Final exam in Advanced Discrete Mathematics, 20 May 2005, 10:00-13:00
Solutions

1. n is either even or odd. If n is even, then both n^2 and $7n$ are even, so $n^2 + 7n + 12$ is even. If n is odd, then both n^2 and $7n$ are odd, so so $n^2 + 7n + 12$ is even again.

2. There are $10^5 = 100000$ five-digit telephone numbers altogether and $10!/5! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30240$ of them have all digits different. So the number of five-digit telephone numbers with a digit that occurs more than once is $100000 - 30240 = 69760$.

3. We need the smallest number n such that combination of 3 out of n is at least 12. We have that $\binom{5}{3} = \frac{5\cdot4\cdot3}{1\cdot2\cdot3} = 10$ and $\binom{6}{3} = \frac{6\cdot5\cdot4}{1\cdot2\cdot3} = 20$. So the right number is 6.

4. $4^{2\cdot0} - 1 = 4^0 - 1 = 1 - 1 = 0$ is a multiple of 15. Assuming that $4^{2\cdot n} - 1$ is a multiple of 15, $4^{2(n+1)} - 1 = 4^{2n+2} - 1 = 16 \cdot 4^{2n} - 1 = 15 \cdot 4^{2n} + (4^{2n} - 1)$ is a multiple of 15 too, since $15 \cdot 4^{2n}$ is trivially a multiple of 15 and $4^{2n} - 1$ is so by the assumption.

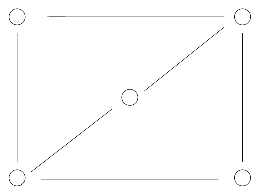
5. The square of sidelength 2 is naturally divided into four squares of sidelength 1. Any two points within any single one of these four squares are not more than $\sqrt{2}$ apart. With 5 points in the big square, one of the four small squares must contain at least two of them.

6. The total amount of integers in the range 1..1000 is 1000. Among these, $\lfloor\frac{1000}{2}\rfloor = 500$ are divisible by 2, $\lfloor\frac{1000}{3}\rfloor = 333$ are divisible by 3, $\lfloor\frac{1000}{5}\rfloor = 200$ are divisible by 5, $\lfloor\frac{1000}{6}\rfloor = 166$ are divisible by 6, $\lfloor\frac{1000}{10}\rfloor = 100$ are divisible by 10, $\lfloor\frac{1000}{15}\rfloor = 66$ are divisible by 15 and $\lfloor\frac{1000}{30}\rfloor = 33$ are divisible by 30. Hence the number of integers divisible by 2 or 3 or 5 is $500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$.

7. We have that $\gcd(325, 26) = \gcd(13, 26) = \gcd(13, 0) = 13$. From this, $13 = 325 - 12 \cdot 26$. Multiplying by 7, we get $91 = 7 \cdot 325 - 84 \cdot 26$.

8. $3^3 = 27 \equiv 1 \pmod{13}$. Hence $3^{50} = 3^{3\cdot12+2} = (3^3)^{12} \cdot 9 \equiv 1^{12} \cdot 9 = 9 \pmod{13}$.

9. The graph is on the picture.



10. Yes, this is possible, see the picture.