Monoidal functors (aka idioms!)

[Symmetric] monoidal functors

- A *lax monoidal functor* between monoidal categories $(\mathcal{C}, I, \otimes)$ and $(\mathcal{C}, I', \otimes')$ is
  - a functor F from $\mathcal{C}$ to $\mathcal{C}'$
  - with natural transformations $\mathrm{e} : I' \to FI$ and $\mathrm{m}_{A,B} : FA \otimes' FB \to F(A \otimes B)$

  such that

$$
\begin{array}{ccccc}
I' \otimes' FA & \xrightarrow{\mathrm{e} \otimes' FA} & FI \otimes' FA & \xrightarrow{\mathrm{m}_{I,A}} & F(I \otimes A) \\
\Big\downarrow {\scriptstyle \lambda'_{FA}} & & & & \Big\downarrow {\scriptstyle F\lambda_A} \\
FA & & = = = = = = & & FA
\end{array}
\qquad
\begin{array}{ccccc}
FA & & = = = = = = & & FA \\
\Big\downarrow {\scriptstyle \rho'_{FA}} & & & & \Big\downarrow {\scriptstyle F\rho_A} \\
FA \otimes' I' & \xrightarrow{FA \otimes' \mathrm{e}} & FA \otimes' FI & \xrightarrow{\mathrm{m}_{A,I}} & F(A \otimes I)
\end{array}
$$

$$
\begin{array}{ccccc}
(FA \otimes' FB) \otimes' FC & \xrightarrow{\mathrm{m}_{A,B} \otimes' FC} & F(A \otimes B) \otimes' FC & \xrightarrow{\mathrm{m}_{A \otimes B, C}} & F((A \otimes B) \otimes' C) \\
\Big\downarrow {\scriptstyle \alpha'_{FA,FB,FC}} & & & & \Big\downarrow {\scriptstyle F\alpha_{A,B,C}} \\
FA \otimes' (FB \otimes' FC) & \xrightarrow[FA \otimes' \mathrm{m}_{B,C}]{} & FA \otimes' F(B \otimes C) & \xrightarrow[\mathrm{m}_{A, B \otimes C}]{} & F(A \otimes (B \otimes C))
\end{array}
$$

- A lax monoidal functors between symmetric monoidal categories is *lax symmetric monoidal*, if also

$$\begin{array}{ccc} FA \otimes' FB & \xrightarrow{m_{A,B}} & F(A \otimes B) \\ {\scriptstyle \sigma'_{FA,FB}} \downarrow & & \downarrow {\scriptstyle F\sigma_{A,B}} \\ FB \otimes' FA & \xrightarrow{m_{B,A}} & F(B \otimes A) \end{array}$$

- An *oplax [symmetric] monoidal functor* is like a lax [symmetric] monoidal functor, but e, m go in the opposite direction.
- A *monoidal [symmetric] functor* is like a lax [symmetric] monoidal functor, but e, m are required to be natural isomorphisms.

- A *lax [symmetric] monoidal natural transformation* between two lax [symmetric] monoidal functors $(F, \mathrm{e}, \mathrm{m})$, $(G, \mathrm{e}', \mathrm{m}')$ is a natural transformation $\tau : F \dot{\to} G$ satisfying

$$\begin{array}{ccc} I' & \xrightarrow{\mathrm{e}} & FI \\ \Big\| & & \Big\downarrow{\scriptstyle \tau_I} \\ I' & \xrightarrow{\mathrm{e}'} & GI \end{array} \qquad \begin{array}{ccc} FA \otimes' FB & \xrightarrow{\mathrm{m}_{A,B}} & F(A \otimes B) \\ {\scriptstyle \tau_A \otimes' \tau_B}\Big\downarrow & & \Big\downarrow{\scriptstyle \tau_{A \otimes B}} \\ GA \otimes' GB & \xrightarrow{\mathrm{m}'_{A,B}} & G(A \otimes B) \end{array}$$

- *Oplax [symmetric] monoidal* and *[symmetric] monoidal natural transformations* are defined similarly.

- Any functor F between Cartesian categories is canonically oplax symmetric monoidal via
  - $\mathsf{e} = F1 \xrightarrow{!} 1$,
  - $\mathsf{m}_{A,B} = F(A \times B) \xrightarrow{\langle F\mathsf{fst}, F\mathsf{snd}\rangle} FA \times FB$.
- Any natural transformation between functors F, G between Cartesian categories is oplax symmetric monoidal for the canonical oplax symmetric monoidalities on F and G.

Lax monoidal functors $\cap$ containers

- Containers whose interpretation carries a lax monoidality are given by a container (S, P) with
  - $\mathsf{e} : S$
  - $\bullet : S \to S \to S$
  - $q_0 : \Pi\{s_0 : S\}. \Pi s_1 : S. P\,(s_0 \bullet s_1) \to P\,s_0$
  - $q_1 : \Pi s_0 : S. \Pi\{s_1 : S\}. P\,(s_0 \bullet s_1) \to P\,s_1$

  where we write
  - $q_0\,\{s_0\}\,s_1\,p$ as $s_1 \nwarrow_{s_0} p$
  - $q_1\,s_0\,\{s_1\}\,p$ as $p \nearrow_{s_1} s_0$

  such that
  - $\mathsf{e} \bullet s = s$
  - $s = s \bullet e$
  - $(s \bullet s') \bullet s'' = s \bullet (s' \bullet s'')$

  and ...

- ... and
  - $\mathrm{e} \nwarrow_s p = p$
  - $p \nearrow_s \mathrm{e} = p$
  - $s' \nwarrow_s (s'' \nwarrow_{s \bullet s'} p) = (s' \bullet s'') \nwarrow_s p$
  - $(s'' \nwarrow_{s \bullet s'} p) \nearrow_{s'} s = s'' \nwarrow_{s'} (p \nearrow_{s' \bullet s''} s)$
  - $p \nearrow_{s''} (s \bullet s') = (p \nearrow_{s' \bullet s''} s) \nearrow_{s''} s'$
- $(S, \mathrm{e}, \bullet)$ make a monoid.
- $(\nwarrow, \nearrow)$ resemble a biaction of $(S, \mathrm{e}, \bullet)$.

- Those containers whose interpretation carries a lax symmetric monoidality satisfy also
  - $s \bullet s' = s' \bullet s$,
  - $s' \nwarrow_s p = p \nearrow_{s'} s$

  i.e., the monoid $(S, \mathrm{e}, \bullet)$ is commutative and one action determines the other.

Monoidal monads

[Symmetric] monoidal monads

- A *lax [symmetric] monoidal monad* on a [symmetric] monoidal category $(\mathcal{C}, I, \otimes)$ is a monad (T, η, μ) with a lax [symmetric] monoidality (e, m) of T for which η and μ are lax [symmetric] monoidal, i.e., satisfy

$$\begin{CD} I @= I \\ @| @VV{\eta_I}V \\ I @>>{\mathrm{e}}> TI \end{CD} \qquad \begin{CD} I @>{\mathrm{e}}>> TI @>{T\mathrm{e}}>> T(TI) \\ @| @. @VV{\mu_I}V \\ I @>>> @>>{\mathrm{e}}> TI \end{CD} \qquad \begin{CD} A \otimes B @= A \otimes B \\ @V{\eta_A \otimes \eta_B}VV @VV{\eta_{A \otimes B}}V \\ TA \otimes TB @>>{\mathrm{m}_{A,B}}> T(A \otimes B) \end{CD}$$

$$\begin{CD} T(TA) \otimes T(TB) @>{\mathrm{m}_{TA,TB}}>> T(TA \otimes TB) @>{T\mathrm{m}_{A,B}}>> T(T(A \otimes B)) \\ @V{\mu_A \otimes \mu_B}VV @. @VV{\mu_{A \otimes B}}V \\ TA \otimes TB @>>> @>>{\mathrm{m}_{A,B}}> T(A \otimes B) \end{CD}$$

(Note that Id is lax [symmetric] monoidal and, if F, G are lax [symmetric] monoidal, then so is $G \cdot F$.)

- The 1st law forces that $\mathsf{e} = \eta_I$ and the 2nd law follows from one of the monad laws, so we only need m and the 3rd and 4th laws.

- On a Cartesian category, every monad is canonically oplax symmetric monoidal.

Lax monoidal monads = Comm. bistrong monads

- There is a bijection of lax [symmetric] monoidalities m on a monad (T, η, μ) on a [symmetric] monoidal category $(\mathcal{C}, I, \otimes)$ and commutative [symmetric] bistrengths (θ, ϑ).
- It is defined by
  - $\mathrm{m}_{A,B} = \mathrm{m}^{lr}_{A,B} = \mathrm{m}^{rl}_{A,B}$

  and
  - $\theta_{A,B} = A \otimes TB \xrightarrow{\eta_A \otimes TB} TA \otimes TB \xrightarrow{\mathrm{m}_{A,B}} T(A \otimes B)$,
  - $\vartheta_{A,B} = TA \otimes B \xrightarrow{TA \otimes \eta_B} TA \otimes TB \xrightarrow{\mathrm{m}_{A,B}} T(A \otimes B)$.
- On $(\mathbf{Set}, 1, \times)$, as any monad has a unique left strength and [symmetric] bistrength, it is lax [symmetric] monoidal in at most one way.

Exception idioms

- Lax [symmetric] monoidalities (e, m) on the exception functor for E
  - $TA = E + A$

  are in a bijection with [commutative] semigroup structures $\otimes$ on E via
  - $\mathrm{e}\,* = \mathrm{inr}\,*$,
    $\mathrm{m}_{A,B}\,(\mathrm{inl}\,e_0, \mathrm{inl}\,e_1) = \mathrm{inl}\,(e_0 \otimes e_1)$,
    $\mathrm{m}_{A,B}\,(\mathrm{inl}\,e, \mathrm{inr}\,b) = \mathrm{inl}\,e$
    $\mathrm{m}_{A,B}\,(\mathrm{inr}\,a, \mathrm{inl}\,e) = \mathrm{inl}\,e$
    $\mathrm{m}_{A,B}\,(\mathrm{inr}\,a, \mathrm{inr}\,b) = \mathrm{inr}\,(a, b)$;
  - $e_0 \otimes e_1 = \mathrm{case}\ \mathrm{m}_{0,0}\ \mathrm{of}\ \mathrm{inl}\,e \mapsto e$.
- Two special cases are $e_0 \otimes e_1 = e_0$ (the left zero semigroup) and $e_0 \otimes e_1 = e_1$ (the right zero semigroup).
- The exception monad for E is not lax [symmetric] monoidal except for the special case $E = 1$.

Writer idioms

- Lax [symmetric] monoidalities (e, m) on the writer functor for a set P
  - $TA = P \times A$

  are in a bijection with [commutative] monoid structures $(\mathsf{i}, \otimes)$ on P.

- Lax [symmetric] monoidalities m on the writer monad for a monoid $(P, \mathsf{o}, \oplus)$ are in a bijection with those [commutative] monoid structures $(\mathsf{i}, \otimes)$ on P that satisfy
  - $\mathsf{i} = \mathsf{o}$
  - $(e_0 \oplus e_1) \otimes (e_2 \oplus e_3) = (e_0 \otimes e_2) \oplus (e_1 \otimes e_3)$ (middle-four interchange)

- Under the 1st condition, the 2nd condition implies

  $$e_0 \otimes e_1 = (e_0 \oplus \mathrm{o}) \otimes (\mathrm{o} \oplus e_1) = (e_0 \otimes \mathrm{i}) \oplus (\mathrm{i} \otimes e_1) = e_0 \oplus e_1$$

  and further

  $$e_0 \oplus e_1 = (\mathrm{o} \oplus e_0) \oplus (e_1 \oplus \mathrm{o}) = (\mathrm{o} \oplus e_1) \oplus (e_0 \oplus \mathrm{o}) = e_1 \oplus e_0$$

  as well as follows from these conditions.
- Hence the writer monad is lax [symmetric] monoidal if and only if $\oplus$ is commutative.