Predikaatloogika Hilberti süsteem

- Sarnaselt lauseloogikaga, saab ka predikaatloogika jaoks defineerida adekvaatse Hilberti süsteemi, mis on korrektne ja täielik (st. valemi tõestatavus on samaväärne tema üldkehtivusega).
- Aksioomid on nagu lauseloogika Hilberti süsteemis pluss:

$$\begin{array}{ll} \forall x(A \supset B) \supset (A \supset \forall x B) & x \text{ tohi ei esineda vabalt } A\text{-s} \\ \forall x A \supset A[t/x] & \\ A[t/x] \supset \exists x A & \\ \forall x(A \supset B) \supset (\exists x A \supset B) & x \text{ tohi ei esineda vabalt } B\text{-s} \end{array}$$

- Reegliteks on Modus Ponens pluss üldistusreegel:

$$\frac{A[y/x]}{\forall x A}$$

Siin y ei tohi esineda vabalt valemis A ega üheski hüpoteesis (loogikavälises aksioomis).

Predikaatloogika loomulik tuletus

- Standardesitus: lauseloogika loomuliku tuletuse standardesituse reeglitele lisanduvad järgmised:

$$\frac{A[y/x]}{\forall x\ A}\ \forall\mathcal{I}^{\star} \qquad \frac{\forall x\ A}{A[t/x]}\ \forall\mathcal{E}$$

$$\frac{A[t/x]}{\exists x\ A}\ \exists\mathcal{I} \qquad \frac{\exists x\ A \qquad \begin{matrix} A[y/x] \\ \vdots \\ C \end{matrix}}{C}\ \exists\mathcal{E}^{\dagger}$$

* y ei tohi vabalt esineda valemis $\forall x\ A$ ja kasutada olevates hüpoteesides
† y ei tohi vabalt esineda valemites $\exists x\ A$, C ja kasutada olevates hüpoteesides

- Näiteid tõestustest:

$$\dfrac{\dfrac{\dfrac{\overset{+1}{\exists x\; \forall y\;\; p(x,y)} \qquad \dfrac{\dfrac{\overset{+2}{\forall y\;\; p(x',y)}}{p(x',y')}\;\forall\mathcal{E}}{\exists x\;\; p(x,y')}\;\exists\mathcal{I}}{\exists x\;\; p(x,y')}\;\exists\mathcal{E}, -2, x' \text{ värske}}{\forall y\; \exists x\;\; p(x,y)}\;\forall\mathcal{I}, y' \text{ värske}}{\exists x\; \forall y\;\; p(x,y) \supset \forall y\; \exists x\;\; p(x,y)}\;\supset\mathcal{I}, -1$$

$$\dfrac{\dfrac{\dfrac{\overset{+1}{\exists x\;\; p(x)} \qquad \dfrac{\dfrac{\overset{+2}{\forall x\; \neg p(x)}}{\neg p(x')}\;\forall\mathcal{E} \qquad \overset{+3}{p(x')}}{\bot}\;\neg\mathcal{E}}{\bot}\;\exists\mathcal{E}, -3, x' \text{ värske}}{\neg\forall x\; \neg p(x)}\;\neg\mathcal{I}, -2}{\exists x\;\; p(x) \supset \neg\forall x\; \neg p(x)}\;\supset\mathcal{I}, -1$$

$$
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        \overset{+1}{\neg\forall x\ \neg p(x)}
        \qquad
        \dfrac{
          \dfrac{
            \dfrac{
              \overset{+2}{\neg\exists x\ p(x)}
              \qquad
              \dfrac{\overset{+4}{p(x')}}{\exists x\ p(x)}\ \exists\mathcal{I}
            }{\bot}\ \neg\mathcal{E}
          }{\neg p(x')}\ \neg\mathcal{I}, -4
        }{\forall x\ \neg p(x)}\ \forall\mathcal{I}, x' \text{ värske}
      }{\bot}\ \neg\mathcal{E}
    }{\exists x\ p(x)}\ \bot\mathcal{E}
    \qquad
    \overset{+3}{\exists x\ p(x)}
  }{\exists x\ p(x)}\ \text{dilemma}, -2, -3
}{\neg\forall x\ \neg p(x) \supset \exists x\ p(x)}\ \supset\mathcal{I}, -1
$$

- Veel näiteid:

$$\dfrac{\dfrac{\dfrac{\overset{+1}{\exists x\, p(x) \supset q} \quad \dfrac{\overset{+1}{p(x')}}{\exists x\, p(x)}\ \exists\mathcal{I}}{q}\ \supset\mathcal{E}}{p(x') \supset q}\ \supset\mathcal{I}, -2}{\dfrac{\forall x\,(p(x) \supset q)}{(\exists x\, p(x) \supset q) \supset \forall x\,(p(x) \supset q)}\ \supset\mathcal{I}, -1}\ \forall\mathcal{I}, x' \text{ värske}$$

$$\dfrac{\dfrac{\overset{+1}{\exists x\, p(f(x))} \quad \dfrac{\overset{+2}{p(f(x'))}}{\exists x\, p(x)}\ \exists\mathcal{I}}{\exists x\, p(x)}\ \exists\mathcal{E}, x' \text{ värske}, -2}{(\exists x\, p(f(x)) \supset \exists x\, p(x)}\ \supset\mathcal{I}, -1$$

- Sekventsiesitus: lauseloogika loomuliku tuletuse sekventsiesituse reeglitele lisanduvad järgmised:

$$\frac{\Gamma \rightarrow A[y/x]}{\Gamma \rightarrow \forall x\ A}\ \forall\mathcal{I}^{\star} \qquad \frac{\Gamma \rightarrow \forall x\, A}{\Gamma \rightarrow A[t/x]}\ \forall\mathcal{E}$$

$$\frac{\Gamma \rightarrow A[t/x]}{\Gamma \rightarrow \exists x\ A}\ \exists\mathcal{I} \qquad \frac{\Gamma \rightarrow \exists x\, A \quad \Gamma, A[y/x] \rightarrow C}{\Gamma \rightarrow C}\ \exists\mathcal{E}^{\dagger}$$

* y ei tohi vabalt esineda valemis $\forall x\ A$ ja valemihulgas Γ

† y ei tohi vabalt esineda valemites $\exists x\ A$, C ja valemihulgas Γ

Predikaatloogika sekventsiarvutus

- Predikaatloogika sekventsiarvutuses lisanduvad lauseloogika sekventsiarvutusele järgmised reeglid kvantorite kohta:

$$\frac{\Gamma \to A[y/x], \Delta}{\Gamma \to \forall x\, A, \Delta}\ \forall\mathcal{R}^{\star} \qquad \frac{\Gamma, (\forall x\, A,)\, A[t/x] \to \Delta}{\Gamma, \forall x\, A \to \Delta}\ \forall\mathcal{L}$$

$$\frac{\Gamma \to A[t/x], (\exists x\, A,)\, \Delta}{\Gamma \to \exists x\, A, \Delta}\ \exists\mathcal{R} \qquad \frac{\Gamma, A[y/x] \to \Delta}{\Gamma, \exists x,\ A \to \Delta}\ \exists\mathcal{L}^{\dagger}$$

* y ei tohi vabalt esineda valemis $\forall x\ A$ ja valemihulkades $\Gamma,\ \Delta$
† y ei tohi vabalt esineda valemis $\exists x\ A$ ja valemihulkades $\Gamma,\ \Delta$

- Näiteid tõestustest:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(x',y') \to p(x',y')}\,\text{hyp.}}{\forall y\, p(x',y) \to p(x',y')}\,\forall\mathcal{L}}{\forall y\, p(x',y) \to \exists x\, p(x,y')}\,\exists\mathcal{R}}{\exists x\, \forall y\, p(x,y) \to \exists x\, p(x,y')}\,\exists\mathcal{L}, x' \text{ värske}}{\exists x\, \forall y\, p(x,y) \to \forall y\, \exists x\, p(x,y)}\,\forall\mathcal{R}, y' \text{ värske}}{\to \exists x\, \forall y\, p(x,y) \supset \forall y\, \exists x\, p(x,y)}\,\supset\mathcal{R}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(x') \to p(x')}\,\text{hyp.}}{p(x'), \neg p(x') \to}\,\neg\mathcal{L}}{p(x'), \forall x\, \neg p(x) \to}\,\forall\mathcal{L}}{\exists x\, p(x), \forall x\, \neg p(x) \to}\,\exists\mathcal{L}, x' \text{ värske}}{\exists x\, p(x) \to \neg\forall x\, \neg p(x)}\,\neg\mathcal{R}}{\to \exists x\, p(x) \supset \neg\forall x\, \neg p(x)}\,\supset\mathcal{R}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p(f(x'),y') \to p(f(x'),y')}\,\text{hyp.}}{p(f(x'),y') \to \exists y\ p(f(x'),y)}\,\exists\mathcal{R}}{p(f(x'),y') \to \exists x\ \exists y\ p(x,y)}\,\exists\mathcal{R} \qquad \dfrac{\dfrac{\dfrac{}{p(y',g(x')) \to p(y',g(x'))}\,\text{hyp.}}{p(y',g(x')) \to \exists y\ p(y',y)}\,\exists\mathcal{R}}{p(y',g(x')) \to \exists x\ \exists y\ p(x,y)}\,\exists\mathcal{R}}{p(f(x'),y') \vee p(y',g(x')) \to \exists x\ \exists y\ p(x,y)}\,\vee\mathcal{L}}{\exists y\ (p(f(x'),y) \vee p(y,g(x'))) \to \exists x\ \exists y\ p(x,y)}\,\exists\mathcal{L}, y' \text{ värske}}{\exists x\ \exists y\ (p(f(x),y) \vee p(y,g(x))) \to \exists x\ \exists y\ p(x,y)}\,\exists\mathcal{L}, x' \text{ värske}}{\to \exists x\ \exists y\ (p(f(x),y) \vee p(y,g(x))) \supset \exists x\ \exists y\ p(x,y)}\,\supset\mathcal{R}
$$