LAI 8721

LOGICS OF AGENCY, 4 cu

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www.cs.ioc.ee/~tarmo/agency/

Tarmo Uustalu

tarmo@cs.ioc.ee
www.cs.ioc.ee/~tarmo/

Chair of Theor. Computer Science
Tallinn Techn. Univ.

INTRO:

WHAT IS IT ALL ABOUT?

③

philosophy (foundations)

cognitive science

logics of agency (mathematical machinery)

(symbolic) artificial intelligence

agent-based computing (applications)

④

How are they made?

WHAT ARE AGENTS, ANYWAY

How to make them?

How can agent systems be spoken / reasoned about?

Some features of agents:

- can reason;
- perception, action;
- may have beliefs, desires;
- deliberation, intentions;
- communication;
- cooperative;
- ...

Typical logics of agency are modal, provide support for speaking / reasoning about a few features only.

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This course:

- PREPARATION:
  - Basic modal logic
- MAIN TRACK:
  - Logics of agency
    - logics of knowledge, belief;
    - logics of ability, action;
    - logics of intention;
    - logics of communication
- SIDE TRACKS:
  - philosophical foundations
  - agent-based computing applications

⑦

Topics for seminar presentations:

- Newell's "knowledge level",
  Dennett's "intentional stance";
- McCarthy and Sloman
  on philosophy and AI;

- Levesque's logic of "only knowing";
- action logics of
  van der Hoek, van Linder, Meyer
- Elgesem's logic of agency
- Israel's action logic

- BDI agent architectures;
- McCarthy's situation calculus
  and Elephant 2000;

⑧

- Shoham's agent-oriented programming (AOP)
  and Agent0

- Genesereth and Ketchpel's
  software agents and
  agent communication languages (ACLs),
  KIF + KQML

- Wellman's market-oriented programming

PREPARATION:

BASIC MODAL LOGIC

Modal logics, what they are?

- logics of qualified truth, typical qualifiers are:

  necessarily, possibly (very abstract)
  it is known (more concrete)
  it ought to be
  after action ..., inevitably
  from now on
  until

- The qualifiers (modal operators) come in pairs; one pair – uni-modal logic, many pairs – multi-modal logic.

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An appetizer from epistemic logic (logic of knowledge)

* Marthe Matics has two children, a, b, both logicians. Marthe puts a spot of mud on the forehead of each child.
* Each child can see the forehead of the other, but neither knows whether her own is muddy.
* Martha announces: At least one of you has a muddy forehead.
* Martha asks each: Do you know whether your forehead is muddy.
* Both say: I don't.
* Martha asks the same question again.
* Both say: I know that it is muddy.

* How did they do that?

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a knew the answer since

A	a's forehead is muddy
B	b's forehead is muddy
$K_a B$	a sees b's forehead is muddy
$K_a(K_b A \vee K_b \neg A)$	a sees b sees whether a's forehead is muddy
$K_b A$	b sees a's forehead is muddy
$K_b(K_a B \vee K_a \neg B)$	b sees a sees whether b's forehead is muddy
$K_a(A \vee B)$, $K_b(A \vee B)$, $K_a K_b(A \vee B)$, $K_b K_a(A \vee B)$, ...	it was announced publicly that at least one of $\{a, b\}$ a muddy forehead
$\neg K_b B \wedge \neg K_b \neg B$	b does not know whether her own forehead is muddy..
$K_a(\neg K_b B \wedge \neg K_b \neg B)$, ...	... and announced it publicly

imply

$K_a A$	a knows her forehead is muddy

$$\dfrac{\dfrac{}{\neg K_b B \wedge \neg K_b \neg B \to \neg K_b B}\,\text{RPL} \qquad \dfrac{\dfrac{\dfrac{}{\neg A \wedge (A \vee B) \to B}\,\text{RPL}}{K_b \neg A \wedge K_b (A \vee B) \to K_b B}\,\text{RK}}{\neg K_b B \wedge K_b (A \vee B) \to \neg K_b \neg A}\,\text{RPL} \qquad \dfrac{}{\neg K_b \neg A \vee (K_b A \vee K_b \neg A) \to K_b A}\,\text{RPL}}{(\neg K_b B \wedge \neg K_b \neg B) \wedge K_b (A \vee B) \wedge (K_b A \vee K_b \neg A) \to K_b A}\,\text{RPL}$$

$$\dfrac{(\neg K_b B \wedge \neg K_b \neg B) \wedge K_b (A \vee B) \wedge (K_b A \vee K_b \neg A) \to K_b A}{K_a(\neg K_b B \wedge \neg K_b \neg B) \wedge K_a K_b (A \vee B) \wedge K_a (K_b A \vee K_b \neg A) \to K_a K_b A}\,\text{RK}$$

$$\dfrac{K_a(\neg K_b B \wedge \neg K_b \neg B) \wedge K_a K_b (A \vee B) \wedge K_a (K_b A \vee K_b \neg A) \to K_a K_b A \qquad \dfrac{}{K_a K_b A \to K_a A}\,\text{T}}{K_a(\neg K_b B \wedge \neg K_b \neg B) \wedge K_a K_b (A \vee B) \wedge K_a (K_b A \vee K_b \neg A) \to K_a A}$$

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<u>Modal logics</u>, formally

- <u>Syntax</u>: Shared, ie, same for all logics.
- <u>Semantics</u>: The definitive account is usually <u>model-theoretic</u>, varies from logic to logic, but almost always a <u>relational</u> [aka Kripke, standard] <u>semantics</u> or a <u>neighbourhood</u> [aka Montague-Scott, minimal] <u>semantics</u>.

Any relational (resp. neighbourhood) semantics is based on an <u>interpretation</u> of the syntax in something called <u>relational</u> (resp. <u>neighbourhood</u>) <u>structures</u>, drawn from some <u>non-empty collection</u> of such structures.

The interpretation is always the same, a particular semantics is determined by the collection from which the structures considered are drawn.

The two kinds of semantics have a lot in common. In both kinds of structures, the elements of the carrying set are called possible worlds.

Relational semantics are the norm.

Neighbourhood semantics are more general; they are needed for the model-theoretic definition of some weaker logics (logics with less tautologies).

(Propositional uni-) modal Logics

Syntax

- Assume a denumerable set $Fma_0 = \{p_0, p_1, p_2, \dots\}$ of symbols, called proposition letters.

- The set Fma of expressions, called formulae, is defined as follows:
  - * if $p \in Fma_0$, then $p \in Fma$;
  - * $\top$ (read: verum), $\bot$ (read: falsum) $\in Fma$;
  - * if $A \in Fma$, then $\neg A$ (read: not A) $\in Fma$;
  - * if $A, B \in Fma$, then $A \wedge B$ (read: A and B), $A \vee B$ (read: A or B), $A \supset B$ (read: A implies B) $\in Fma$;
  - * if $A \in Fma$, then $\Box A$ (read: necessarily A, box A), $\Diamond A$ (read: possibly A, diamond A) $\in Fma$.

- Examples of formulae:

  $\bot$, $p \supset p \vee q$, $\Diamond p \wedge \Box(q \vee r)$, $\Box(p \supset \Diamond p)$

Relational [aka Kripke, standard] structures

- A <u>relational frame</u> is a pair $F = (W, R)$ where
  * W is a non-empty set of whatever; its elements are called <u>(possible) worlds</u> [points];
  * $R \in \mathcal{P}(W \times W)$ is a relation between W and W called the <u>accessibility</u> [alternativeness] <u>relation</u>;

    for $w, w' \in W$, ie, w, w' worlds, if wRw' holds, we say that w' is <u>accessible from</u> w [w' is <u>an alternative of</u> w].
- A <u>relational structure</u> [model] is a triple $M = (W, R, V)$ where
  * (W, R) is a relational frame;
  * $V \in [\,Fma_0 \to \mathcal{P}(W)\,]$ is a function from prop. letters to subsets of W, called the <u>valuation</u>.

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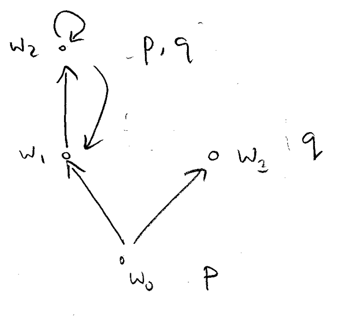
- It is helpful to think of frames and structures <u>diagrammatically</u>.

  A frame is a <u>directed graph</u> (with at most one arc between any two vertices).

  A structure is a directed graph plus, for every prop. letter, a marking of the vertices.
- <u>Example:</u>

M:



Here:

$$W = \{w_0, w_1, w_2, w_3\}$$

$$R = \{(w_0, w_1), (w_1, w_2), (w_2, w_1), (w_2, w_2), (w_0, w_3)\}$$

$$V(p) = \{w_0, w_2\}$$

$$V(q) = \{w_2, w_3\}$$

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- Intuition:

  possible worlds – consistent states-of-affairs (either real in some sense or imagined)

  accessibility relation –
  tells which states-of-affairs are the alternatives of the given one, eg,
  * the states-of-affairs a system could end up in after one clock-tick, if starting in the given one;
  * the states-of-affairs an agent could imagine he is in (with a limited perception, but perfect reasoning ability), if actually being in the given one;
  * ...

  valuation –
  tells in which states-of-affairs each prop. letter is true

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- Hint:

A (uni-) relational frame is nothing else than an (unlabelled) transition system:

possible worlds – states

(the) accessibility relation – (the) transition relation

Satisfaction in relational structures

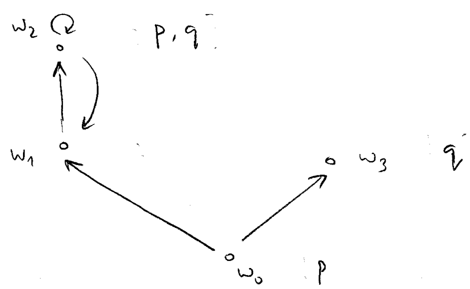
- Given a relational structure $M = (W, R, V)$, let $[\![-]\!]^M \in [Fma \to \mathcal{P}(W)]$ be the function from formulae to subsets of W defined as follows ($\models^M_w A$ is short for $w \in [\![A]\!]^M$):

  * if $p \in Fma_0$, then: $\models^M_w p$ iff $w \in V(p)$;
  * $\models^M_w \top$ is true;
  * $\models^M_w \bot$ is false;
  * $\models^M_w \neg A$ iff not $\models^M_w A$;
  * $\models^M_w A \wedge B$ iff $\models^M_w A$ and $\models^M_w B$;
  * $\models^M_w A \vee B$ iff $\models^M_w A$ or $\models^M_w B$;
  * $\models^M_w A \supset B$ iff $\models^M_w A$ implies $\models^M_w B$;
  * $\models^M_w \Box A$ iff, for any w' such that wRw', $\models^M_{w'} A$;
  * $\models^M_w \Diamond A$ iff, for some w' such that wRw', $\models^M_{w'} A$.

For $A \in Fma$, $w \in W$, if $\models^M_w A$, we say that A is <u>true in</u> M <u>at</u> w, or A <u>holds in</u> M at <u>w</u>, or M <u>satisfies</u> A <u>at</u> w.

- Example:

M:



$\models^M_{w_2} p,\ q,\ p\wedge q,\ \top$
$\quad \Diamond p,\ \Diamond q,\ \Diamond\neg p,\ \Diamond\neg q, \ldots$

$\not\models^M_{w_2} \bot,\ \Box p,\ \Box q, \ldots$

$\models^M_{w_3} q,\ \Box q, \ldots$

$\not\models^M_{w_3} \Diamond q,\ \Diamond\top$ (sic!), ...

$\models^M_{w_0} p,\ \Diamond q,\ \Diamond\Diamond\Diamond^n q\ \ (n \geq 0),$
$\quad \Diamond\neg q,\ \Box(\Box(p\wedge q)\vee q), \ldots$

$\not\models^M_{w_0} \Box q,\ \Box\Diamond\top, \ldots$

Why?

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Intuition:

$\Box A$, true in a state-of-affairs

- A true in all alternative states-of-affairs, eg,
  - in all states-of-affairs a system could end up after one clock-tick, ie, A true after one clock-tick, necessarily;
  - in all states-of-affairs an agent could imagine he is in, ie, A true according to the agent necessarily (the agent <u>believes</u> A);
  - ...

Validity in (collections of) relational structures

- Given a relational *structure* $M = (W, R, V)$, a formula A is *valid* in M (notation $\models^M A$), if $\models^M_w A$, for any $w \in W$.

- Given a relational *frame* F, a formula A is *valid* in F (notation $\models^F A$), if A is valid in all structures based on F.

- Given a (non-empty) *collection* $\mathbb{C}$ of relational structures (resp. frames), a formula A is *valid* in $\mathbb{C}$ (notation $\models^{\mathbb{C}} A$), if A is valid in all structures (resp. frames) from this collection.

Any collection of structures (resp. frames) can be thought of as *determining* a *modal logic*. The tautologies of this logic are the formulae valid in the given collection.

Some observations

- For any relational structure M, the following holds:
  * $\models^M A$, for any A that is a uniform substitution instance of a tautology of prop. logic;
  * if $\models^M A \supset B$ and $\models^M A$, then $\models^M B$.
- For any relational frame F, this is true:
  * if $\models^F A$ and A' is a uniform substitution instance of A, then $\models^F A'$.

- For any relational structure M, this is true:
  * $\models^M \neg\Box A \equiv \Diamond\neg A$.

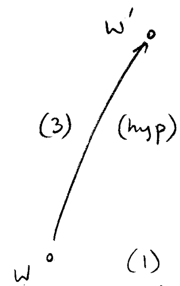
- For any relational structure M, also this is true:
  * $\models^M \Box(A \supset B) \supset (\Box A \supset \Box B)$;
  * if $\models^M A$, then $\models^M \Box A$.

Let us verify the first claim:

Assume $M = (W, R, V)$. Pick any $w \in W$. Assume (1) $\models^M_w \Box(A \supset B)$ and (2) $\models^M_w \Box A$.

We have to show that $\models^M_w \Box B$, but this is the same as showing that $\models^M_{w'} B$, for any w' s.t. wRw'.

Pick any such w'. Because of (1), (2), we can be sure that $\models^M_{w'} A \supset B$ and $\models^M_{w'} A$; from this, $\models^M_{w'} B$ follows immediately.



(4) $A \supset B$ (from (1))
(5) A (from (2))
(6) B (from (4), (5))

(1) $\Box(A \supset B)$ (hyp)
(2) $\Box A$ (hyp)
(7) $\Box B$ (from (6) disch. (3))
(8) $\Box A \supset \Box B$ (from (7) disch. (2))
(9) $\Box(A \supset B) \supset (\Box A \supset \Box B)$ (from (8) disch. (1))

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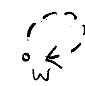
- For any relational frame $F = (W, R)$, the following is a pair of equivalent conditions:
  - * for any $w \in W$, wRw (ie, F is reflexive);
  - * $\models^F \Box A \supset A$, for any $A \in Fma$.



Here is a verification:

($\Rightarrow$) Pick any structure M based on F and any $w \in W$. Assume $\models^M_w \Box A$. Since wRw, this immediately gives us that $\models^M_w A$.

($\Leftarrow$) Suppose that, for some $w_0 \in W$, it is not the case that $w_0 R w_0$. Pick a prop. letter p_0 and a valuation V_0 such that $V_0(p_0) = W - \{w_0\}$. For $M_0 = (W, R, V_0)$ then $\models^{M_0}_{w_0} \Box p_0$, but $\not\models^{M_0}_{w_0} p_0$, from where $\not\models^{M_0}_{w_0} \Box p_0 \supset p_0$.

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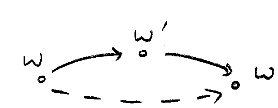
- For any relational frame $F = (W, R)$, the following is also a pair of equivalent conditions:
  - * for any $w, w', w'' \in W$, if wRw', $w'Rw''$, then wRw'' (ie, F is transitive);
  - * $\models^F \Box A \supset \Box\Box A$, for any $A \in Fma$.



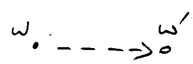
A verification is here:

($\Rightarrow$) Pick any structure M based on F and any $w \in W$. Assume $\models^M_w \Box A$. We have to show that $\models^M_w \Box\Box A$, which means showing that for any $w', w'' \in W$ st. wRw', $w'Rw''$, we have $\models^M_{w''} A$. But this is immediate from $\models^M_w \Box A$, as transitivity gives us wRw''.

($\Leftarrow$) Suppose that, for some $w_0, w_0', w_0'' \in W$, we have $w_0 R w_0'$, $w_0' R w_0''$, but not $w_0 R w_0''$. Pick a prop. letter p_0 and a valuation V_0 such that $V_0(p_0) = W - \{w_0''\}$. For $M_0 = (W, R, V_0)$ then $\models^{M_0}_{w_0} \Box p_0$, but $\not\models^{M_0}_{w_0} \Box\Box p_0$, from where $\not\models^{M_0}_{w_0} \Box p_0 \supset \Box\Box p_0$.

- For any relational frame $F = (W, R)$, even the following are pairs of equivalent conditions:

  * for any $w \in W$, there exists a $w' \in W$ st wRw' (ie, F <u>serial</u>)

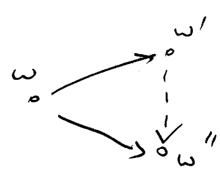
    * $\models^F \Box A \supset \Diamond A$, for any $A \in Fma$

  * for any $w, w' \in W$, if wRw', then $w'Rw$ (ie, F <u>symmetric</u>)

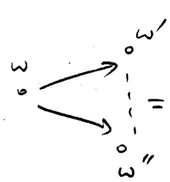
    * $\models^F A \supset \Box\Diamond A$, for any $A \in Fma$

  * for any $w, w', w'' \in W$, if wRw', wRw'', then $w'Rw''$ (ie, F <u>euclidean</u>)

    * $\models^F \Diamond A \supset \Box\Diamond A$, for any $A \in Fma$

  * for any $w, w', w'' \in W$, if wRw', wRw'', then $w' = w''$ (ie, F <u>partial-functional</u>)

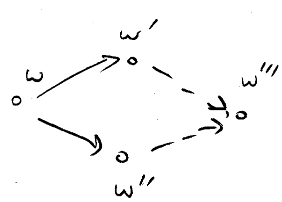
    * $\models^F \Diamond A \supset \Box A$, for any $A \in Fma$

  * for any $w, w' \in W$, if wRw', then there exists a $w'' \in W$ st wRw'', $w''Rw'$ (ie, F <u>weakly dense</u>)

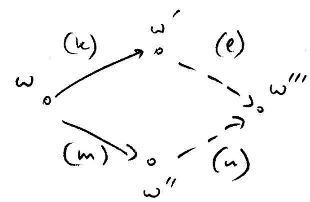
    * $\models^F \Box\Box A \supset \Box A$, for any $A \in Fma$

  * for any $w, w', w'' \in W$, if wRw', wRw'', then there exists a $w''' \in W$ st $w'Rw'''$, $w''Rw'''$ (ie, F <u>weakly directed</u>)

    * $\models^F \Diamond\Box A \supset \Box\Diamond A$, for any $A \in Fma$

* for any $w, w', w'', w''' \in W$,
  if $wR^k w'$, $wR^m w''$,
  then there exists a
  $w''' \in W$ st $w' R^l w'''$, $w'' R^n w'''$
  (ie, F k,l,m,n - <u>incestual</u>)



* $\models^F \Diamond^k \Box^l A \supset \Box^m \Diamond^n A$,
  for any $A \in Fma$

($wR^k w'$ iff
there exist $w_0, \ldots, w_k$
st $w = w_0$, $w_0 R w_1$, ..., $w_{k-1} R w_k$, $w_k = w'$)

($\Box^k A = \underbrace{\Box \ldots \Box}_{k \text{ times}} A$

$\Diamond^k A = \underbrace{\Diamond \ldots \Diamond}_{k \text{ times}} A$)

We say that the formula schemes on the right hand side <u>define</u> the frame conditions on the left hand side.

Not every 1st-order frame condition is definable. Some formula schemes define 2nd-order frame conditions.

Axiomatization of modal logics with relational semantics

Introduce the following axiom. schemata and rules:

(PL) A, where this is a uniform subst. instance of a tautology of prop. logic

(MP) $\dfrac{A \supset B \quad A}{B}$

(RPL) $\dfrac{A_1 \; \dots \; A_n}{B}$ $(n \geq 0)$

where ~~this~~ is a uniform subst. instance of a consequence of prop. logic

(Df◇) $\neg \Box A \equiv \Diamond \neg A$

(K) $\Box(A \supset B) \supset (\Box A \supset \Box B)$

(RN) $\dfrac{A}{\Box A}$

(RK) $\dfrac{A_1 \wedge \dots \wedge A_n \to B}{\Box A_1 \wedge \dots \wedge \Box A_n \to \Box B}$ $(n \geq 0)$

(T) $\Box A \supset A$

(D) $\Box A \supset \Diamond A$

(4) $\Box A \supset \Box\Box A$

(B) $A \supset \Box\Diamond A$

(5) $\Diamond A \supset \Box\Diamond A$

(T◇) $A \supset \Diamond A$

(D◇) $\Box A \supset \Diamond A$

(4◇) $\Diamond\Diamond A \supset \Diamond A$

(B◇) $\Diamond\Box A \supset A$

(5◇) $\Diamond\Box A \supset \Box A$

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Define

$\mathbb{K}$ $=_{df}$ the system given by
PL+ MP (or RPL), Df$\Diamond$, K+RN (or RK)

$\mathbb{K}X_1 \dots X_m$
$=_{df}$ system $\mathbb{K}$ plus axioms $X_1 \dots X_m$
(where $\{X_1, \dots, X_m\} \subseteq \{T, 4, D, B, 5\}$

$\mathbb{T}$ $=_{df}$ $\mathbb{K}T$

$\mathbb{S}4$ $=_{df}$ $\mathbb{K}T4$

$\mathbb{S}5$ $=_{df}$ $\mathbb{K}T45$

$\mathbb{B}$ $=_{df}$ $\mathbb{K}TB$

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- $\mathbb{K}$ is a (sound and complete) axiomatization of the logic determined by the collection of all frames

- $\mathbb{K}$[T][D][4][B][5] is a (sound and complete) axiomatization of the logic determined by the coll. all frames that are [reflexive], [serial], [transitive], [symmetric] and [euclidean].

Neighbourhood [aka Montague-Scott, minimal] structures

- A neighbourhood frame is a pair $F = (W, N)$ where
  - W is a non-empty set of whatever, its elements being called (possible) worlds [points]; — as in a rel. frame
  - $N \in \mathcal{P}(W \times \mathcal{P}(W))$ is a relation between worlds and sets of worlds, called the neighbourhood relation;
    for $w \in W$, $X \subseteq W$, if wNX holds, we say that X is in the neighbourhood of w.
- A neighbourhood structure [model] is a triple $M = (W, N, V)$ where
  - (W, N) is a neighbourhood frame,
  - $V \in [Fma_0 \to \mathcal{P}(W)]$ is a function from prop. letters to sets of worlds, called the valuation. — as in a rel. struct.

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Satisfaction in neighbourhood structures

- Given a neighbourhood structure $M = (W, N, V)$ we let $[\![-]\!]^M \in [\mathrm{Fma} \to \mathcal{P}(W)]$ be the function from formulae to subsets of worlds defined as follows ($\models^M_w A$ is short for $w \in [\![A]\!]^M$):

* if $p \in \mathrm{Fma}_0$, then: $\models^M_w p$ iff $w \in V(p)$;
* $\models^M_w \top$ is true;
* $\models^M_w \bot$ is false;
* $\models^M_w \neg A$ iff not $\models^M_w A$;
* $\models^M_w A \wedge B$ iff $\models^M_w A$ and $\models^M_w B$;
* $\models^M_w A \vee B$ iff $\models^M_w A$ or $\models^M_w B$;
* $\models^M_w A \supset B$ iff $\models^M_w A$ implies $\models^M_w B$;
* $\models^M_w \Box A$ iff $w \, N \, [\![A]\!]^M$;
* $\models^M_w \Diamond A$ iff not $w \, N \, (W - [\![A]\!]^M)$.

as in a rel. struct.

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Validity in (collections of) neighbourhood structures

is defined as validity in (collections of) relational structures.

As in the case of collections of relational structures, every collection of relations can be said to determine a modal logic. The tautologies of this logic are the formulae that are valid in this collection.

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Some observations

- Given a relational structure $M = (W, R, V)$, define a neighbourhood structure $M' = (W, N, V)$ by letting, for any $w \in W$, $X \subseteq W$,

$$wNX \quad \text{iff} \quad \{w' \in W \mid wRw'\} \subseteq X.$$

The structures M, M' are equivalent in the sense that, for any $w \in W$, $A \in Fma$,

$$\models^M_w A \quad \text{iff} \quad \models^{M'}_w A.$$

Hence, neighbourhood structures are at least as general as relational structures.

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- For any neighbourhood structure M, the following holds:
  - * $\models^M A$, for any A that is a uniform subst. instance of a tautology of prop. logic;
  - * if $\models^M A \supset B$ and $\models^M A$, then $\models^M B$.

- For any neighbourhood frame F, this is true:
  - * If $\models^F A$ and A' is a uniform subst. instance of A, then $\models^F A'$.

- For any neighbourhood structure M, this is true:
  - * $\models^M \neg\Box A \equiv \Diamond\neg A$.

— as for rel. structures and frames

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- For any neighbourhood structure M, also this is true:

 * if $\models^M A \equiv B$, then $\models^M \Box A \equiv \Box B$.

Verification:

Assume $M = (W, N, V)$. Assume also $\models^M A \equiv B$. The latter means that, for any $w \in W$, $\models^M_w A$ iff $\models^M_w B$, ie, $[\![A]\!]^M = [\![B]\!]^M$. We shall now show that $\models^M \Box A \supset \Box B$ (one direction of the equiv.). Assume $\models^M \Box A$. We have to show that $\models^M \Box B$, ie, that for any $w \in W$, $wN[\![B]\!]^M$. Pick any $w \in W$. We have that $\models^M \Box A$, therefore $\models^M_w \Box A$, therefore $wN[\![A]\!]^M$. But $[\![A]\!]^M = [\![B]\!]^M$, so $wN[\![B]\!]^M$, trivially.

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- For any neighbourhood frame $F = (W, N)$, the following are pairs of equivalent conditions:

* for any $w \in W$, $X, Y \subseteq W$, if wNX and $X \subseteq Y$, then wNY (ie, F is supplemented)	* if $\models^F A \supset B$, then $\models^F \Box A \supset \Box B$, for any $A, B \in Fma$
* for any $w \in W$, $X, Y \subseteq W$, if wNX, wNY, then $wN(X \cap Y)$ (ie, F closed under intersections)	* $\models^F \Box A \wedge \Box B \supset \Box(A \wedge B)$, for any $A, B \in Fma$
* for any $w \in W$, wNW (ie, F contains the universal set)	* $\models^F \Box\top$

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Axiomatization of modal logics with neighbourhood semantics

Introduce some more axiom schemata and rules:

(RE) $\dfrac{A \equiv B}{\Box A \equiv \Box B}$

(M) $\Box(A_1 \wedge A_2) \supset \Box A_1 \wedge \Box A_2$

(RM) $\dfrac{A \supset B}{\Box A \supset \Box B}$

(C) $\Box A_1 \wedge \Box A_2 \supset \Box(A_1 \wedge A_2)$

(RR) $\dfrac{A_1 \wedge A_2 \supset B}{\Box A_1 \wedge \Box A_2 \supset \Box B}$

(N) $\Box \top$

(RK) $\dfrac{A_1 \wedge \ldots \wedge A_n \rightarrow B}{\Box A_1 \wedge \ldots \wedge \Box A_n \rightarrow \Box B}$ $(n \geq 0)$

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Define

$\mathbb{E}$ $=_{df}$ the system given by PL + MP, Df $\Diamond$, RE

$\mathbb{E}$[M][C][N] $=_{df}$ system $\mathbb{E}$ plus [M], [C], [N]

$\mathbb{M}$ $=_{df}$ $\mathbb{E}$M

$\mathbb{R}$ $=_{df}$ $\mathbb{E}$MC

$\mathbb{K}$ $=_{df}$ $\mathbb{E}$MCN

- $\mathbb{E}$ is a (sound and complete) axiomatization of the logic determined by the collection of all neighbourhood frames.

- $\mathbb{E}[M][C][N]$ is a (sound and complete) axiomatization of the logic determined by the collection of neighbourhood frames that are [supplemented], [closed under intersections] and [contain the unit].

(Propositional) multi-modal logics

Syntax

- Assume a denumerable set $Fma_0 = \{p_0, p_1, p_2, \dots\}$ of symbols, called proposition letters, and a (non-empty) set I of indices, called accessibility labels.

- The set Fma of expressions, called formulae, is defined as follows:
  - * if $p \in Fma_0$, then $p \in Fma$;
  - * $\top, \bot \in Fma$;
  - * if $A \in Fma$, then $\neg A \in Fma$;
  - * if $A, B \in Fma$, then $A \wedge B$, $A \vee B$, $A \supset B \in Fma$;
  - * if $i \in I$, and $A \in Fma$, then $[i]A$ (read: i-necessarily A), $\langle i \rangle A$ (read: i-possibly A) $\in Fma$.

- Examples of formulae:

  $p \wedge q$, $[i](p \supset \langle j \rangle p)$, ...

Multi-relational structures

- A multi-relational frame is a pair $F = (W, R)$ where
  - W is a non-empty set, its elements being called (possible) worlds;
  - $R \in I \to \mathcal{P}(W \times W)$ is an I-indexed set of relations between W and W, $R(i)$ being called the i-accessibility relation, for $i \in I$.
- A multi-relational structure is a triple $M = (W, R, V)$ where
  - (W, R) is a multi-relational frame;
  - $V \in [Fma_0 \to \mathcal{P}(W)]$ is a function from prop. letters to subsets of W, called the valuation.

Diagrammatically, a multi-relational frame is a directed multi-graph [labelled directed graph].

Multi-relational frames are the same as labelled transition systems.

Satisfaction in multi-relational structures

- Given a multi-relational structure $M = (W, R, V)$, let $[\![-]\!]^M \in [Fma \to \mathcal{P}(W)]$ be the function from formulae to subsets of W defined as follows ($\models^M_w A$ is short for $w \in [\![A]\!]^M$):

  * if $p \in Fma_0$, then: $\models^M_w p$ iff $w \in V(p)$;
  * $\models^M_w \top$ is true;
  * $\models^M_w \bot$ is false;
  * $\models^M_w \neg A$ iff not $\models^M_w A$;
  * $\models^M_w A \wedge B$ iff $\models^M_w A$ and $\models^M_w B$;
  * $\models^M_w A \vee B$ iff $\models^M_w A$ or $\models^M_w B$;
  * $\models^M_w A \supset B$ iff $\models^M_w A$ implies $\models^M_w B$;
  * $\models^M_w [i]A$ iff, for any w' st $w R(i) w'$, $\models^M_{w'} A$;
  * $\models^M_w \langle i \rangle A$ iff, for some w' st $w R(i) w'$, $\models^M_{w'} A$.

Some observations

- For any multi-relational frame $F = (W, R)$, the following is a pair of equivalent conditions:
  - * for any $w, w' \in W$,
    if $w\, R(a)\, w'$, then $w\, R(b)\, w'$
    ($R(b)$ contains $R(a)$);

$$\begin{tikzcd} w \arrow[r, bend left, "a"] \arrow[r, dashed, "b"'] & w' \end{tikzcd}$$

  - * $\models^F [b] A \supset [a] A$, for any $A \in Fma$.

Verification:

($\Rightarrow$) Pick any structure M based on F and any $w \in W$. Assume $\models^M_w [b] A$. We have to show that $\models^M_w [a] A$, ie that $\models^M_{w'} A$ for any $w' \in W$ st $w\, R(a)\, w'$. But this is immediate from $\models^M_w [b] A$, as for any such w', we also have that $w\, R(b)\, w'$.

($\Leftarrow$) Suppose that, for some $w_0, w_0' \in W$, we have $w\, R(a)\, w'$, but not $w\, R(b)\, w'$. Pick a prop letter p_0 and a valuation V_0 st $V_0(p_0) = W - \{w_0'\}$. For $M_0 = (W, R, V_0)$ then $\models^{M_0}_{w_0} [b]\, p_0$, but not $\models^{M_0}_{w_0} [a]\, p_0$.

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- For any multi-relational frame $F = (W, R)$, the following, too is a pair of equivalent conditions:
  - * for any $w, w' \in W$,
    $w\, R(b)\, w'$ iff $w\, R(a_1)\, w'$ or $w\, R(a_2)\, w'$
    ($R(b)$ is the <u>union</u> of $R(a_1)$ and $R(a_2)$);
  - * $\models^F [b] A \equiv [a_1] A \wedge [a_2] A$, for any $A \in Fma$.

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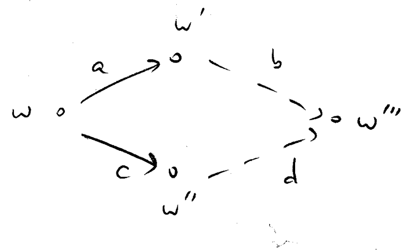
- For any multi-relational frame $F = (W, R)$, also this is a pair of equiv. conditions:

  * for any $w, w', w'' \in W$, if $w R(a) w'$, $w R(c) w''$, then there exists a $w''' \in W$ s.t. $w' R(b) w'''$, $w'' R(d) w'''$;



  * $\models^{F} \langle a \rangle [b] A \supset [c] \langle d \rangle A$, for any $A \in Fma$.

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Axiomatization

- Introduce the following axiom schemata and rules:

($Df\,\Diamond_i$) $\quad \neg[i] A \equiv \langle i \rangle \neg A$

(K_i) $\quad [i](A \supset B) \supset [i] A \supset [i] B$

(RN_i) $\quad \dfrac{A}{[i] A}$

($G_{a,b,c,d}$) $\quad \langle a \rangle [b] A \supset [c] \langle d \rangle A$

- Define
  $\mathbb{K}_I =_{df}$ the system given by $PL + MP$, $Df\,\Diamond_i$ $(i \in I)$, $K_i + RN_i$ $(i \in I)$

  $\mathbb{K}_I X_1 \dots X_m =_{df}$ system $\mathbb{K}$ plus axiom schemata $X_1, \dots, X_m$

- $\mathbb{K}$ is a (sound and complete) axiomatization of the logic of the coll. of all multi-rel. frames for I.
- $\mathbb{K}\{G_{a_j, b_j, c_j, d_j}\}_{j=1,\dots,m}$ is a (sound and complete) ax. of the logic of the coll. of multi-rel. frames for I that are a_j, b_j, c_j, d_j - incestual (for all $j = 1, \dots, m$)

Propositional Dynamic Logic, PDL

Syntax

- Assume a denumerable set $Fma_0 = \{p_0, p_1, p_2, \dots\}$ of symbols called proposition letters, and a denumerable set $Prg_0 = \{a_0, a_1, a_2, \dots\}$ of symbols called action letters.
- The set Fma of expressions, called formulae, and the set Prg of expressions, called programs are defined as follows:
  - if $a \in Prg_0$, then $a \in Prg$;
  - if $P, Q \in Prg$, then $P;Q$, $P \cup Q \in Prg$;
  - if $P \in Prg$, then $P^* \in Prg$;
  - if $A \in Fma$, then $A? \in Prg$;
  - if $p \in Fma_0$, then $p \in Fma$;
  - $\top, \bot \in Fma$;
  - if $A \in Fma$, then $\neg A \in Fma$;
  - if $A, B \in Fma$, then $A \wedge B$, $A \vee B$, $A \supset B \in Fma$;
  - if $P \in Prog$, $A \in Fma$, then $[P]A$, $\langle P \rangle A \in Fma$.

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Intended meanings:

$[P]A$	after P, A holds, necessarily
$P;Q$	do first P, then Q
$P \cup Q$	do either P or Q, non-deterministically
P^*	repeat P some finite number (≥ 0) times
$A?$	test A; if A holds, do nothing (skip), otherwise, fail (abort)

Further constructs can be introduced as abbreviations:

$$\text{if } A \text{ then } P \text{ else } Q =_{df} (A?; P) \cup (\neg A?; Q)$$

$$\text{while } A \text{ do } P =_{df} (A?; P)^*; \neg A?$$

$$\text{repeat } P \text{ until } A =_{df} P; (\neg A?; P)^*; A?$$

$$\text{skip} =_{df} \top?$$

$$\text{abort} =_{df} \bot?$$

$$P^0 =_{df} \text{skip}$$

$$P^{n+1} =_{df} P; P^n$$

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Dynamic structures

- A <u>dynamic structure</u> is a triple $M = (W, R, V)$ where
  - W is a non-empty set and its elements are called <u>states</u>;
  - $R \in Prg_0 \to \mathcal{P}(W \times W)$ is a function from <u>action letters</u> to relations between W and W, called <u>action valuation</u>;
  - $V \in Fma_0 \to \mathcal{P}(W)$ is a function from <u>proposition letters</u> to subsets of W, called <u>proposition valuation</u>.

Satisfaction in dynamic structures

- Given a dynamic structure $M = (W, R, V)$, let $\{\!|-|\!\}^M \in [\mathrm{Prg} \to \mathcal{P}(W \times W)]$ be the function from programs to relations between W and W and $[\![-]\!]^M \in [\mathrm{Fma} \to \mathcal{P}(W)]$ be the function from formulae to subsets of W, defined as follows ($\models^M_w A$ is short for $w \in [\![A]\!]^M$):
  * if $a \in \mathrm{Prg}_0$, then: $w \{\!|a|\!\}^M w'$ iff $w\, R(a)\, w'$;
  * $w \{\!|P;Q|\!\}^M w'$ iff, for some $w'' \in W$, $w \{\!|P|\!\} w''$ and $w'' \{\!|Q|\!\} w'$;
  * $w \{\!|P \cup Q|\!\}^M w'$ iff $w \{\!|P|\!\}^M w'$ and $w \{\!|Q|\!\}^M w'$;
  * $w \{\!|P^*|\!\}^M w'$ iff, for some $n \geq 0$, $w_0, \ldots, w_n \in W$, $w = w_0$, $w_0 \{\!|P|\!\} w_1, \ldots, w_{n-1} \{\!|P|\!\} w_n$, $w_n = w'$;
  * $w \{\!|A?|\!\}^M w'$ iff $\models^M_w A$ and $w = w'$;
  * if $p \in \mathrm{Fma}_0$, then: $\models^M_w p$ iff $w \in V(p)$;
  * $\models^M_w \top$ is true;
  * $\models^M_w A \wedge B$ iff $\models^M_w A$ and $\models^M_w B$;
  * ...
  * $\models^M_w [P] A$ iff, for any $w' \in W$ st $w \{\!|P|\!\} w'$, $\models^M_{w'} A$;
  * $\models^M_w \langle P \rangle A$ iff, for some $w' \in W$ st $w \{\!|P|\!\} w'$, $\models^M_{w'} A$.

Validity in (collections) of dynamic structures

- A formula A is <u>valid</u> in a dynamic structure $M = (W, R, V)$, if $\models^M_w A$, for any $w \in W$.

- The collection of all dynamic structures determines PDL in the sense that the tautologies of PDL are the formulae valid in the collection of all dynamic structures, ie, valid in every dynamic structure.

Axiomatization

PDL is (soundly & completely) axiomatized by $\mathbb{K}_{\mathrm{Prg}_0}$ plus

$$[P;Q]\,A \equiv [P][Q]\,A$$

$$[P \cup Q]\,A \equiv [P]\,A \wedge [Q]\,A$$

$$[B?]\,A \equiv B \rightarrow A$$

$$[P^*]\,A \supset A \wedge [P][P^*]\,A$$

$$\frac{B \supset A \wedge [P]\,B}{B \supset [P^*]\,A}$$