

Predikaatloogika Hilberti süsteem

- Sarnaselt lauseloogikaga, saab ka predikaatloogika jaoks defineerida adekvaatse Hilberti süsteemi, mis on korrektne ja täielik (st. valemi tõestatavus on samaväärne tema üldkehtivusega).
- Aksiomid on nagu lauseloogika Hilberti süsteemis pluss:

$$\forall x(A \supset B) \supset (A \supset \forall xB) \quad x \text{ tohi ei esineda vabalt } A\text{-s}$$

$$\forall xA \supset A[t/x]$$

$$A[t/x] \supset \exists xA$$

$$\forall x(A \supset B) \supset (\exists xA \supset B) \quad x \text{ tohi ei esineda vabalt } B\text{-s}$$

- Reegliteks on Modus Ponens pluss üldistusreegel:

$$\frac{A[y/x]}{\forall xA}$$

Siin y ei tohi esineda vabalt valemis A ega üheski hüpoteesis (loogikavälises aksiomis).

Predikaatloogika loomulik tuletus

- Standardesitus: lauseloogika loomuliku tuletuse standardesituse reeglitele lisanduvad järgmised:

$$\frac{A[y/x]}{\forall x A} \forall I^*$$

$$\frac{\forall x A}{A[t/x]} \forall E$$

$$\frac{A[t/x]}{\exists x A} \exists I$$

$$\frac{\exists x A \quad \begin{array}{c} A[y/x] \\ \vdots \\ C \end{array}}{C} \exists E^\dagger$$

* y ei tohi vabalt esineda valemis $\forall x A$ ja kasutada olevates hüpoteesides

† y ei tohi vabalt esineda valemites $\exists x A$, C ja kasutada olevates hüpoteesides

- Näiteid tõestustest:

$$\begin{array}{c}
 \frac{\frac{\frac{\exists x \forall y p(x, y)}{+1} \quad \frac{\frac{\forall y p(x', y)}{+2} p(x', y')}{\forall \mathcal{E}}}{\exists x p(x, y')} \forall \mathcal{I}}{\exists x \forall y p(x, y) \quad \exists x p(x, y')} \exists \mathcal{I} \\
 \frac{\exists x p(x, y')}{\forall y \exists x p(x, y)} \forall \mathcal{I}, y' \text{ v\"arske} \\
 \frac{\exists x \forall y p(x, y) \supset \forall y \exists x p(x, y)}{\supset \mathcal{I}, -1} \exists \mathcal{E}, -2, x' \text{ v\"arske}
 \end{array}$$

$$\begin{array}{c}
 \frac{\frac{\frac{\exists x p(x)}{+1} \quad \frac{\frac{\forall x \neg p(x)}{+2} \neg p(x')}{\forall \mathcal{E}}}{\perp} \perp}{\exists x p(x) \quad \neg p(x')} \neg \mathcal{E} \\
 \frac{\perp}{\neg \forall x \neg p(x)} \neg \mathcal{I}, -2 \\
 \frac{\exists x p(x) \supset \neg \forall x \neg p(x)}{\supset \mathcal{I}, -1} \exists \mathcal{E}, -3, x' \text{ v\"arske}
 \end{array}$$

$$\begin{array}{c}
\begin{array}{c}
+4 \\
\frac{\frac{+2}{\neg \exists x p(x)} \quad \frac{p(x')}{\exists x p(x)}}{\exists x p(x)} \exists \mathcal{I} \\
\neg \mathcal{E}
\end{array} \\
\frac{\perp}{\neg p(x')} \neg \mathcal{I}, -4 \\
\frac{\frac{+1}{\neg \forall x \neg p(x)} \quad \frac{\frac{\perp}{\neg p(x')} \quad \forall \mathcal{I}, x' \text{ värsket}}{\forall x \neg p(x)} \forall \mathcal{I}}{\forall x \neg p(x)} \forall \mathcal{I} \\
\neg \mathcal{E} \\
\frac{\frac{\perp}{\exists x p(x)} \quad \perp \mathcal{E}}{\exists x p(x)} \perp \mathcal{E} \\
\frac{\frac{\frac{\perp}{\exists x p(x)} \quad \perp \mathcal{E}}{\exists x p(x)} \quad \frac{\frac{+3}{\exists x p(x)}}{\exists x p(x)}}{\exists x p(x)} \text{dilemma, -2, -3} \\
\frac{\frac{\exists x p(x)}{\neg \forall x \neg p(x) \supset \exists x p(x)} \supset \mathcal{I}, -1}
\end{array}$$

- Veel näiteid:

$$\frac{\frac{\frac{\frac{\exists x p(x) \supset q}{+1}}{q}}{p(x') \supset q} \supset \mathcal{I}, -2}{\forall x (p(x) \supset q)} \forall \mathcal{I}, x' \text{ värske}}{\frac{\exists x p(x) \supset q \supset \forall x (p(x) \supset q)}{\supset \mathcal{I}, -1}} \supset \mathcal{E}$$

$$\frac{\frac{\frac{\frac{\exists x p(f(x))}{+1}}{\exists x p(x)} \supset \mathcal{E}, x' \text{ värske}, -2}{\exists x p(x)} \supset \mathcal{I}, -1}{\frac{\exists x p(f(x)) \supset \exists x p(x)}{\supset \mathcal{I}, -1}} \supset \mathcal{I}, -1} \supset \mathcal{E}$$

- Sekventsiesitus: lauseloogika loomuliku tuletuse sekventsiesituse reeglitele lisanduvad järgmised:

$$\frac{\Gamma \rightarrow A[y/x]}{\Gamma \rightarrow \forall x A} \forall\mathcal{I}^*$$

$$\frac{\Gamma \rightarrow \forall x A}{\Gamma \rightarrow A[t/x]} \forall\mathcal{E}$$

$$\frac{\Gamma \rightarrow A[t/x]}{\Gamma \rightarrow \exists x A} \exists\mathcal{I}$$

$$\frac{\Gamma \rightarrow \exists x A \quad \Gamma, A[y/x] \rightarrow C}{\Gamma \rightarrow C} \exists\mathcal{E}^\dagger$$

* y ei tohi vabalt esineda valemis $\forall x A$ ja valemihulgas Γ

† y ei tohi vabalt esineda valemities $\exists x A$, C ja valemihulgas Γ

Predikaatloogika sekvensiarvutus

- Predikaatloogika sekvensiarvutuses lisanduvad lauseloogika sekvensiarvutusele järgmised reeglid kvantorite kohta:

$$\frac{\Gamma \rightarrow A[y/x], \Delta}{\Gamma \rightarrow \forall x A, \Delta} \forall \mathcal{R}^*$$
$$\frac{\Gamma \rightarrow A[t/x], (\exists x A,) \Delta}{\Gamma \rightarrow \exists x A, \Delta} \exists \mathcal{R}$$

$$\frac{\Gamma, (\forall x A,) A[t/x] \rightarrow \Delta}{\Gamma, \forall x A \rightarrow \Delta} \forall \mathcal{L}$$
$$\frac{\Gamma, A[y/x] \rightarrow \Delta}{\Gamma, \exists x A \rightarrow \Delta} \exists \mathcal{L}^\dagger$$

* y ei tohi vabalt esineda valemis $\forall x A$ ja valemihulkades Γ, Δ

† y ei tohi vabalt esineda valemis $\exists x A$ ja valemihulkades Γ, Δ

- Näiteid tõestustest:

$$\begin{array}{l}
 \frac{}{p(x', y') \rightarrow p(x', y')} \text{ hyp.} \\
 \frac{}{\forall y p(x', y) \rightarrow p(x', y')} \forall \mathcal{L} \\
 \frac{}{\forall y p(x', y) \rightarrow \exists x p(x, y')} \exists \mathcal{R} \\
 \frac{}{\exists x \forall y p(x, y) \rightarrow \exists x p(x, y')} \exists \mathcal{L}, x' \text{ värske} \\
 \frac{}{\exists x \forall y p(x, y) \rightarrow \forall y \exists x p(x, y)} \forall \mathcal{R}, y' \text{ värske} \\
 \frac{}{\rightarrow \exists x \forall y p(x, y) \supset \forall y \exists x p(x, y)} \supset \mathcal{R}
 \end{array}$$

$$\begin{array}{l}
 \frac{}{p(x') \rightarrow p(x')} \text{ hyp.} \\
 \frac{}{p(x'), \neg p(x') \rightarrow} \neg \mathcal{L} \\
 \frac{}{p(x'), \forall x \neg p(x) \rightarrow} \forall \mathcal{L} \\
 \frac{}{\exists x p(x), \forall x \neg p(x) \rightarrow} \exists \mathcal{L}, x' \text{ värske} \\
 \frac{}{\exists x p(x) \rightarrow \neg \forall x \neg p(x)} \neg \mathcal{R} \\
 \frac{}{\rightarrow \exists x p(x) \supset \neg \forall x \neg p(x)} \supset \mathcal{R}
 \end{array}$$

$$\begin{array}{c}
\frac{\overline{p(f(x'), y') \rightarrow p(f(x'), y')}}{\overline{p(f(x'), y') \rightarrow \exists y p(f(x'), y)}} \text{hyp.} \\
\frac{\overline{p(f(x'), y') \rightarrow \exists y p(f(x'), y)}}{\overline{p(f(x'), y') \rightarrow \exists x \exists y p(x, y)}} \exists \mathcal{R} \\
\frac{\overline{p(f(x'), y') \rightarrow \exists x \exists y p(x, y)}}{\overline{p(f(x'), y') \vee p(y', g(x')) \rightarrow \exists x \exists y p(x, y)}} \exists \mathcal{R} \\
\frac{\overline{p(f(x'), y') \vee p(y', g(x')) \rightarrow \exists x \exists y p(x, y)}}{\overline{\exists y (p(f(x'), y) \vee p(y, g(x')))) \rightarrow \exists x \exists y p(x, y)}} \vee \mathcal{L} \\
\frac{\overline{\exists y (p(f(x'), y) \vee p(y, g(x')))) \rightarrow \exists x \exists y p(x, y)}}{\overline{\exists x \exists y (p(f(x), y) \vee p(y, g(x))) \rightarrow \exists x \exists y p(x, y)}} \exists \mathcal{L}, y' \text{ v\u00e4rske} \\
\frac{\overline{\exists x \exists y (p(f(x), y) \vee p(y, g(x))) \rightarrow \exists x \exists y p(x, y)}}{\overline{\rightarrow \exists x \exists y (p(f(x), y) \vee p(y, g(x))) \supset \exists x \exists y p(x, y)}} \exists \mathcal{L}, x' \text{ v\u00e4rske} \\
\rightarrow \exists x \exists y (p(f(x), y) \vee p(y, g(x))) \supset \exists x \exists y p(x, y) \supset \mathcal{R}
\end{array}$$