A system- and language-theoretic outlook on cellular automata

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Overview

- Cellular automata (CA) are descriptions of global dynamics in terms of local transformations, applied at all points at the same time.
- By their own nature, they are easy to implement on a computer, and useful as tools for qualitative analysis of dynamical systems.
- Their properties are also a very vast research field.

Applications

- Population dynamics.
- Economics.
- Fluid dynamics.
- Simulations of geological phenomena.
- Symbolic dynamics.
- Approximation of differential equations.
- Screen savers.
- And many more...

Section 1

Introduction

History

- von Neumann, 1950s:
  mechanical model of self-reproduction
- Moore, 1962:
  the Garden of Eden problem
- Hedlund, 1969:
  shift dynamical system
- Richardson, 1972:
  d-dimensional cellular automata
- Hardy, de Pazzis, Pomeau 1976:
  lattice gas automata
- Amoroso and Patt, 1972; Kari, 1990:
  the invertibility problem

John von Neumann's model of self reproduction

- An infinite square grid
- A finite number of states for each point of the grid
- A finite number of neighbors for each point of the grid
- An evolution law where the next state of each point only depends on the current states of its neighbors

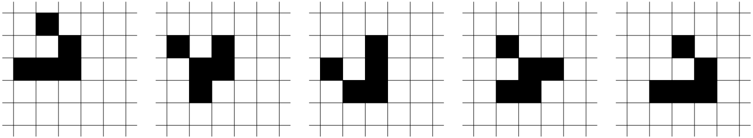
Life is a Game

Ideated by John Horton Conway (1960s) popularized by Martin Gardner.
The checkboard is an infinite square grid.
Each case of the checkboard is "surrounded" by those within a chess' king's move, and can be "living" or "dead".

1. A "dead" case surrounded by exactly three living cases, becomes living.
2. A living case surrounded by two or three living cases, survives.
3. A living case surrounded by less than two living cases, dies of isolation.
4. A living case surrounded by more than three living cases, dies of overpopulation.

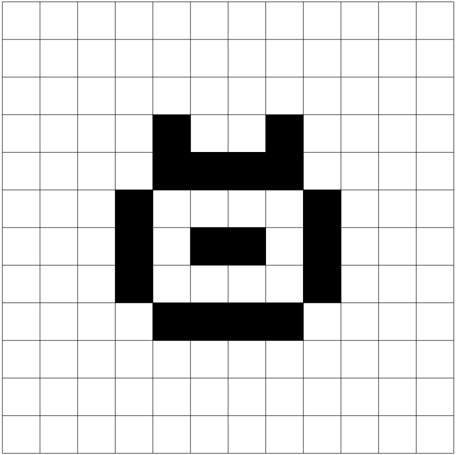
Simple rule, complex behavior

The structures of the Game of Life can exhibit a wide range of behaviors.
This is a glider, which repeats itself every four iterations, after having moved:

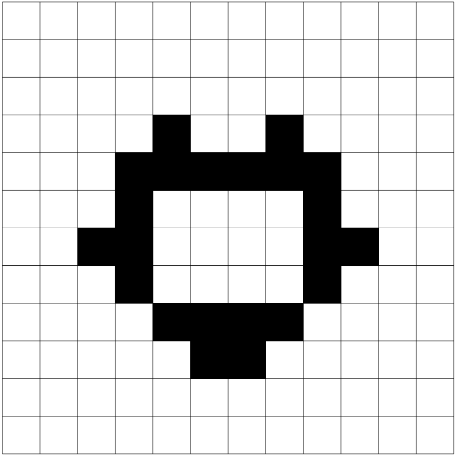


Gliders can transmit information between regions of the checkboard.
Actually, using gliders and other complex structures, any planar circuit can be simulated inside the Game of Life.

On a more funny side, this is called the Cheshire cat:



... because it vanishes...



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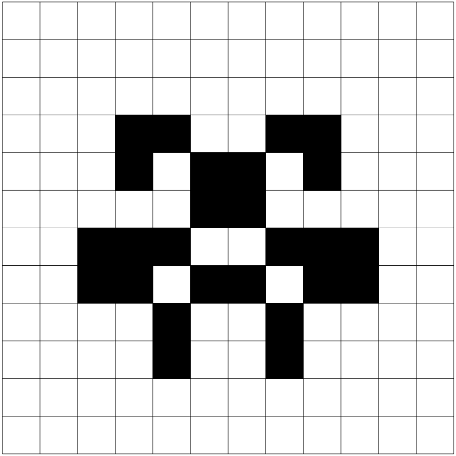

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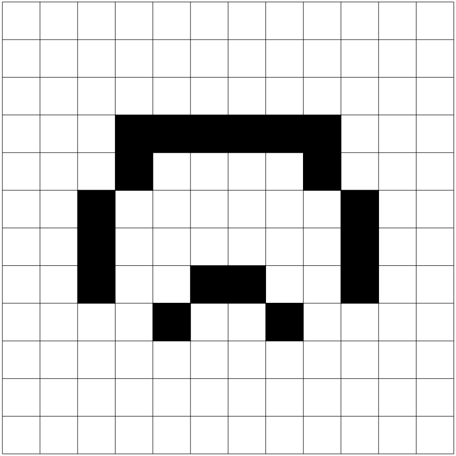

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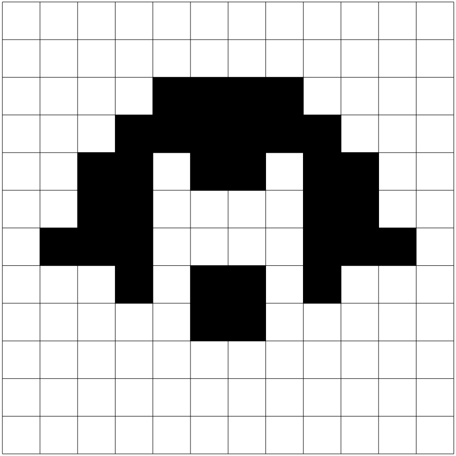

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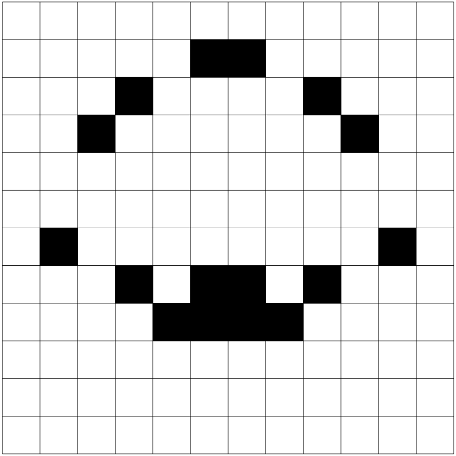
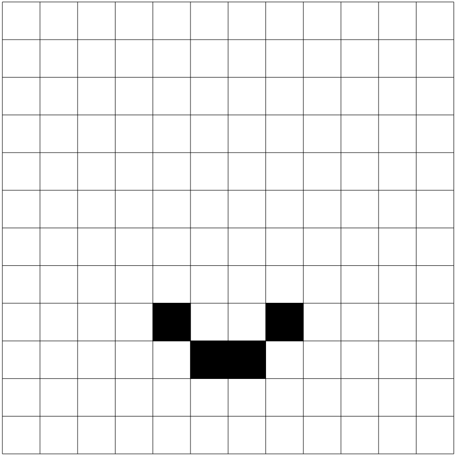
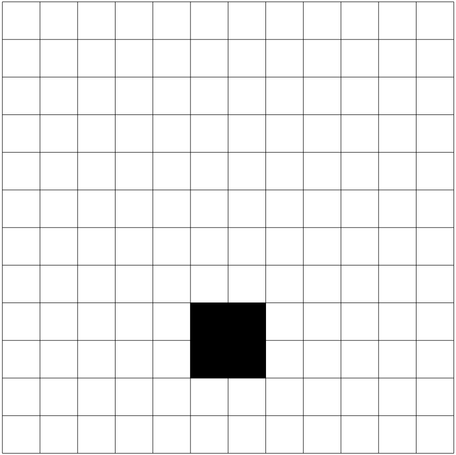


... until the smile alone cheers at us...



... and at last, only a pawprint remains to tell it was there!



The ingredients of a recipe

A cellular automaton (CA) is a quadruple $\mathcal{A} = \langle d, A, \mathcal{N}, f \rangle$ where

- $d > 0$ is an integer—dimension
- $A = \{q_1, \ldots, q_n\}$ is a finite set—alphabet
- $\mathcal{N} = \{n_1, \ldots, n_k\}$ is a finite subset of $\mathbb{Z}^d$—neighborhood
- $f : A^{\mathcal{N}} \to A$ is a function—local map

Special neighborhoods are:

- the von Neumann neighborhood of radius r
  $\mathrm{vN}(r) = \{x \in \mathbb{Z}^d : \sum_{i=1}^{d} |x_i| \leq r\}$
- the Moore neighborhood of radius r
  $\mathrm{M}(r) = \{x \in \mathbb{Z}^d : \max_{1 \leq i \leq d} |x_i| \leq r\}$

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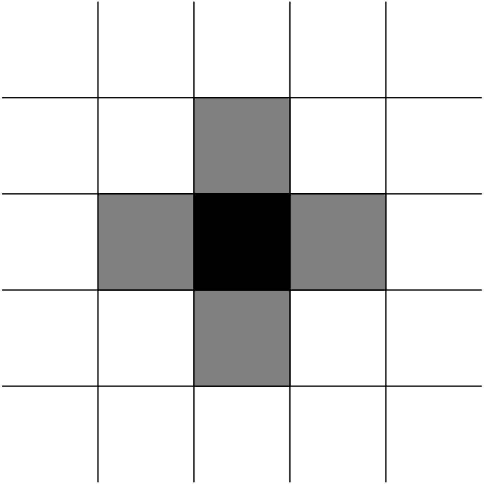
For $d = 2$, this is von Neumann's neighborhood $\mathrm{vN}(1)$...



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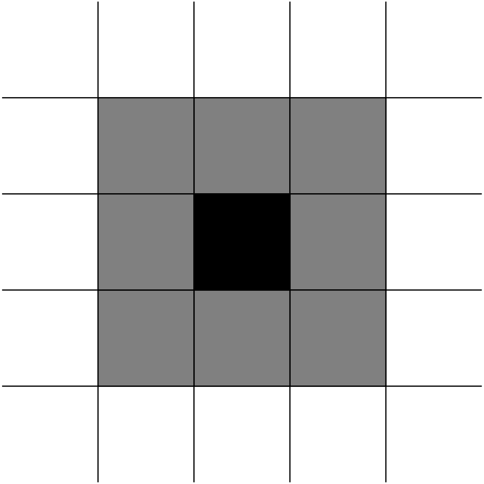

From local to global

A d-dimensional configuration is a map $c : \mathbb{Z}^d \to A$.
Let $\mathcal{A} = \langle d, A, \mathcal{N}, f \rangle$ be a CA.
The map $F_{\mathcal{A}} : A^{\mathbb{Z}^d} \to A^{\mathbb{Z}^d}$ defined by

$$(F_{\mathcal{A}}(c))(x) = f(c(x + n_1), \ldots, c(x + n_k))$$

is the global evolution function.
We say that $\mathcal{A}$ is injective, surjective, etc. if $F_{\mathcal{A}}$ is.

Implementations

Given their distinctive features, CA are straightforward to implement on a computer.

More difficult is to provide a general framework for CA.

Such frameworks often work on a torus instead of the full plane.

- Hardware
  - CAM6 (Toffoli and Margolus, ca. 1985; expansion card for PC)
  - CAM8 (Toffoli and Margolus, ca. 1990; external device for SparcStation)
- Software
  - JCASim (Weimar; in Java)
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Wolfram's classification of CA

Stephen Wolfram: pioneering and influential work on 1-D CA.
Four classes, depending on dynamics:

1. evolution leads to homogenous state
2. evolution leads to periodic structures
3. evolution leads to chaotic space-time patterns
4. evolution leads to complex localized structures

Wolfram's classification is much of an "appeal to common sense" and cannot, for example, identify universal computation.
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Wolfram's enumeration of 1D CA rules

Given a 1-dimensional, 2-state rule with neighborhood $\mathrm{vN}(1)$,

1. identify the sequence $(x, y, z) \in \{0, 1\}^{\mathrm{vN}(1)}$ with the the binary number xyz, and
2. associate to the rule f the number $\sum_{j=0}^{7} 2^j f(j)$.

Exercise: compute Wolfram's number for $f(x, y, z) = x \oplus z$.
Hint:

x	1	1	1	1	0	0	0	0
y	1	1	0	0	1	1	0	0
z	1	0	1	0	1	0	1	0
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Symbolic dynamics

Origins

- Hadamard, 1898:
  geodesic flows on surfaces of negative curvature
- Morse and Hedlund, 1938:
  trajectories as infinite words

Key ideas

- given a continuous dynamics on a space
- identify finitely many "gross-grained" aggregates
- and consider evolution of these via iteration of the dynamics
- Then infer properties of original dynamics via those of the new one

Symbolic dynamics also considers CA—usually, calling them "sliding block codes"—though possibly with different start and end alphabets.

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Subshifts

The shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is given by

$$(\sigma(w))(x) = w(x+1) \quad \forall x \in \mathbb{Z}$$

The shift is continuous w.r.t. the distance defined as

if $(x_1)_{[-r,r]} = \neq (x_2)_{[-r,r]}$ and $(x_1)_{[-r+,r-1]} = (x_2)_{[-r-1,r+1]}$
then $d(x_1, x_2) = 2^{-r}$

A shift space (subshift) is an $X \subseteq A^{\mathbb{Z}}$ which is

1. closed—in the sense that sequences in X converging in $A^{\mathbb{Z}}$ have their limit in X
2. shift-invariant

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Characterization of subshifts

The language of a subshift X is

$$\mathcal{L}(X) = \{w \in A^* \mid \exists x \in X \mid x = lwr\}$$

Given $\mathcal{F} \subseteq A^*$, let $\mathrm{X}_{\mathcal{F}}$ be the set of bi-infinite words that have no factor in $\mathcal{F}$.

1. $\mathrm{X}_{\mathcal{F}}$ is a subshift.
2. For every $X \subseteq A^{\mathbb{Z}}$ there exists $\mathcal{F} \subseteq A^*$ s.t. $X = \mathrm{X}_{\mathcal{F}}$.

A shift of finite type (SFT) is a subshift for which $\mathcal{F}$ can be chosen finite.

Applications: data storage

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Sofic shifts

Fact For a subshift $X \subseteq A^{\mathbb{Z}}$ the following are equivalent:

1. X is the image of a SFT via a CA
2. X is the set of labelings of bi-infinite paths on some finite labeled graph
3. $\mathcal{L}(X)$ has finitely many successor sets

$$F(w) = \{u \in A^* \mid wu \in \mathcal{L}(X)\} \ , \ w \in \mathcal{L}(X)$$

4. $\mathcal{L}(X)$ is a factorial closed regular language

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… and for $d > 1$?

Many of these concept extend naturally to higher dimension:

- patterns—i.e., d-dimensional words (rectangular, etc.)
- translations—i.e., shifts in several directions
- multi-dimensional subshifts
- finiteness of type in dimension d
- images of subshifts via CA
- multi-dimensional SFT
- sofic shifts as images of SFT via CA
- and many more…

though not all (e.g. sofic shifts presentations by labeled graphs)

Section 2
Facts

Hedlund's theorem (1969)

Let $X \subseteq A^{\mathbb{Z}}$, $Y \subseteq B^{\mathbb{Z}}$ be subshifts. Let $F : X \to Y$.
The following are equivalent:

1. F is a CA global map
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Reason why: $A^{\mathbb{Z}}$ is compact w.r.t. metric d.
Note: true in arbitrary dimension, even if dynamics restricted to subshift

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Cellular automata and Turing machines

Let $\mathcal{T}$ be a Turing machine with output alphabet Σ and set of states Δ.

Construct $\mathcal{A}$ as follows:

1. $d = 1$
2. $A = \Sigma \times (\Delta \cup \{\mathrm{no-head}\})$
3. $\mathcal{N} = \{-1, 0, 1\}$
4. f so that it reproduces
   - the write operation of $\mathcal{T}$ on the first component, and
   - the state update of $\mathcal{T}$ and the movement of $\mathcal{T}$'s head on the right component.

Then $\mathcal{A}$ simulates $\mathcal{T}$ in real time, so that

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One after another

1. Given $\mathcal{A}_j = \langle d, A, \mathcal{N}_j, f_j \rangle$, $j = 1, 2$
2. put $\mathcal{N} = \{x_1 + x_2 \mid x_1 \in \mathcal{N}_1, x_2 \in \mathcal{N}_2\}$
3. and define $f : Q^{\mathcal{N}} \to Q$ as

$$f(\alpha) = f_1 \left(\ldots, f_2(\ldots, \alpha_{n_{1,i}+n_{2,j}}, \ldots,), \ldots \right)$$

Then $\mathcal{A} = \langle d, A, \mathcal{N}, f \rangle$ satisfies $F_{\mathcal{A}} = F_{\mathcal{A}_1} \circ F_{\mathcal{A}_2}$, so that

the class of CA with given dimension and alphabet is a monoid under composition.

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Reversibility

A CA $\mathcal{A}$ is reversible if

1. $\mathcal{A}$ is invertible, and
2. $F_{\mathcal{A}}^{-1}$ is the global evolution function of some CA.

Equivalently, $\mathcal{A}$ is reversible iff there exists $\mathcal{A}'$ s.t. both $\mathcal{A}' \circ \mathcal{A}$ and $\mathcal{A} \circ \mathcal{A}'$ are the identity cellular automaton.

This seems more than just existence of inverse global evolution function.

Reversible CA are important in physical modelization because Physics, at microscopical scale, is reversible.

Fact CA reversibility is r.e.

Reason why: try all CA until a composition of local functions returns the "identity" $f(c(x+n_1),\ldots,c(x+nj_k)) = c(x)$

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Richardson's reversibility principle(1972)

The following are equivalent:

1. $\mathcal{A}$ is reversible
2. $\mathcal{A}$ is bijective

Reason why: compactness and Hedlund's theorem.
Thus, existence of inverse CA comes at no cost from existence of inverse global evolution, so that

the class of reversible CA with given dimension and alphabet is a group under composition.

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Reversible CA are universal

Theorem (Toffoli, 1977)
Every d-dimensional cellular automaton can be simulated by a $(d+1)$-dimensional reversible cellular automaton.

Theorem (Morita and Harao, 1989)
Reversible Turing machines can be simulated by 1-dimensional reversible cellular automata.

Gardens of Eden

A Garden of Eden (GoE) for a CA $\mathcal{A}$ is an object that has no predecessor according to the global law of $\mathcal{A}$.
This applies to both configurations and patterns, even if global law is restricted to a subshift.
A GoE pattern is allowed for X and forbidden for $F_{\mathcal{A}}(X)$.
Lemma Suppose $F_{\mathcal{A}} : X \to X$. The following are equivalent:

1. $\mathcal{A}$ has a GoE configuration
2. $\mathcal{A}$ has a GoE pattern

Reason why: compactness.
Corollary: CA surjectivity is co-r.e.
Reason why: try all patterns until one has no predecessors.
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Gardens of Eden

A Garden of Eden (GoE) for a CA $\mathcal{A}$ is an object that has no predecessor according to the global law of $\mathcal{A}$.
This applies to both configurations and patterns, even if global law is restricted to a subshift.
A GoE pattern is allowed for X and forbidden for $F_{\mathcal{A}}(X)$.
Lemma Suppose $F_{\mathcal{A}} : X \to X$. The following are equivalent:

1. $\mathcal{A}$ has a GoE configuration
2. $\mathcal{A}$ has a GoE pattern

Reason why: compactness.
Corollary: CA surjectivity is co-r.e.
Reason why: try all patterns until one has no predecessors.
Note: still true if CA dynamics restricted to a subshift

Moore-Myhill's theorem (1962)

Two distinct patterns p_1, p_2 on the same support E are mutually erasable (m.e.) for $\mathcal{A}$ if $F_{\mathcal{A}}(c_1) = F_{\mathcal{A}}(c_2)$ whenever $(c_i)|_E = p_i$ and $(c_1)|_{\mathbb{Z}^d \setminus E} = (c_2)|_{\mathbb{Z}^d \setminus E}$.

The following are equivalent:

1. $\mathcal{A}$ has a GoE pattern on $A^{\mathbb{Z}^d}$
2. $\mathcal{A}$ has two m.e. pattern on $A^{\mathbb{Z}^d}$

Reason why: the boundary of a hypercube grows “slower” than the hypercube

Corollary: (Richardson's lemma, 1972) injective CA are surjective

Caution: not true if CA dynamics restricted to arbitrary subshift (Fiorenzi, 2000 even for $d = 1$)

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Wolfram's rule 90 is surjective but not injective

Non-injectivity: put

$$c_0(x) = 0 \ \ \forall x \in \mathbb{Z} \ \ ; \ \ c_1(x) = 1 \ \ \forall x \in \mathbb{Z}$$

then $F_{90}(c_0) = F_{90}(c_1) = c_0$.

Surjectivity:

1. for every a and k, the equation $a \oplus x = k$ has a unique solution
2. for every b and k, the equation $x \oplus b = k$ has a unique solution

Thus every configuration has exactly four predecessors for Wolfram's rule 90.

Is this just a case?

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The balancement theorem

Given $\mathcal{A} = \langle d, A, \mathcal{N}, f \rangle$, $U \subseteq \mathbb{Z}^d$, define $F_U : A^{U+\mathcal{N}} \to A^U$ as

$$(F_U(p))(z) = f(p(z+n_1), \ldots, p(z+n_k))$$

Theorem (Maruoka and Kimura, 1976) The following are equivalent:

1. $\mathcal{A}$ is surjective
2. for every $U \subseteq \mathbb{Z}^d$, any $p : U \to A$ has the same number of F_U-preimages

Reason why: Moore-Myhill's theorem

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The invertibility problem

Let $\mathcal{C}$ be a class of cellular automata.
The invertibility problem for $\mathcal{C}$ states:

given an element $\mathcal{A}$ of $\mathcal{C}$,
determine whether $F_{\mathcal{A}}$ is invertible.

Meaning: invertibility of the global dynamics of any CA in $\mathcal{C}$ can be inferred algorithmically by looking at its local description.

Decidability of the invertibility problem

Theorem (Amoroso and Patt, 1972)

The invertibility problem for 1D CA is decidable.

Proof: rather convoluted, "should be adaptable to $d > 1$".

Theorem (Kari, 1990)

The invertibility problem for 2D CA is undecidable.

Proof: by reduction from Hao Wang's Tiling Problem.

Corollary: The invertibility problem for dD CA is undecidable for all $d \geq 2$.

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Section 3
Results

Which dynamics are CA dynamics?

Let $F : X \to X$ be a continuous dynamics on a compact space X.
Question: Can that dynamics be described by a CA?
That is:
Are there

- a one-to-one and onto correspondance θ between X and (a subshift of) A^G
- a CA $\mathcal{A}$ on A^G

such that $\theta \circ F = F_{\mathcal{A}} \circ \theta$?

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Conjecture (Levin and Toffoli, 1980)

The following are equivalent:

1. (X, F) has a presentation as a d-dimensional CA;
2. there exists a continuous action ϕ of $\mathbb{Z}^d$ on X such that
   2.1 F commutes with ϕ and
   2.2 a map $\pi : X \to A$ exists such that

   if $x_1 \neq x_2$
   then $\pi(\phi_z(x_1)) \neq \pi(\phi_z(x_2))$ for some $z \in \mathbb{Z}^d$

Rationale: evaluation at a point acts as an "observation at the microscope"

Theorem (Capobianco, 2004)

The following are equivalent:

1. (X, F) has a presentation as a d-dimensional CA on some subshift
2. the hypotheses of Levin and Toffoli's conjecture hold.

Reason why: ϕ would take the role of the natural action.
But the natural action cannot tell $A^{\mathbb{Z}^d}$ from an arbitrary subshift.
Thus, the "completeness" requirement may not be satisfied.

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Other kinds of finitary descriptions

- Lattice gas automata operate via a two-phase discipline:
  1. a many-to-many collision in the nodes
  2. a reversible propagation along lines
- Block automata
  1. subdivision of the space in blocksat each step
  2. local map operates on the states of single blocks

Advantages: allow realizations with greater thermodynamical efficiency

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Are CA dynamics block automata dynamics?

1. Kari, 1996:
   YES for reversible CA if $d \leq 2$
2. Durand-Lôse, 2001:
   YES for reversible CA but a larger alphabet is required
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Theorem (Toffoli, Capobianco and Mentrasti, *TCS* 2008)

Any 1D non-surjective CA can be rewritten as a block automaton.

Reason why:

- non-surjective CA have GoE patterns
- by Fekete's lemma, the number of GoE patterns grows unbounded
- then, the state of large enough blocks can be compressed to encode that of the boundary

... but what if $d > 1$?

Conjecture (TCM) YES

Reason to believe: by a generalization of Fekete's lemma (Capobianco, *DMTCS* 2008) the number of GoE patterns grows faster than the number of patterns on the boundary

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A chaotic issue—and a possible solution

No translation invariant distance can induce the product topology.
Reason why: for that topology, the shift is a chaotic map
Idea: change the topology! (with some loss)
Define d_B on $\{0,1\}^{\mathbb{Z}}$ as

$$d_B(c_1, c_2) = \limsup_{n \to +\infty} \frac{|\{z \in [-n, n] \mid c_1(z) \neq c_2(z)\}|}{2n+1}$$

and

$$c_1 \sim c_2 \Leftrightarrow d_B(c_1, c_2) = 0$$

Then consider the Besicovitch space $X_B = A^{\mathbb{Z}} / \sim$.
This corresponds to the ultimate point of view of an observer getting *farther and farther* from the grid.

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CA in Besicovitch space

If $\mathcal{A}$ is a CA, then

$$F_B\left([c]_\sim\right) = [F(c)]_\sim$$

is well defined.
Moreover (Blanchard, Formenti, and Kurka, 1999) several properties of $\mathcal{A}$ can be inferred from those of F_B.
In particular, $F_{\mathcal{A}}$ is surjective iff F_B is.

Besicovitch spaces in arbitrary dimension

Let $\{U_n\}_{n\in\mathbb{N}}$ satisfy

1. $U_n \subseteq U_{n+1}$ for all n
2. $\bigcup_{n\in\mathbb{N}} U_n = \mathbb{Z}^d$

The quotient space X_B of $A^{\mathbb{Z}^d}$ w.r.t.

$$c_1 \sim c_2 \Leftrightarrow \lim_{n\to\infty} \frac{|\{z \in U_n \mid c_1(z) \neq c_2(z)\}|}{|U_n|} = 0$$

is the Besicovitch space associate to $\{U_n\}$.
X_B is a metric space w.r.t. the Besicovitch distance

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A Richardson-like theorem (Capobianco, *JCA* 2009)

Let $\mathcal{A}$ be a d-dimensional CA with alphabet A.
Let $\{U_n\}$ be the sequence of either von Neumann or Moore neighborhoods of radius n.

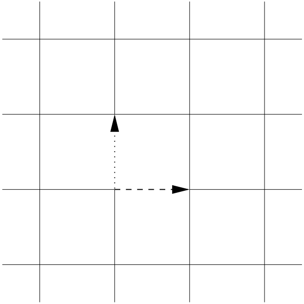
1. The classes of d_B are the same in either case.
2. d_B is invariant by translations.
3. $F_{\mathcal{A}}$ induces a Lipschitz continuous $F_B : X_B \to X_B$
4. $\mathcal{A}$ is surjective iff F_B is.
5. If F_B is injective, then it is surjective.

Cayley graphs

Instead of $\mathbb{Z}^d$, one can use such grids.

- Take a group G—even non-commutative
- together with a finite set S
- such that every $g \in G$ "is" a word on $S \cup S^{-1}$
- and construct a graph $\mathrm{Cay}(G, S)$
- whose nodes are the elements of G
- and an arc (g, h) exists iff $g^{-1}h \in S \cup S^{-1}$

Example with $G = \mathbb{Z}^2$, $S = \{(1,0),(0,1)\}$



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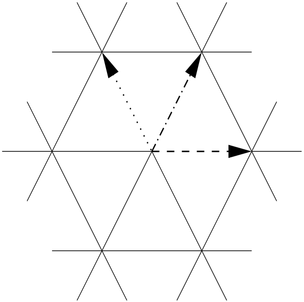

CA on Cayley graphs

Then

1. one can define translations as (beware of order!)

$$(c^g)(h) = c(gh)$$

2. each node has finitely many one-step neighbors
3. the "shape" of one-step neighborhood is the same for all nodes

and it's possible to define CA on such groups, via

$$(F(c))(g) = f\left(c(gn_1), \ldots, c(gn_k)\right)$$

... and subshifts still exist

Simply define a pattern as a map $p : E \longrightarrow A$ for some finite $E \subseteq G$.
p occurs in c iff $(c^g)|_E = p$ for some g.

Changes with respect to the "classical" setting

- Characterization of subshifts: holds
- Hedlund's theorem: holds
- Reversibility principle: holds
- Translations are CA: holds only for some elements of the group!
- Characterization of CA dynamics: holds
- Richardson's lemma for the Besicovitch space: holds if group and sequence are "good enough"

Subshift extensions to larger groups

Suppose $G \subseteq \Gamma$.

Consider a set $\mathcal{F}$ of patterns over G.

Question: is there any relation between the subshifts defined by $\mathcal{F}$ on A^G and A^Γ?

Question: and does the induced shift depend on $\mathcal{F}$

Theorem (Capobianco, LATA 2008) Suppose $\mathcal{F}_i$ induce subshifts X_i and Ξ_i and local maps f_i induce CA F_i and Φ_i when considered on G and Γ,respectively. Then

$$F_1(X_1) \subseteq F_2(X_2) \Leftrightarrow \Phi_1(\Xi_1) \subseteq \Phi_2(\Xi_2)$$

Reason why: since the $\mathcal{F}_i$ and f_i are "based on" G, dynamics on A^Γ can be "sliced" w.r.t. the left cosets of G.

Corollary: induced depends on subshift not on description

Corollary: a subshift induced by a sofic shift is sofic

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CA extensions to larger groups

As a consequence, CA extension to alarger group is always well defined.
(Easier to visualize in d and $d + d'$ dimensions.)
... but the abstract dynamics is usually not the same!
(Immediate if Γ is finite and G is proper.)

Theorem (Capobianco, LATA 2008)

1. The following properties are shared by original and induced CA:
   - injectivity
   - surjectivity
   - existence of m.e. patterns
2. Induced CA contains a copy of original

Corollary: by increasing the group (even up to isomorphisms) and/or the alphabet, the class of CA dynamics grows.

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CA and semi-direct products

The semi-direct product of groups H and K by group homomorphism $\tau : H \to \mathrm{Aut}(K)$ is the group $H \ltimes_\tau K$ of pairs (h, k) with the product

$$(h_1, k_1)(h_2, k_2) = (h_1 h_2, \tau_{h_2}(k_1) k_2)$$

Direct product is a special case when $\tau_h = \mathrm{id}_K \forall h$.
Example: the semi-direct product of $\mathbb{Z}_2$ and $\mathbb{Z}$ by

$$\tau_0(x) = x \; ; \; \tau_1(x) = -x \; .$$

is isomorphic to the infinite dihedral group

$$D_\infty = \langle a, b \mid a^2 = (ab)^2 = e \rangle$$

Note: $H \ltimes_\tau K$ is f.g. if H and K are both.

CA and semi-direct products

The semi-direct product of groups H and K by group homomorphism $\tau : H \to \mathrm{Aut}(K)$ is the group $H \ltimes_\tau K$ of pairs (h, k) with the product

$$(h_1, k_1)(h_2, k_2) = (h_1 h_2, \tau_{h_2}(k_1) k_2)$$

Direct product is a special case when $\tau_h = \mathrm{id}_K \forall h$.
Example: the semi-direct product of $\mathbb{Z}_2$ and $\mathbb{Z}$ by

$$\tau_0(x) = x \; ; \; \tau_1(x) = -x \; .$$

is isomorphic to the infinite dihedral group

$$D_\infty = \langle a, b \mid a^2 = (ab)^2 = e \rangle$$

Note: $H \ltimes_\tau K$ is f.g. if H and K are both.

A "splitting" theorem (Capobianco, *IJAC* 2006)

Let H and K be f.g., $G = H \ltimes_\tau K$.

1. If K is finite, any CA with alphabet A and group G can be rewritten with alphabet A^K and group H.
2. If H is finite, any CA with alphabet A and group G can be rewritten with alphabet A^H and group K.
3. Finiteness of type and soficityare preserved.
4. The transformations above are computable if the word problem is decidable for both H and K.

Reason why: moving in a direction from the finite component cannot take too far

Noteworthy because: the role of H and K is not symmetrical

Corollary: invertibility problem for complete CA on the group of previous slide is decidable.

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Section 4
Conclusions

Personal projects for the future

- Characterize dynamics presented by "complete" CA.
- Extend the "splitting" theorem to group extensions.
  (Or: find a counterexample)
- Study the topological properties of X_B and CA in many dimensions.
- Explore feasibility of a CA variant of Noether's theorem in classical mechanics.

For the interested

On the Web

- Cellular automata FAQ www.cafaq.com
- Jörg R. Weimar's JCASim www.jweimar.de/jcasim/
- Stephen Wolfram's articles www.stephenwolfram.com/publications/articles/ca/

Compendia

- T. Toffoli, N. Margolus. Invertible cellular automata: A review. *Physica D* **45** (1990) 229–253.
- J. Kari. Theory of cellular automata: A survey. *Theor. Comp. Sci.* **334** (2005) 3–33.

Thank you for your attention!

Any questions?