Comonads

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Directed containers

Directed containers

- A *directed container* is given by
  - S : **Set** *(shapes)*
  - $P : S \to$ **Set** *(positions)*

  and

  - $\downarrow : \Pi s : S.\, P\, s \to S$ *(subshape)*
  - $o : \Pi\{s : S\}.\, P\, s$ *(root position)*
  - $\oplus : \Pi\{s : S\}.\, \Pi p : P\, s.\, P\,(s \downarrow p) \to P\, s$ *(subshape positions)*

  such that

  1. $\forall\{s\}.\, s \downarrow o = s$,
  2. $\forall\{s, p, p'\}.\, s \downarrow (p \oplus p') = (s \downarrow p) \downarrow p'$,
  3. $\forall\{s, p\}.\, p \oplus\{s\}\, o = p$,
  4. $\forall\{s, p\}.\, o\{s\} \oplus p = p$,
  5. $\forall\{s, p, p', p''\}.\, (p \oplus\{s\}\, p') \oplus p'' = p \oplus (p' \oplus p'')$.
- Laws 3-5 resemble those of a monoid, laws 1-2 those of an action.

Directed container morphisms

- A *directed container morphism* t, q between $(S, P, \downarrow, \mathsf{o}, \oplus)$ and $(S', P', \downarrow', \mathsf{o}', \oplus')$ is given by
  - $t : S \to S'$
  - $q : \Pi\{s : S\}.P'\,(t\,s) \to P\,s$

  such that

  1. $\forall\{s, p\}.\, t\,(s \downarrow q\,p) = t\,s \downarrow' p$
  2. $\forall\{s\}.\, \mathsf{o}\,\{s\} = q\,(\mathsf{o}'\,\{t\,s\})$
  3. $\forall\{s, p, p'\}.\, q\,p \oplus \{s\}\, q\,p' = q\,(p \oplus' \{t\,s\}\, p')$
- Laws 2-3 are like those of a monoid morphism, law 1 that of an action morphism.
- Directed containers form a category **DCont**.

Interpretation of directed containers

- Any directed container $(S, P, \downarrow, \mathsf{o}, \oplus)$ defines a comonad $[\![S, P, \downarrow, \mathsf{o}, \oplus]\!]^{\mathrm{dc}} = (D, \varepsilon, \delta)$ where

  - $D : \mathbf{Set} \to \mathbf{Set}$

    $D\,X = \Sigma s : S.\, P\,s \to X$

    $D\,f\,(s, v) = (s, f \circ v)$

  - $\varepsilon : \forall\{X\}.(\Sigma s : S.\, P\,s \to X) \to X$

    $\varepsilon\,(s, v) = v\,(\mathsf{o}\,\{s\})$

  - $\delta : \forall\{X\}.\,(\Sigma s : S.\, P\,s \to X) \to \Sigma s : S.\, P\,s \to \Sigma s' : S.\, P\,s' \to X$

    $\delta\,(s, v) = (s, \lambda p.\,(s \downarrow p, \lambda p'.\, v\,(p \oplus \{s\}\,p')))$

Interpretation of directed container morphisms

- Any directed container morphism t, q between $(S, P, \downarrow, o, \oplus)$ and $(S', P', \downarrow', o', \oplus')$ defines a comonad morphism $[\![t, q]\!]^{\mathrm{dc}}$ between $[\![S, P, \downarrow, o, \oplus]\!]^{\mathrm{dc}}$ and $[\![S', P', \downarrow', o', \oplus']\!]^{\mathrm{dc}}$
  - $[\![t, q]\!]^{\mathrm{dc}} : \forall\{X\}. (\Sigma s : S. P\, s \to X) \to \Sigma s' : S'. P'\, s' \to X$

    $[\![t, q]\!]^{\mathrm{dc}}\, (s, v) = (t\, s, v \circ q\, \{s\})$

- $[\![-]\!]^{\mathrm{dc}}$ is a fully faithful functor from **DCont** to **Cmnds**(**Set**).

Streams

- $S = 1$
- $P* = \text{Nat}$
- $s \downarrow p = s$
- $\text{o} = 0$
- $p \oplus p' = p + p'$

Nonempty lists

- with the comultiplication structure of suffixes
  - $S = \mathrm{Nat}$
  - $P\, s = [0..s]$
  - $s \downarrow p = s - p$
  - $\mathrm{o} = 0$
  - $p \oplus p' = p + p'$
- with the comultiplication structure of cyclic shifts
  - $S = \mathrm{Nat}$
  - $P\, s = [0..s]$
  - $s \downarrow p = s$
  - $\mathrm{o} = 0$
  - $p \oplus \{s\}\, p' = (p + p') \mod s$

Comonads $\cap$ containers = directed containers

- Directed containers account for all those containers whose interpretation carries a comonad structure.

- More precisely, the following is a pullback in **CAT**:

$$\begin{array}{ccc}
\mathbf{DCont} \cong \mathbf{Comonoid}(\mathbf{Cont}) & \xrightarrow{U} & \mathbf{Cont} \\
{\scriptstyle [\![-]\!]^{\mathrm{dc}}} \Big\downarrow {\scriptstyle \text{f.f.}} & & {\scriptstyle [\![-]\!]^{\mathrm{c}}} \Big\downarrow {\scriptstyle \text{f.f.}} \\
\mathbf{Cmnd}(\mathbf{Set}) \cong \mathbf{Comonoid}([\mathbf{Set},\mathbf{Set}]) & \xrightarrow{U} & [\mathbf{Set},\mathbf{Set}]
\end{array}$$

Compatible compositions of comonads

Distributive laws of comonads

- A *distributive law* between comonads $(D_0, \varepsilon, \delta)$ and $(D_1, \varepsilon, \delta)$ is a natural transformation $\theta : D_0 \cdot D_1 \dot{\to} D_1 \cdot D_0$ such that

$$
\begin{array}{ccc}
D_0 \cdot D_1 & \xrightarrow{\theta} & D_1 \cdot D_0 \\
{\scriptstyle \varepsilon_0 \cdot D_1}\downarrow & & \downarrow{\scriptstyle D_1 \cdot \varepsilon_0} \\
D_1 & = & D_1
\end{array}
\qquad
\begin{array}{ccc}
D_0 \cdot D_1 & \xrightarrow{\theta} & D_1 \cdot D_0 \\
{\scriptstyle D_0 \cdot \varepsilon_1}\downarrow & & \downarrow{\scriptstyle \varepsilon_1 \cdot D_0} \\
D_0 & = & D_0
\end{array}
$$

$$
\begin{array}{ccccc}
D_0 \cdot D_1 & & \xrightarrow{\theta} & & D_1 \cdot D_0 \\
{\scriptstyle \delta_0 \cdot D_1}\downarrow & & & & \downarrow{\scriptstyle D_1 \cdot \delta_0} \\
D_0 \cdot D_0 \cdot D_1 & \xrightarrow[D_0 \cdot \theta]{} & D_0 \cdot D_1 \cdot D_0 & \xrightarrow[\theta \cdot D_0]{} & D_1 \cdot D_0 \cdot D_0
\end{array}
\qquad
\begin{array}{ccccc}
D_0 \cdot D_1 & & \xrightarrow{\theta} & & D_1 \cdot D_0 \\
{\scriptstyle D_0 \cdot \delta_1}\downarrow & & & & \downarrow{\scriptstyle \delta_1 \cdot D_0} \\
D_0 \cdot D_1 \cdot D_1 & \xrightarrow[\theta \cdot D_1]{} & D_1 \cdot D_0 \cdot D_1 & \xrightarrow[D_1 \cdot \theta]{} & D_1 \cdot D_1 \cdot D_0)
\end{array}
$$

- Comonad structures on $D_0 \cdot D_1$ compatible with $(D_0, \varepsilon_0, \delta_0)$ and $(D_1, \varepsilon_1, \delta_1)$ are in a bijection with distributive laws.

Distributive laws of directed containers

- A *distributive law* between directed containers $(S_0, P_0, \downarrow_0, \mathsf{o}_0, \oplus_0)$ and $(S_1, P_1, \downarrow_1, \mathsf{o}_1, \oplus_1)$ is given by 3 operations
    - $t_1 : \Pi\, s : S_0.\, \Pi\, v : P_0\, s \to S_1.\, P_1\, (v\, (\mathsf{o}_0\, \{s\})) \to S_0$
    - $q_0 : \Pi\, \{s : S_0\}.\, \Pi\, \{v : P_0\, s \to S_1\}.\, \Pi\, p_1 : P_1\, (v\, (\mathsf{o}_0\, \{s\})).$ $P_0\, (t_1\, s\, v\, p_1)) \to P_0\, s$
    - $q_1 : \Pi\, \{s : S_0\}.\, \Pi\, \{v : P_0\, s \to S_1\}.\, \Pi\, p_1 : P_1\, (v\, (\mathsf{o}_0\, \{s\}).$ $\Pi\, p_0 : P_0\, (t_1\, s\, v\, p_1).\, P_1\, (v\, (q_0\, \{s\}\, \{v\}\, p_1\, p_0))$

  subject to 11 laws (on next slides).
- They induce a container morphism $(t, q : (S_0, P_0) \cdot^{\mathrm{c}} (S_1, P_1) \to (S_1, P_1) \cdot^{\mathrm{c}} (S_0, P_0)$
    - $t\, (s, v) = (v\, (\mathsf{o}_0\, \{s\}), t_1\, s\, v)$
      $q\, \{s, v\}\, (p_1, p_0) = (q_0\, \{s\}\, \{v\}\, p_1\, p_0, q_1\, \{s\}\, \{v\}\, p_1\, p_0)$

Distributive law laws

- A distribute law is required to satisfy these 11 laws.
- (Shape equations)

1. $\forall\{s, v, p_1, p_0\}.\ t_1\ s\ v\ p_1 \downarrow_0 p_0 =$
   $t_1\,(s \downarrow_0 q_0\ p_1\ p_0)\,(\lambda p_0'.\ v\,(q_0\ p_1\ p_0 \oplus_0 p_0'))\,(q_1\ p_1\ p_0)$
2. $\forall\{s, v\}.\ t_1\ s\ v\ \mathrm{o}_1 = s$
3. $\forall\{s, v, p_1, p_1'\}.\ t_1\ s\ v\,(p_1 \oplus_1 p_1') =$
   $t_1\,(t_1\ s\ v\ p_1)\,(\lambda p_0.\ v\,(q_0\ p_1\ p_0) \downarrow_1 q_1\ p_1\ p_0)\ p_1'$

Distributive law laws

- (Position equations)

  4. $\forall\{s, v, p_1\}.\ q_0\ \{s\}\ \{v\}\ p_1\ o_0 = o_0$
  5. $\forall\{s, v, p_1, p_0, p_0'\}.\ q_0\{s\}\ \{v\}\ p_1\ (p_0 \oplus_0 p_0') =$
     $q_0\ p_1\ p_0 \oplus_0 q_0\ (q_1\ p_1\ p_0)\ p_0'$
  6. $\forall\{s, v, p_0\}.\ q_0\ \{s\}\ \{v\}\ o_1\ p_0 = p_0$
  7. $\forall\{s, v, p_1, p_1', p_0\}.\ q_0\{s\}\ \{v\}\ (p_1 \oplus_1 p_1')\ p_0 =$
     $q_0\ p_1\ (q_0\ p_1'\ p_0)$
  8. $\forall\{s, v, p_1\}.\ q_1\ \{s\}\ \{v\}\ p_1\ o_0 = p_1$
  9. $\forall\{s, v, p_1, p_0, p_0'\}.\ q_1\{s\}\ \{v\}\ p_1\ (p_0 \oplus_0 p_0') =$
     $q_1\ (q_1\ p_1\ p_0)\ p_0'$
  10. $\forall\{s, v, p_0\}.\ q_1\ \{s\}\ \{v\}\ o_1\ p_0 = o_1$
  11. $\forall\{s, v, p_1, p_1', p_0\}.\ q_1\{s\}\ \{v\}\ (p_1 \oplus_1 p_1')\ p_0 =$
      $q_1\ p_1\ (q_0\ p_1\ p_0) \oplus_1 q_1\ p_1'\ p_0$

- The laws 4-11 resemble the conditions of matching pairs of mutual actions.

Composed directed container

- Given two directed containers and a distributive laws between them, the composed directed container is given by
  - $S = \Sigma s : S_0.\, P_0\, s \to S_1$
  - $P\,(s, v) = \Sigma p : P_0\, s.\, P_1\,(v\, p_0)$
  - $o\{s, v\} = (o_0\{s\}, o_1\{v\,(o_0\{s\})\})$
  - $(s, v) \downarrow (p_0, p_1) = (t_1\,(s \downarrow_0 p_0)\,(\lambda p.\, v\,(p_0 \oplus_0 p))\, p_1,$
    $\lambda p.\, v\,(p_0 \oplus_0 q_0\, p_1\, p) \downarrow_1 q_1 p_1 p)$
  - $(p_0, p_1) \oplus (p'_0, p'_1) = (p_0 \oplus_0 q_0\, p_1\, p'_0, q_1\, p_1\, p'_0 \oplus_1 p'_1)$

Product comonad and any comonad

- Two directed containers
  - $S_0, P_0, \downarrow_0, o_0, \oplus_0$ arbitrary *for any comonad*
  - S_1 arbitary, $P_1\,s = 1$, *for the product comonad*
    $s \downarrow_1 * = s$, $o_1 = *$, $* \oplus_1 * = *$
- Distributive law
  - $t_1\,s\,v\,p = s$
  - $q_0\,p_1\,p_0 = p_0$
  - $q_1\,p_1\,p_0 = p_1$
- Composed directed container
  - $S = \Sigma s : S_0 . P_0\,s \to E$
  - $P\,(s, v) = \Sigma p_0 : P_0\,s . 1$
  - $(s, v) \downarrow (p_0, p_1) = (s \downarrow_0 p_0\,, \lambda p\,.v(p_0 \oplus_0 p))$
  - $o\{s, v\} = (o_0, *)$
  - $(p_0, *) \oplus (p_0', *) = (p_0 \oplus_0 p_0', *)$

Streams with suffixes and sampling

- Two directed containers
  - $S_0 = 1$, $P_0 * = \mathsf{Nat}$,
    $* \downarrow_0 p = *$, $\mathsf{o}_0 = 0$, $p \oplus_0 p' = p + p'$
  - $S_1 = 1$, $P_1 * = \mathsf{Nat}$,
    $* \downarrow_1 p = *$, $\mathsf{o}_1 = 1$, $p \oplus_1 p' = p \times p'$
- Distributive law
  - $t_1 * (\lambda \_ . *) \, p = *$
  - $q_0 \, p_1 \, p_0 = p_1 \times p_0$
  - $q_1 \, p_1 \, p_0 = p_1$
- Composed directed container
  - $S = \Sigma * : 1. \, P_0 * \to 1$
  - $P \, (*, \lambda \_ . *) = \Sigma p_0 : P_0. \, P_1 *$
  - $(*, \lambda \_ . *) \downarrow (p_0, p_1) = (*, \lambda \_ . *)$
  - $\mathsf{o} = (0, 1)$
  - $(p_0, p_1) \oplus (p'_0, p'_1) = (p_0 + p_1 \times p'_0, p_1 \times p'_1)$