q^G \mid \forall A \in \Sigma, \lim_{n \to \infty} \frac{1}{n}\left|A \cap \{F^t(x)|0 \leq t < n\}\right| = \mu(A)\right\}\right) = 1 . \tag{4}$$

Let $\mu$ and $\nu$ be $F$-ergodic measures; suppose they have a typical point $x$ in common. Then for any measurable set $A$, $\mu(A) = \lim_{n \to \infty} \frac{1}{n}\left|A \cap \{F^t(x)|0 \leq t < n\}\right| = \nu(A)$ : we have thus

**Lemma 4** *Any two distinct $F$-ergodic measures have no typical point in common.*

Let $\phi : \mathbb{N} \to G$ be a total computable enumeration. It is easy to see that it induces a computable enumeration of the cylinders, which we call $B' = \{B_i'\}_{i \geq 0}$ in accordance with [CHJW01].

Given any two sequences of open sets $\mathcal{U} = \{U_i\}_{i \geq 0}$, $\mathcal{V} = \{V_j\}_{j \geq 0}$, we say that $\mathcal{U}$ is $\mathcal{V}$-*computable* if there is a *recursively enumerable* set $A \subseteq \mathbb{N}$ s.t.

$$U_i = \bigcup_{j \in \mathbb{N}: \pi(i,j) \in A} V_j \ \ \forall i \geq 0 , \tag{5}$$

where $\pi(i,j) = (i+j)(i+j+1)/2 + i$ is the standard primitive recursive bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. A $B'$-computable family $\mathcal{U}$ of open sets is a *Martin-Löf $\mu$-test* (briefly, a M-L $\mu$-test) if $\mu(U_n) \leq 2^n$ for every $n \geq 0$. A configuration $c \in Q^G$ *fails* a M-L $\mu$-test $\mathcal{U}$ if $c \in \bigcap_{n \geq 0} U_n$. $c \in Q^G$ is $\mu$-*random* (in the sense of Martin-Löf) if it does not fail any M-L $\mu$-test. Note that, since the number of M-L $\mu$-tests is countable, the set of $\mu$-random configurations has full measure.

Given any pattern $p$, the set of configurations where $p$ has no occurrence is an intersection of a countably infinite, computable family of cylinders $U_i$ having equal product measure $\mu_\Pi(U_i) = m < 1$. It is then straightforward to construct a M-L $\mu_\Pi$-test that every such configuration fails. If we call *rich* a configuration in which any pattern occurs (or, equivalently, whose orbit under the shift action is dense), we then have the following.

**Remark 5** *Any $\mu_\Pi$-random configuration is rich.*

Note that $\phi : \mathbb{N} \to G$ induces $\phi^* : Q^G \to Q^\mathbb{N}$ defined by $\phi^*(c)(n) = c(\phi(n))$. If $\phi$ is a computable bijection, then so is $\phi^*$: in this case (cf. [GHR10, Proposition 2.5.2]) $\phi^*$ is continuous and preserves the product measure. In particular, $c$ is random for the product measure on $Q^G$ if and only if $\phi^*(c)$ is random for the product measure on $Q^\mathbb{N}$, and the set of random configurations has measure 1.

## 3 Results

According to Maruoka and Kimura [MK76], a $d$-dimensional CA with neighborhood a hypercube of radius $r$ is $n$-*balanced* if each pattern on a hypercube of side $n$ has $|Q|^{(n+2r)^d - n^d}$ pre-images. The authors then prove that a $d$-dimensional CA is surjective if and only if it is $n$-balanced for *every* $n$. On the other hand, the majority rule is 1-balanced but has the Garden-of-Eden pattern 01001.

The balancedness condition means that each pattern on a *given shape* has the *same number* of pre-images. (Just "patch" arbitrary shapes to "fill" a hypercube.) This works for CA over arbitrary groups.

**Definition 6** *Let $G$ be a group and let $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ be a CA on $G$. $\mathcal{A}$ is* balanced *if for every finite nonempty $E \subseteq G$ and pattern $p : E \to Q$,*

$$|f^{-1}(p)| = |Q|^{|E\mathcal{N}| - |E|}. \tag{6}$$

Since the r.h.s. in (6) is always positive, no pattern is an orphan for a balanced CA. In [CSMS99], two CA on the free group on two generators are shown, one being surjective but not pre-injective, the other pre-injective but not surjective: both have an unbalanced local function. Therefore, balancedness in general groups is *strictly stronger* than surjectivity, and possibly uncorrelated with pre-injectivity.

**Remark 7** *A cellular automaton is balanced if and only if it preserves the uniform product measure.*

The proof is similar to that in [CHJW01]. In fact, let $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ and $p : E \to Q$: then $\mu_\Pi(F_{\mathcal{A}}^{-1}(C(E,p))) = \sum_{f(p')=p} |Q|^{-|E\mathcal{N}|}$. But balancedness means r.h.s. has $|Q|^{|E\mathcal{N}|-|E|}$ summands whatever $p$ is, while preservation of $\mu_\Pi$ means l.h.s. equals $|Q|^{-|E|}$ whatever $p$ is.

By [CSC09, Theorem 1.2] several important properties, including injectivity and surjectivity, are preserved by induction and restriction: this is also true for balancedness.

**Remark 8** *Let $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ be a CA on $G \leq \Gamma$ and $\mathcal{A}'$ the CA induced by $\mathcal{A}$ on $\Gamma$. Then $\mathcal{A}$ is balanced if and only if $\mathcal{A}'$ is balanced.*

**Proof:** If $\mathcal{A}'$ is balanced, then $\mathcal{A}$ clearly is. Suppose then $\mathcal{A}$ is balanced; let $J$ be a set of representatives of the left cosets of $G$ in $\Gamma$. Let $E \subseteq \Gamma$: put $J_E = \{ j \in J \mid jG \cap E \neq \emptyset \}$. Then $E = \bigsqcup_{j \in J_E} (jG \cap E)$ and, since $\mathcal{N} \subseteq G$, $E\mathcal{N} = \bigsqcup_{j \in J_E} (jG \cap E\mathcal{N})$, with $J_E$ finite since $E$ is. Given $p : E \to Q$, call $p_j = p|_{jG \cap E}$ for $j \in J_E$. Then, since $\mathcal{A}'$ operates slicewise and $\mathcal{A}$ is balanced,

$$|f^{-1}(p)| = \prod_{j \in J_E} |f^{-1}(p_j)| = \prod_{j \in J_E} |Q|^{|jG \cap E\mathcal{N}| - |jG \cap E|} = |Q|^{\sum_{j \in J_E} |jG \cap E\mathcal{N}| - \sum_{j \in J_E} |jG \cap E|} ,$$

which is precisely $|Q|^{|E\mathcal{N}|-|E|}$. Since $E$ and $p$ are arbitrary, $\mathcal{A}'$ is balanced. $\square$

With the next statement, we strengthen [Wei00, Theorem 1.3], which states that injective CA on r.f. groups are surjective. We rely on a lemma which is immediate to prove.

**Lemma 9** *If $F : Q^G \to Q^G$ commutes with translations, then $\mathrm{st}(c) \subseteq \mathrm{st}(F(c))$ for every $c \in Q^G$. In particular, if $F$ is bijective then $\mathrm{st}(c) = \mathrm{st}(F(c))$.*

**Theorem 10** *Let $G$ be a residually finite group and $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ an injective CA over $G$. Then $\mathcal{A}$ is balanced.*

**Proof:** Let $E$ be a finite subset of $G$: it is not restrictive to suppose $1 \in E \cap \mathcal{N}$, so that $E, \mathcal{N} \subseteq E\mathcal{N}$. Suppose, for the sake of contradiction, that $p : E \to Q$ satisfies $|F_{\mathcal{A}}^{-1}(p)| = M > |Q|^{|E\mathcal{N}|-|E|}$. Since $G$ is residually finite, by Lemma 2 there exists a subgroup $H \leq G$ of finite index such that $H \cap E\mathcal{N} = H \cap \mathcal{N} = \{1\}$ : if $J$ is a set of representatives of the right cosets of $H$ such that $E\mathcal{N} \subseteq J$, then

$$|\{\pi : J \to Q \mid F_{\mathcal{A}}(\pi)|_E = p\}| = M \cdot |Q|^{[G:H]-|E\mathcal{N}|} > |Q|^{[G:H]-|E|} . \tag{7}$$

The r.h.s. in (7) is the number of $H$-periodic configurations that coincide with $p$ on $E$. Since $\mathcal{A}$ is injective and $G$ is r.f., by [Wei00, Theorem 1.3] $\mathcal{A}$ is reversible, and by Lemma 9, $F_{\mathcal{A}}$ sends $H$-periodic configurations into $H$-periodic configurations. But because of (7) and the pigeonhole principle, there must exist two $H$-periodic configurations with the same image according to $F_{\mathcal{A}}$: which contradicts injectivity of $\mathcal{A}$. $\square$

The proof of Moore's and Myhill's theorems for CA on amenable groups given in [CSMS99] is based on the following lemma.

**Lemma 11 ([CSMS99, Step 1 in proof of Theorem 3])** *Let $G$ be an amenable group, $q \geq 2$, and $n > r > 0$. For $L = D_n$ there exist $m > 0$ and $B \subseteq G$ such that $B$ contains $m$ disjoint copies of $L$ and*

$$(q^{|L|} - 1)^m \cdot q^{|B|-m|L|} < q^{|B^{-r}|} . \tag{8}$$

We use Lemma 11 to get a combinatorial proof of the equivalence between surjectivity and balancedness, that was already essentially stated in [Bar10].

**Theorem 12** *Let $G$ be an amenable group and let $\mathcal{A}$ a CA on $G$. If $\mathcal{A}$ is surjective then $\mathcal{A}$ is balanced.*

**Proof:** Put $L = D_n$, $L' = D_{n-r}$, $q = |Q|$. Suppose, for the sake of contradiction, that $\mathcal{A}$ is not balanced. Then, for suitable $n$, there is a pattern $p : L' \to Q$ that has at most $q^{|L|-|L'|} - 1$ pre-images. Let $m$ and $B$ be as by Lemma 11. Consider the patterns on $B$ whose image under the global rule of $\mathcal{A}$ coincides with $p$ on each of the $m$ copies of $L'$ contained in those of $L$: their number $t$ is at most

$$\left( q^{|L|-|L'|} - 1 \right)^m q^{|B|-m|L|} .$$

However, $\left( q^{|L|-|L'|} - 1 \right) \le q^{-|L'|} \left( q^{|L|} - 1 \right)$, so that, by Lemma 11,

$$t \le q^{-m|L'|} \left( q^{|L|} - 1 \right)^m q^{|B|-m|L|} < q^{|B^{-r}|-m|L'|} .$$

But the last term is precisely the number of patterns on $B^{-r}$ that coincide with $p$ on each of the given $m$ copies of $L'$. There are more of these than available pre-images, so one of them must be an orphan. $\square$

Thanks again to Lemma 11, [CHJW01, Point 1 of Theorem 4.4] generalizes to amenable groups.

**Proposition 13** *Let $G$ be an amenable group and let $\mathcal{A} = \langle Q, D_r, f \rangle$, $r > 0$, be a CA on $G$. If $c$ is not rich then $F_{\mathcal{A}}(c)$ is not rich.*

**Proof:** Suppose there is a pattern with support $L = D_n$, $n > r$, that does not occur in $c$. Choose $m$ and $B$ according to Lemma 11. By hypothesis, the number of patterns with support $B$ that occur in $c$ is at most $(q^{|L|} - 1)^m q^{|B|-m|L|}$, with $q = |Q|$; therefore, the number of patterns with support $B \setminus \partial_r B$ which occur in $F_{\mathcal{A}}(c)$ cannot exceed this number too. By Lemma 11, this is strictly less than $q^{|B|-|\partial_r B|}$, which is the total number of patterns with support $B \setminus \partial_r B$: hence, some of those patterns do not occur in $F_{\mathcal{A}}(c)$. $\square$

We now consider another property that, for CA on $\mathbb{Z}^d$, is equivalent to surjectivity: sending $\mu_\Pi$-random configurations into $\mu_\Pi$-random configurations. Before going ahead, we must remember that, according to [CHJW01], the definition of a random configuration on $\mathbb{Z}^d$ depends on the existence (and choice!) of a total computable bijection from $\mathbb{N}$ to $\mathbb{Z}^d$. This is still ensured for a general group $G$ when it has a decidable word problem: we thus can first enumerate $D_0 = \{1_G\}$, then $D_1 \setminus D_0$, then $D_2 \setminus D_1$, and so on.

The proofs of the following two statements are then similar to the original ones in [CHJW01]

**Lemma 14** *Let $G$ be a group with decidable word problem, $\mathcal{U}$ a $B'$-computable sequence, and $\mathcal{A}$ a CA on $G$. Then $F_{\mathcal{A}}^{-1}(\mathcal{U})$ is a $B'$-computable sequence.*

**Proof:** Let $A$ be a r.e. set such that $U_i = \bigcup_{\pi(i,j) \in A} B'_j$ for every $i \ge 0$, where the $B'_j$ are cylinders. Since $\mathcal{A}$ is a CA, $F_{\mathcal{A}}^{-1}(U_i)$ is itself a union of cylinders: such union is computable because $G$ has decidable word problem. By exploiting these facts and the primitive recursive functions $L, K : \mathbb{N} \to \mathbb{N}$ such that $\pi(L(n), K(n)) = n$ for every $n \ge 0$, we can construct a r.e. set $Z$ such that $F_{\mathcal{A}}^{-1}(U_i) = \bigcup_{\pi(i,j) \in Z} B'_j$ for every $i \ge 0$. $\square$

**Proposition 15** *Let $G$ be a group with decidable word problem and $\mathcal{A}$ a* CA *over $G$. If $F_{\mathcal{A}}(c)$ is $\mu_\Pi$-random whenever $c$ is, then $\mathcal{A}$ is surjective. If $\mathcal{A}$ preserves $\mu_\Pi$, then $F_{\mathcal{A}}(c)$ is $\mu_\Pi$-random when $c$ is.*

**Proof:** Since $\mu_\Pi$-random configurations form a set of measure 1 and contain occurrences of any pattern, the first part is immediate. For the second part, if $F_{\mathcal{A}}\mu_\Pi = \mu_\Pi$, then by Lemma 14 the preimage of a M-L $\mu_\Pi$-test is still a M-L $\mu_\Pi$-test: but if $F_{\mathcal{A}}(c)$ fails $\mathcal{U}$, then $c$ fails $F_{\mathcal{A}}^{-1}(\mathcal{U})$.                             □

From Proposition 15 combined with Theorem 12 follows

**Corollary 16** *Let $G$ be an amenable group with decidable word problem and $\mathcal{A}$ be a surjective* CA *on $G$. If $c$ is $\mu_\Pi$-random then $F_{\mathcal{A}}(c)$ is $\mu_\Pi$-random.*

What is the role of amenability in all this? Could this happen on non-amenable groups as well? The following counterexample shows that this is not the case.

**Example 17 (Surjective CA with a spreading state)** *Let $G$ be a non-amenable group; let $\phi$ be a bounded-propagation $2 : 1$ compressing map with propagation set $S$. Let $\preceq$ be a total ordering of $S$ and let $Q = S \times \{0, 1\} \times S \sqcup \{q_0\}$, where $q_0 \notin S \times \{0, 1\} \times S$. Let $\mathcal{A} = \langle Q, S, f \rangle$ with:*

$$f : \quad Q^S \quad \to \quad Q$$
$$u \quad \mapsto \quad \begin{cases} q_0 & \text{if } \exists s \in S, u_s = q_0, \\ (p, \alpha, q) & \text{if } \exists!(s, t) \in S \times S, s \prec t, u_s = (s, \alpha, p), u_t = (t, 1, q), \\ q_0 & \text{otherwise.} \end{cases}$$

*Then $\mathcal{A}$ admits the spreading state $q_0$, and at least one other state, hence it is not nonwandering. Nevertheless, it is surjective.*

**Proof:** Let $x \in Q^G$, $i \in G$, $j = \phi(i)$: then $i = js$ for some $s \in S$, and there exists a unique $t \in S \setminus \{s\}$ such that $\phi(jt) = j$. If $x_j = q_0$, then set $y_i = (s, 0, s)$: otherwise, we can write $x_j = (p, \alpha, q)$. If $s \prec t$, then set $y_i = (s, \alpha, p)$; otherwise set $y_i = (s, 1, q)$. This definition has the property that for any $i \in G$, $y_i \in \{\phi(i)^{-1}i\} \times \{0, 1\} \times S$. Let us prove that the configuration $y$ is a preimage of $x$ by the global map of the CA. Let $j \in G$ and $s, t \in S$ such that $s \prec t$, $y_{js} \in \{s\} \times \{0, 1\} \times S$, and $y_{jt} \in \{t\} \times \{0, 1\} \times S$. Then $s = \phi(js)^{-1}js$ and $t = \phi(jt)^{-1}jt$, and $\phi(js) = \phi(jt) = j$: hence, there exists *exactly one* such pair $(s, t)$. If $x_j = q_0$, then the definition of $y$ gives $y_{jt} = (t, 0, t)$, and $f$ will apply its third subrule. If $x_j$ is written $(p, \alpha, q)$, then $y_{js} = (s, \alpha, p)$ and $y_{jt} = (t, 1, q)$, and $f$ will apply its second subrule.    □

Now, let $G$ be a non-amenable group with decidable w.p., $\mathcal{A}$ the CA from Example 17, and $c$ a $\mu_\Pi$-random configuration. By Remark 5, there are some points $g \in G$ where $c(g) = q_0$: since $|S| \geq 2$, $F_{\mathcal{A}}(c)$ cannot have isolated $q_0$'s, and by the same Remark 5, it cannot be $\mu_\Pi$-random. On the other hand, as a consequence of the Poincaré recurrence theorem, a CA that preserves $\mu_\Pi$ is nonwandering: we have thus yet another characterization of amenable groups as those where surjective CA are nonwandering.

A general scheme of the implications is provided by Figure 1. By joining Bartholdi's theorem, Remark 7, Corollary 16, Example 17, and the observations above we get Theorem 1.

We conclude this section with some results involving general measures for the configuration space.

**Proposition 18** *Let $\mathcal{A} = \langle Q, \mathcal{N}, f \rangle$ be a* CA *over group $G$, and $\mu$ a $\sigma_k$-ergodic Borel probability measure on $Q^G$ for some $k \in G$. Then for $t \geq 1$, $F_{\mathcal{A}}^t \mu$ is also $\sigma_k$-ergodic. Moreover, $F_{\mathcal{A}}$ is $\mu$-recurrent if and only if $F_{\mathcal{A}}^t$ preserves $\mu$ for some $t \geq 1$.*
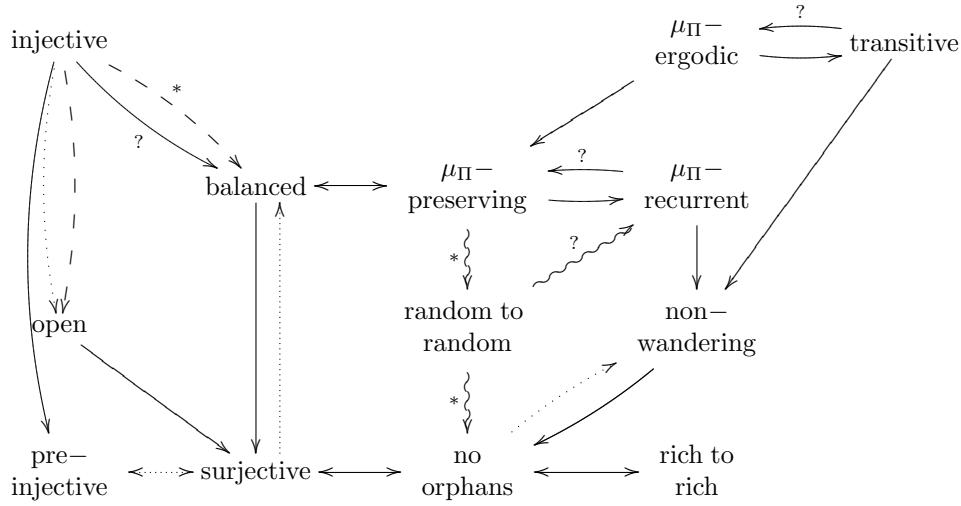
**Figure 1:** A diagram of implications between cellular automata properties. Full lines hold for every group; dotted lines hold for amenable groups; dashed lines hold for residually finite groups; wavy lines hold for countable groups with decidable word problem. Starred implications are proved in the present paper. Implications with a question mark are conjectured.

**Proof:** Since $\sigma_k$ and $F_\mathcal{A}$ commute, if $\sigma_k^{-1}(U) = U$ then $\sigma_k^{-1}(F_\mathcal{A}^{-t}(U)) = F_\mathcal{A}^{-t}(U)$ as well, hence $F_\mathcal{A}^t\mu(U) \in \{0, 1\}$; also, for any Borel set $U$, $F_\mathcal{A}^t\mu(\sigma_k^{-1}(U)) = \mu(\sigma_k^{-1}(F^{-t}(U))) = F_\mathcal{A}^t\mu(U)$.

By the Poincaré recurrence theorem, if $F_\mathcal{A}^t$ preserves $\mu$ then it is $\mu$-recurrent, and this trivially implies that $F$ also is. For the converse implication, let $U$ be the set of $\mu$-typical configurations for $\sigma_k$: then $\mu(U) = 1$, so $t \geq 1$ exists such that $\mu(U \cap F_\mathcal{A}^t(U)) > 0$. But since $\sigma_k$ and $F_\mathcal{A}$ commute, if $x$ is $\mu$-typical for $\sigma_k$, then $F_\mathcal{A}^t(x)$ is $(F_\mathcal{A}^t\mu)$-typical for $\sigma_k$: thus, $\mu$ and $F_\mathcal{A}^t\mu$ are two $\sigma_k$-ergodic measures having a common typical point for $\sigma_k$, so they are equal by Lemma 4. $\qquad\square$

If $F$ is a $\mu$-recurrent system where $\mu$ is $\sigma_k$-ergodic for some $k \in G$, then for suitable $t \geq 1$ the *mean* of $F^i\mu$ for $0 \leq i < t$ is $F$-invariant. Note that this does *not* imply that $F$ is $\mu$-invariant: a simple counter-example is a CA performing a simple state permutation, over a non-uniform Bernoulli measure.

**Example 19** *Let $Q = \{0, 1\}$ and let $\mu$ be a product of independent identical measures $\mu(0) = 1/3$, $\mu(1) = 2/3$; let $\mathcal{A} = \langle Q, \{1_G\}, f \rangle$ with $f(z) = 1 - z$. Then $F_\mathcal{A}^2\mu = \mu$ but $F_\mathcal{A}\mu \neq \mu$. However, if $\bar{\mu}_2 = (\mu + F_\mathcal{A}\mu)/2$, then $F_\mathcal{A}\bar{\mu}_2 = \bar{\mu}_2$.*

## 4 Conclusions

We have shown that several characterizations of surjective CA which are known to hold on Euclidean groups also hold in the more general case of amenable groups.

This is a work in progress, and many more questions arise. Among those:

1. Does Myhill's theorem only hold on amenable groups?

2. What is the actual role of the word problem in Lemma 14 and Proposition 15? Can we find some amenable groups with undecidable word problem but where surjective CA still send $\mu_\Pi$-random to $\mu_\Pi$-random?

3. For the uniform product measure, is every recurrent CA invariant?

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