# A depth-first implementation of Geometric Logic in Prolog

(extended overview)

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# GL as a fragment of FOL

- Geometric formula: C => D
- $C = A_1 \wedge \ldots \wedge A_n$ ($n \geq 0$, $A_i$ atoms)
- $D = E_1 \vee \ldots \vee E_m$ ($m \geq 0$)
- $E_j = (E\ x_1 \ldots x_k)\ C_j$ ($k \geq 0$, $C_j$ like C)
- Implicit universal closure
- No function symbols (yet), only constants

# Examples

- Lattices (meet is associative, Horn clause):
  $x \cap y = u \wedge u \cap z = v \wedge y \cap z = w \Rightarrow x \cap w = u$
- Projective unicity (resolution clause):
  $p|l \wedge p|m \wedge q|l \wedge q|m \Rightarrow p=q \vee l=m$
- Diamond property (geometric clause):
  $a \rightarrow b \wedge a \rightarrow c \Rightarrow (E\ d)\ (b \rightarrow d \wedge c \rightarrow d)$
- In general: $A_1 \wedge \ldots \wedge A_n \Rightarrow$
  $((E\ \mathbf{x})\ A_{11} \wedge \ldots \wedge A_{1i}) \vee \ldots \vee ((E\ \mathbf{y})\ A_{k1} \wedge \ldots \wedge A_{kj})$

# Rationale

- Horn clauses: DCG and Prolog
- Resolution: ATP
- Geometric logic: ATP and ?
  - Less skolemization
  - Direct proofs
  - Constructive logic
  - Natural proof theory/objects

# Inductive definition of $X \vdash_{(T)} D$

- (base) X |- D if X ↓ D
- (step) X,<u>C1</u> |- D, … , X,<u>Cn</u> |- D
  (%) ---------------------------------
  X |- D

X a finite set of facts (= closed atoms)

D closed geometric disjunction (parameters in D must occur in X)

X ↓ D iff D = ... ∨ (E **x**) C ∨ … and X contains all facts in C[**x**:=**a**]
for suitable parameters **a**

(%) there exists a *closed* instance $C_0$=>$D_0$ of an axiom in T with
$C_0$ *included in* X (X contains all facts in $C_0$) and
$D_0$ = …∨ (E **x**) Ci ∨… and each <u>Ci</u> a *fresh* instance of Ci ($1 \leq i \leq n$)

# Examples of derivations

- T={true=>p, p=>q}, ø |- q
- T={p∨q, p=>r, q=>r}, ø |- r
- T={p, p=>q, q=>false}, ø |- r
- T={(E x)p(x), p(x)=>q}, ø |- q
- T={s(a,b), s(x,y)=>(E z) s(y,z)},
  ø |- (E x y)(s(a,x)/\s(x,y))
- Forward reasoning (cf. Prolog)!

# Metaproperties

- Soundness
- Completeness
- Constructivity
- Conservativity
- Semidecidability
- Automation (SATCHMO!)

# Samples of ATP

- exist.in
- or.in
- nijm.in

# Case studies

- Confluence theory: induction steps in Newman’s Lemma, Hindley-Rosen, Self-lengthening Thm, ..
- Lattice theory: $x \cap (y \cup z) \leq (x \cap y) \cup (x \cap z)$ for all x,y,z implies $(x \cup y) \cap (x \cup z) \leq x \cup (y \cap z)$ for all x,y,z
- Projective geometry: equivalence of two versions of Pappus’ Axiom (1 minute, 1MB proof)

# Semantics and completeness

- Geometric logic: no proof by contradiction (= EM, TND, $A \vee \sim A$, $\sim\sim A \Rightarrow A$)
- Digression: constructivism in mathematics
  - $p \vee q$ stronger than $\sim(\sim p \wedge \sim q)$
  - $(E\ x)\ p(x)$ stronger than $\sim(A\ x) \sim p(x)$
  - more strict on ontology of objects
  - EM only in specific cases, f.e., for integers $(A\ x)(x=0 \vee x \neq 0)$, but not for reals

# Example of non-constructivism

- Do there exist irrational real numbers $x$ and $y$ such that $x^y$ is rational ?
- Greek constructivists: √2 is irrational
- Non-constructivist: take $x = y = \sqrt{2}$. If $x^y$ is rational, then I'm done. If $x^y$ is not rational, then I'm also done: $(x^y)^y = x^{y \cdot y} = x^2 = 2$ is rational. Next problem, please.
- Constructivist: what do you mean?

# Tarskian semantics

- Truth values from a complete Boolean algebra, without loss of generality ({0,1}, max, min, not($x$) =1-$x$), [|p∨q|]= max([|p|] [|q|]) etc.
- Thus p∨q is true iff p is true or q is true (Girard: ``what a discovery!´´)
- Sound but not complete for constructive logic, not sound for some forms of constructive mathematics
- Constructive logic is more expressive (∨,E) and requires a more refined semantics …

# Semantics for constructivism (digr.)

- Algebraic: complete Heyting algebra**s** (plural!)
- Topological: open sets as truth values
- Kripke semantics: tree-structured Tarski models (graph-stuctured for modal logic), creative subject
- Curry-Howard interpretation: [|$\varphi$|] is the set of proofs of $\varphi$
- Kleene, Beth, Joyal, …
- Different aspects, counter models, metatheory, …

# Semantics for GL

- Tarskian (non-constructive completeness)
- Beth-Joyal-Coquand (fully constructive, extra information, but highly non-trivial)
- Curry-Howard (for proof objects)
- Other semantics unexplored …

# Completeness wrt Tarskian models

- Given D true in all models of T, how do you find a proof ? Try them all !
- Breadth-first derivability on the blackboard
- Herbrand models along the branches
- König’s Lemma to get the tree finite
- Finite tree => breadth-first proof => |- proof