Introduction to Symbolic Dynamics

Part 1: The basics

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Overview

- Historical introduction
- Shift subspaces
- Basic constructions on shift subspaces
- Sliding block codes
- A parallel with coding theory

A short history of symbolic dynamics

- 1898:
  Hadamard's work on geodetic flows.
- 1930s:
  Morse and Hedlund's work.
- 1960s:
  Smale introduces the word "subshift".
- 1990s:
  Boyle and Handelman make a crucial step towards characterization of nonzero eigenvalues of nonnegative matrices.

Hadamard's problem

Geodesic flows on surfaces of negative curvature

Generally hard problem, but...

What if...

- Partition the space into finitely many regions.
- Discretize time.
- Check the region instead of the exact position.

Discovery!

The complicated dynamics can be described in terms of finitely many forbidden pairs of symbols!

Sequences and blocks

Full shifts

Let A be a finite alphabet. The full A-shift is the set

$$A^{\mathbb{Z}} = \{\text{bi-infinite words on } A\}.$$

The full r-shift is the full A-shift for $A = \{0, \dots, r-1\}$.

Blocks

- A block, or word, over A is a finite sequence u of elements of A.
- If $u = a_1 \dots a_k$ then $k = |u|$ is the length of u. If $|w| = 0$ then $w = \varepsilon$.
- A subblock of $u = a_1 \dots a_k$ has the form $v = a_i \dots a_j$, $1 \leq i, j \leq k$. If $x \in A^{\mathbb{Z}}$ then $x_{[i,j]}$ is the subblock $x_i \dots x_j$.
- A block u occurs in a sequence x if $x_{[i,j]} = u$ for some $i, j \in \mathbb{Z}$.

The shift map

$$\sigma(x)_i = x_{i+1} \text{ for all } x \in A^{\mathbb{Z}},\ i \in \mathbb{Z}.$$

Periodic points

- $x \in A^{\mathbb{Z}}$ is periodic if $\sigma^n(x) = x$ for some $n > 0$.
- Any such n is called a period of x.
- x is a fixed point for σ if $\sigma(x) = x$.

Consequences

- Definition above is the same as $x_{i+n} = x_i \ \ \forall i \in \mathbb{Z}$.
- If x has a period, then it also has a least period.

Interpretation

- The group $\mathbb{Z}$ represents time.
- (Bi-infinite) sequences represent (reversible) trajectories.
- The shift represent the passing of time.
- Periodic sequences represent periodic (closed) trajectories.

Shift subspaces

Definition

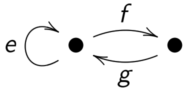
Let $\mathcal{F}$ be a set of blocks over A and let

$$\mathsf{X}_{\mathcal{F}} = \left\{ x \in A^{\mathbb{Z}} \mid x_{[i,j]} \neq u \ \forall i,j \in \mathbb{Z} \, \forall u \in \mathcal{F} \right\}$$

A shift subspace, or subshift, over A is a subset of $A^{\mathbb{Z}}$ of the form $X = \mathsf{X}_{\mathcal{F}}$ for some set of blocks $\mathcal{F}$.

Examples of subshifts

1. The full shift.
2. The golden mean shift $X = \mathsf{X}_{\{11\}}$.
3. The even shift $X = \mathsf{X}_{\mathcal{F}}$ with $\mathcal{F} = \left\{10^{2k+1}1 \mid k \in \mathbb{N}\right\}$.
4. For $S \subseteq \mathbb{N}$, the S-gap shift $\mathsf{X}(S)$ with $\mathcal{F} = \{10^n1 \mid n \in \mathbb{N} \setminus S\}$.
   For $S = \{d, \dots, k\}$ we have the (d,k) run-length limited shift $\mathsf{X}(d, k)$.
5. The set of labelings of bi-infinite paths on the graph



6. The charge constrained shift over $\{+1, -1\}$ s.t. $x \in X$ iff $\sum_{i=j}^{j+n} x_i \in [-c, c]$ for every $j \in \mathbb{Z}$, $n \geq 0$.
7. The context free shift over $\{a, b, c\}$ with

$$\mathcal{F} = \{ab^m c^k a \mid m \neq k\}$$

Basic facts on subshifts

1. Suppose $X_1 = \mathsf{X}_{\mathcal{F}_1}$ and $X_2 = \mathsf{X}_{\mathcal{F}_2}$.
   Then $X_1 \cap X_2 = \mathsf{X}_{\mathcal{F}_1 \cup \mathcal{F}_2}$.
2. Suppose $\mathcal{F}_1 \subseteq \mathcal{F}_2$.
   Then $\mathsf{X}_{\mathcal{F}_1} \supseteq \mathsf{X}_{\mathcal{F}_2}$.
   In particular, $X_1 \cup X_2 \subseteq \mathsf{X}_{\mathcal{F}_1 \cap \mathcal{F}_2}$.
3. In general, $X_1 \cup X_2 \neq \mathsf{X}_{\mathcal{F}_1 \cap \mathcal{F}_2}$.
4. Let $\{X_i\}_{i \in I}$ be a family of subshifts s.t. $\bigcup_{i \in I} X_i = A^{\mathbb{Z}}$.
   Then $X_i = A^{\mathbb{Z}}$ for some $i \in I$.
5. If X is a subshift over A and Y is a subshift over B, then

$$X \times Y = \{z : \mathbb{Z} \to A \times B \mid \exists\, x \in X, y \in Y \mid \forall\, i \in \mathbb{Z}. z_i = (x_i, y_i)\}$$

is a subshift over $A \times B$.

Shift invariance

Definition

$X \subseteq A^{\mathbb{Z}}$ is shift invariant if $\sigma(X) \subseteq X$.

Subshifts are shift invariant

Write σ_X for the restriction of the shift to X.

Shift invariance is not enough to make a subshift!

$$X = \left\{ x \in \{0,1\}^{\mathbb{Z}} \mid \exists! i \mid x_i = 1 \right\}$$

- X is shift invariant.
- And no block of the form 0^n is forbidden.
- Then, if X were a subshift, it would contain $0^{\mathbb{Z}}$—which it doesn't.

Languages

Definition

Let $X \subseteq A^{\mathbb{Z}}$, not necessarily a subshift.
Let $\mathcal{B}_n(X)$ be the set of subblocks of length n of elements of X.
The language of X is

$$\mathcal{B}(X) = \bigcup_{n \geq 0} \mathcal{B}_n(X).$$

Characterization of subshift languages

1. Let X be a subshift. Let $L = \mathcal{B}(X)$.
   1. For every $w \in L$, if u is a factor of w, then $u \in L$.
   2. For every $w \in L$ there exist nonempty $u, v \in L$ s.t. $uwv \in L$.
2. Suppose $L \subseteq A^*$ satisfies points 1 and 2 above.
   Then $L = \mathcal{B}(X)$ for some subshift X over A.
3. In fact, if X is a subshift and $L = \mathcal{B}(X)$, then $X = \mathsf{X}_{A^* \setminus L}$.
   In particular, the language of a subshift determines the subshift.
4. Subshifts over A are precisely those $X \subseteq A^{\mathbb{Z}}$ s.t.

$$\begin{gathered}\text{for every } x \in A^{\mathbb{Z}}, \\ \text{if } x_{[i,j]} \in \mathcal{B}(X) \text{ for every } i, j \in \mathbb{Z}, \\ \text{then } x \in X.\end{gathered}$$

5. In particular, a finite union of subshifts is a subshift.

Irreducibility

Definition

A subshift X is irreducible if for every $u, v \in \mathcal{B}(X)$ there exists $w \in \mathcal{B}(X)$ s.t. $uwv \in \mathcal{B}(X)$.

Meaning

X is irreducible iff the dynamical system (X, σ) is not made of two parts not joined by any orbit.

Examples

- The golden mean shift is irreducible.
- The subshift $X = \{0^{\mathbb{Z}}, 1^{\mathbb{Z}}\}$ is not irreducible.

Higher block shifts

Let X be a subshift over A. Consider $A_X^{[N]} = \mathcal{B}_N(X)$ as an alphabet.

The N-th higher block code

It is the map $\beta_N : X \to (A_X^{[N]})^{\mathbb{Z}}$ defined by

$$(\beta_N(x))_i = x_{[i,i+N-1]}$$

The N-th higher block shift

It is the subshift $X^{[N]} = \beta_N(X)$.

Higher block shifts are subshifts

Let $X = \mathsf{X}_{\mathcal{F}}$. It is not restrictive to suppose $|u| \geq N$ for every $u \in \mathcal{F}$. For $|w| \geq N$ put $w_i^{[N]} = w_{[i:i+N-1]}$. Let

$$\mathcal{F}_1 = \{w^{[N]} \mid w \in \mathcal{F}\}.$$

Then put

$$\mathcal{F}_2 = \{uv \mid u, v \in A^N, \exists i > 1 \mid u_i \neq v_{i-1}\}$$

Then clearly $X^{[N]} \subseteq \mathsf{X}_{\mathcal{F}_1 \cup \mathcal{F}_2}$. On the other hand, any $x \in \mathsf{X}_{\mathcal{F}_1 \cup \mathcal{F}_2}$ reconstructs some $y \in X$, so that $x = \beta_N(y) \in X^{[N]}$.

Higher power shifts

Let X be a subshift over A. Consider $A_X^{[N]} = \mathcal{B}_N(X)$ as an alphabet.

The N-th higher power code

It is the map $\gamma_N : X \to (A_X^{[N]})^{\mathbb{Z}}$ defined by

$$(\gamma_N(x))_i = x_{[Ni, N(i+1)-1]}$$

The N-th higher power shift

It is the subshift $X^N = \gamma_N(X)$.

Higher block shifts and other operations

Properties

1. $(X \cap Y)^{[N]} = X^{[N]} \cap Y^{[N]}$.
2. $(X \cup Y)^{[N]} = X^{[N]} \cup Y^{[N]}$.
3. $(X \times Y)^{[N]} = X^{[N]} \times Y^{[N]}$.
4. $\beta_N \circ \sigma_X = \sigma_{X^{[N]}} \circ \beta_N$

A note on higher power shifts

$\gamma_N \circ \sigma_X^N = \sigma_{X^N} \circ \gamma_N$.

Sliding block codes

- Let X be a subshift over A. Let $\mathfrak{A}$ be another alphabet.
- Let $\Phi : \mathcal{B}_{m+n+1}(X) \to \mathfrak{A}$.
- Then $\phi : X \to \mathfrak{A}^{\mathbb{Z}}$ defined by

$$\phi(x)_i = \Phi\left(x_{[i-m,i+n]}\right)$$

  is a sliding block code (SBC) with memory m and anticipation n.
- We then write $\phi = \Phi_{\infty}^{[-m,n]}$, or just $\phi = \Phi_{\infty}$.
- We may also write $\phi : X \to Y$ if Y is a subshift over $\mathfrak{A}$ and $\phi(X) \subseteq Y$.
- It is always possible to increase both memory and anticipation.
- We speak of 1-block code when $m = n = 0$.

Examples of sliding block codes

1. The shift.
2. The identity.
3. The converse of the shift.
4. The N-th higher block code map β_N.
5. The XOR, induced by $\Phi(x_0 x_1) = x_0 + x_1 \mod 2$.
6. The map defined by

$$\phi(00) = 1 \ , \ \phi(01) = 0 \ , \ \phi(10) = 0$$

is a SBC from the golden mean shift to the even shift.

The key property of SBC

Let X and Y be shift spaces, and let $\phi : X \to Y$ be a SBC. Then

$$\begin{array}{ccc} X & \xrightarrow{\sigma_X} & X \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ Y & \xrightarrow{\sigma_Y} & Y \end{array}$$

Meaning

- SBC are shift-commuting.
- SBC represent stationary processes.
- A SBC from X to Y is a morphism from (X, σ) to (Y, σ).

Shift-commutativity is not enough to make a SBC

Counterexample

Let $\phi(x) : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$ be defined by

$$\phi(x)_i = \begin{cases} 1 - x_i & \text{if } \exists j > i \mid x_j = 1\,, \\ x_i & \text{otherwise}\,. \end{cases}$$

Theorem

Let $\phi : X \to Y$ be a map between shift spaces.
Then ϕ is a SBC if and only if:

1. ϕ is shift-commuting, and
2. there exists $N \geq 0$ s.t. $\phi(x)_0$ is a function of $x_{[-N:N]}$.

Consequently, compositions of SBC are SBC.

Factors, embeddings, conjugacies

Let X and Y be subshifts, $\phi : X \to Y$ a SBC.

Factors

- ϕ is a factor code if it is surjective.
- Y is a factor of X if there is a surjective SBC from X to Y.

Embeddings

- ϕ is an embedding if it is injective.

Conjugacies

- ϕ is a conjugacy if it is bijective.
- The Nth higher block code is a conjugacy from X to $X^{[N]}$, with converse

$$\beta_N^{-1}(y)_i = (y_i)_0 \, .$$

Every SBC can be recoded as a 1-SBC

Theorem

For every SBC $\phi : X \to Y$ there exist an integer $N > 0$, a conjugacy $\psi : X \to X^{[N]}$, and a 1-block code $\omega : X^{[N]} \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{\psi} & X^{[N]} \\ {\scriptstyle \phi}\downarrow & \swarrow {\scriptstyle \omega} & \\ Y & & \end{array}$$

Reason why

- Suppose $\phi = \Phi_\infty^{[-m,n]}$.
- Put $N = m + n + 1$, $\psi = \sigma^{-m} \circ \beta_N$.
- Then $\omega = \phi \circ \psi^{-1} = \phi \circ \beta_N^{-1} \circ \sigma^m$ is a 1-SBC.

Image of a subshift through a SBC is a subshift

Theorem

Let X be a shift space over A
Let $\phi : X \to \mathfrak{A}^{\mathbb{Z}}$ be a SBC.
Then $Y = \phi(X)$ is a shift space over $\mathfrak{A}$.

Reason why

- It is not restrictive that ϕ is a 1-block code induced by Φ.
- Put $\mathfrak{L} = \{\Phi(w) \mid w \in \mathcal{B}(X)\}$. Clearly $\phi(X) \subseteq \mathsf{X}_{\mathfrak{A}^* \setminus \mathfrak{L}}$.
- Let $y \in \mathsf{X}_{\mathfrak{A}^* \setminus \mathfrak{L}}$. Then $y_{[-n,n]} = \Phi(x^{(n)}_{[-n,n]})$ for some $x^{(n)} \in X$.
- Since $\mathcal{B}_{2k+1}(X)$ is finite for every k, a single $x \in X$ can be constructed s.t. $y_{[-n,n]} = \Phi(x_{[-n,n]})$ for every n.
- Then $y = \phi(x)$.

Interlude: How to extract x from the $x^{(n)}$'s

1. Take an infinite $S_0 \subseteq \mathbb{N}$ s.t. $x_0^{(n)} = x_0^{(n')}$ for every $n, n' \in S_0$.
2. Take an infinite $S_1 \subseteq S_0$ s.t. $x_{[-1,1]}^{(n)} = x_{[-1,1]}^{(n')}$ for every $n, n' \in S_1$.
3. Take an infinite $S_2 \subseteq S_1$ s.t. $x_{[-2,2]}^{(n)} = x_{[-2,2]}^{(n')}$ for every $n, n' \in S_2$.
4. ... and so on, and so on...
5. Then

$$x_i = x_i^{(n)} \text{ for } n \in S_{|i|}$$

is well defined.

The converse of a bijective SBC is a SBC

Theorem

Let X be a subshift over A, Y a subshift over $\mathfrak{A}$.
Let $\phi : X \to Y$ be a bijective SBC.
Then $\phi^{-1} = \Psi_\infty^{[-N,N]}$ for some $N \geq 0$ and $\Psi : \mathcal{B}_{2N+1}(Y) \to A$.

Reason why

- Again, it is not restrictive that ϕ is a 1-SBC.
- Suppose $\phi^{-1}(y)_0$ is not a function of $y_{[-n,n]}$ whatever n is.
- Then, for every n, there are $x^{(n)}, \tilde{x}^{(n)} \in X$ s.t. $x_0^{(n)} \neq \tilde{x}_0^{(n)}$ but $\Phi(x^{(n)})_{[-n,n]} = \Phi(\tilde{x}^{(n)})_{[-n,n]}$.
- Similar to the previous theorem, $x \neq \tilde{x}$ can be found s.t. $\Phi(x)_{[-n,n]} = \Phi(\tilde{x})_{[-n,n]}$ for every $n \in \mathbb{N}$.
- Then $\phi(x) = \phi(\tilde{x})$, against bijectivity of ϕ.

A parallel with coding theory

In symbolic dynamics

- A subshift is a special subspace of a full shift.
- A code is a special map between subshifts.

In coding theory

- A code is a special submonoid of a free monoid.
- An encoder is a special map between codes.

Laurent series and polynomials

- A Laurent series on a field $\mathbb{F}$ is an expression

$$f(t) = \sum_{i=-\infty}^{+\infty} a_i t^i = \sum_{i=-\infty}^{+\infty} (f)_i t^i$$

  with $a_i \in \mathbb{F}$ for all $i \in \mathbb{Z}$.
- A Laurent polynomial is a Laurent series where only finitely many a_i's are non-zero.
- Laurent series can be multiplied by Laurent polynomials through

$$(f \cdot g)_i = \sum_{j=-\infty}^{+\infty} (f)_j (g)_{i-j}$$

Convolutional encoders and codes

- Let $\mathbb{F}$ be a finite field.
- Identify the Laurent series $\sum_i a_i t^i$ with coefficients in $\mathbb{F}$ with the bi-infinite word $\dots a_{-1} a_0 a_1 \dots$ over $\mathbb{F}$.
- Let $G(t) = [g_{i,j}(t)]$ be a $k \times n$ matrix where each $g_{i,j}(t)$ is a Laurent polynomial over $\mathbb{F}$.
- A (k, n)-convolutional encoder is a transformation from the full $\mathbb{F}^k$-shift to the full $\mathbb{F}^n$-shift of the form

$$O(t) = E(I(t)) = I(t) \cdot G(t)$$

  where the elements of $I(t)$ and $O(t)$ are Laurent series over $\mathbb{F}$.
- A (k, n)-convolutional code is the image of a convolutional encoder.

Example

Let $I(t) = [I_1(t), I_2(t)]$ and

$$G(t) = \begin{bmatrix} 1 & 0 & 1+t \\ 0 & t & t \end{bmatrix}$$

Then

$$O(t) = [I_1(t), tI_2(t), (1+t)I_1(t) + tI_2(t)]$$

so that

$$(O)_i = [(I_1)_i, (I_2)_{i-1}, (I_1)_i + (I_1)_{i-1} + (I_2)_{i-1}]$$

From convolutions to sliding blocks

1. Let $O(t) = E(I(t)) = I(t) \cdot G(t)$ be a (k, n)-convolutional encoder.
2. Let M and N be the maximum and minimum power of t in $G(t)$.
3. Identify the array of Laurent series over $\mathbb{F}$

$$[S_1(t), \dots, S_r(t)]$$

with the bi-infinite word over $\mathbb{F}^r$

$$\dots [(S_1)_{-1}, \dots, (S_r)_{-1}][(S_1)_0, \dots, (S_r)_0][(S_1)_1, \dots, (S_r)_1] \dots$$

4. Then $E = \Phi_{\infty}^{[-M,N]}$ with

$$(\Phi((I)_{-M} \dots (I)_N))_s = \sum_{j=-M}^{N} \sum_{i=1}^{k} (I_i)_j ((G)_{i,s})_{-j}$$

And there is more...

Convolutional encoders are linear SBC

- Dependence of $O(t)$ from $I(t)$ is given by a set of linear equations.

Convolutional codes are linear irreducible subshifts

- Images of a full shift under a SBC.
- Subspaces of the (infinite-dimensional) $\mathbb{F}$-vector space $(\mathbb{F}^n)^{\mathbb{Z}}$ through a linear application.
- It is always possible to join u and v through a long enough w.

The converse also holds

There is a one-to-one correspondence between:

- Linear SBC and convolutional encoders.
- Linear irreducible subshifts and convolutional codes.

. . . and there shall be more. . .

- Shifts of finite type.
- Graphs and their shifts.
- Graphs as representations of shifts of finite type.
- State splitting.
- Shifts of finite type and data storage.

Thank you for attention!