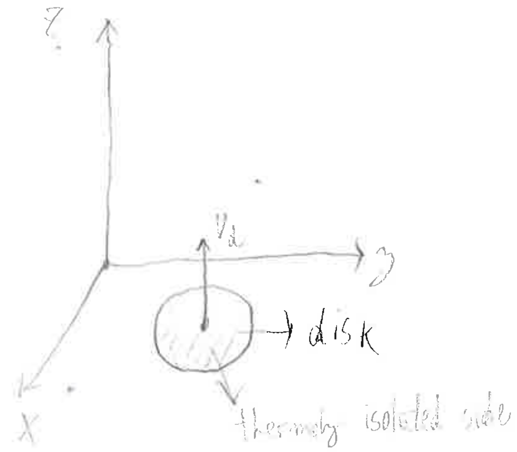
EuPhO 2017

Theory Solution Sheet

Problem Number: 2 **Student Code:** BUL 1

We may assume that there is a coordinate system $Oxyz$:



Since we want to estimate the order of the variables them we can assume that the disk is in the xy plane and it accelerates towards the z axis. Also all molecules are moving in 6 directions:

$\frac{1}{6}$ of the molecules towards z axis.

$\frac{1}{6}$ of the molecules against z axis.

$\frac{1}{6}$ of the molecules towards y axis.

$\frac{1}{6}$ of the molecules against y axis.

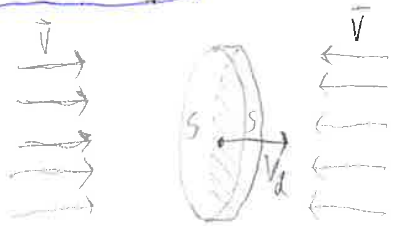
$\frac{1}{6}$ of the molecules towards x axis.

$\frac{1}{6}$ of the molecules against y axis.

At the initial moment the disk have velocity $V=0$ and is affected only by the molecules moving in z direction.

Lets say that all molecules have an average speed of motion $\bar{V}$. If $\bar{V}$ is the average quadratic speed then $\boxed{\frac{\bar{m}.\bar{V}^2}{2} = \frac{3}{2} K_B T_1}$

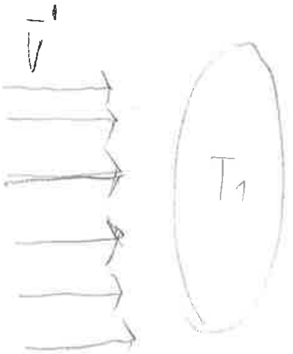
$\Rightarrow \boxed{\bar{V} = \sqrt{\frac{3 K_B T_1}{\bar{m}}}}$ where $\bar{m}$ is the mass of a single gas molecule



For time Δt there will be a number of molecules $\boxed{\Delta N_m = \frac{1}{6}.n.S.(\bar{V}.\Delta t)}$ that will hit the disk on each side. In this case $(S.\bar{V}.\Delta t)$ is the volume of the gas that will reach the surface of the disk and n is the molecular density. If the disk moves with a speed V_d then that changes.

Relative to the disk the gas behind is moving with speed $\boxed{\bar{V}' = \bar{V} - V_d}$

Also we have a similar situation on the other side of the disk but then some of the molecules moving in x and y direction become important.

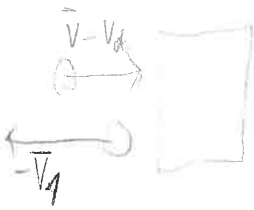


We have for time Δt number of molecules:

$\boxed{\Delta N_m' = \frac{1}{6} n S (\bar{V} - V_d)\Delta t}$ relative to the disk.

They all have momentum $\boxed{\Delta p = \bar{m} \Delta N_m' \cdot (\bar{V} - V_d)}$

We get $\boxed{\Delta p = \bar{m} \cdot \frac{1}{6} n S (\bar{V} - V_d)^2 \cdot \Delta t}$



the difference in momentum before and after the collision is $\boxed{\Delta p_0 = (\bar{V} - V_d - (-\bar{V}_1)) \cdot \bar{m} \cdot \Delta N_m'}$

V_1 can be calculated:

$\boxed{V_1 = \sqrt{\frac{3 k_B T_1}{\bar{m}}}}$ because the molecules heat to T_1.

The energy needed to accelerate the molecules is $\boxed{\Delta E = \Delta N_m' \left(\frac{\bar{m} \bar{V}_1^2}{2} - \frac{\bar{m} (\bar{V} - V_d)^2}{2} \right)}$

Then the difference in the temperature of the disk is:

$\boxed{\Delta E = C_d \cdot \Delta T_1 = N k_B \cdot \Delta T_1 = \frac{M}{\bar{m}_d} \cdot k_B \cdot \Delta T_1}$ where $\bar{m}_d$ is the average mass of a molecule on the disk. We assume $\boxed{\bar{m}_d \approx \bar{m}}$

$\Rightarrow \boxed{\Delta E = \frac{M k_B}{\bar{m}} \cdot \Delta T_1}$. Now in the formula for ΔE we can say that in every moment $\boxed{V_1^2 \gg (\bar{V} - V_d)^2} \Rightarrow \boxed{\Delta E = \Delta N_m' \cdot \frac{\bar{m} \cdot V_1^2}{2}}$

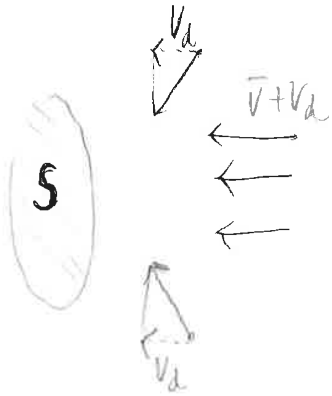
We have $\boxed{\Delta T_1 = \Delta E \cdot \frac{\bar{m}}{M k_B} = \frac{\bar{m}^2}{M k_B} \cdot \frac{V_1^2}{2} \cdot \Delta N_m' = \frac{\bar{m} \cdot 3 k_B T_1}{2 M k_B} \cdot \Delta N_m' = \frac{3}{2} \cdot \frac{T_1 \bar{m}}{M} \cdot \Delta N_m'}$

We get force $\boxed{F = \frac{\Delta p_0}{\Delta t} = (\bar{V} + \bar{V}_1 - V_d) \cdot \bar{m} \cdot \frac{1}{6} n S (\bar{V} - V_d) \cdot \frac{\Delta t}{\Delta t}}$

$\boxed{\bar{m} \cdot n = \rho} \Rightarrow \boxed{F = \frac{(\bar{V} + \bar{V}_1 - V_d)(\bar{V} - V_d)}{6} \cdot \rho \cdot S}$

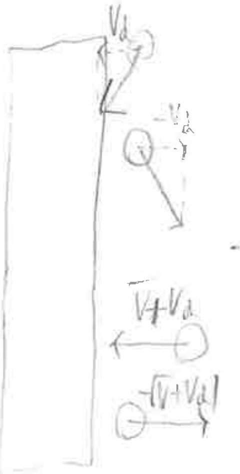
Also $\boxed{\frac{\Delta T_1}{\Delta t} = \frac{3}{2} \cdot \frac{T_1 \cdot \bar{m}}{M} \cdot \frac{1}{6} \cdot n \cdot S \cdot (\bar{V} - V_d) \cdot \frac{\Delta t}{\Delta t} = \frac{1}{4} \cdot \frac{T_1 \rho S (\bar{V} - V_d)}{M}}$

On the other side of the disk we can assume that the molecules experience absolutely elastical collisions.



The molecules moving against the z axis relative to the disk will have speed $\bar{V}+V_d$ and these moving in x and y direction will have radial speed V_d.

For the z axis we have for Δt number of molecules $\boxed{\Delta N_z = \frac{1}{6} n S (\bar{V}+V_d) . \Delta t}$



For the x and y axis we have $\boxed{\Delta N_{xy} = 4 . \frac{1}{6} . n S V_d . \Delta t}$

For the z axis the force is:

$$\boxed{\begin{aligned} F_z &= \frac{\Delta p_z}{\Delta t} = \left(\bar{V}+V_d-(-\bar{V}-V_d)\right) . \frac{\bar{m} . \Delta N_z}{\Delta t} = \\ &= \frac{2(\bar{V}+V_d)^2 . \bar{m} . n . S}{6} = \frac{(\bar{V}+V_d)^2}{3} \rho S \end{aligned}}$$

For the x and y axis

$$\boxed{\begin{aligned} \frac{\Delta p_{xy}}{\Delta t} &= \left(V_d-(-V_d)\right) . \bar{m} . \frac{\Delta N_{xy}}{\Delta t} = \\ &= \frac{4}{3} V_d^2 \rho S = F_{xy} \end{aligned}}$$

The acceleration of the disk is:

$$\boxed{a = \frac{F - F_z - F_{xy}}{M}} \Rightarrow \boxed{a = \frac{\rho S}{M}\left[\frac{(\bar{V}+\bar{V}_1-V_d)(\bar{V}-V_d)}{6} - \frac{4}{3} V_d^2 - \frac{1}{3}(\bar{V}+V_d)^2\right]}$$

for $\boxed{V_d = 0}$ we get $\boxed{a_0 = \frac{\rho S}{M} . \left[\frac{(\bar{V}+\bar{V}_1)\bar{V}}{6} - \frac{1}{3}\bar{V}^2\right] = \frac{\rho S}{6M} . (\bar{V}_1 - \bar{V})\bar{V}}$

Now $\bar{m} = m$ (mass of the molecules in the text).

$$\boxed{Q_0 = \frac{\rho S}{6M} \cdot \frac{3k_B}{m}\left(\sqrt{T_1'} - \sqrt{T_0'}\right)\cdot\sqrt{T_0} = \frac{\rho S k_B}{2Mm}\cdot\left(\sqrt{T_1 T_0} - T_0\right)}$$

$$\boxed{Q_0 \approx 15{,}31 \cdot \frac{\rho S k_B T_0}{Mm}}$$

The maximal speed V_{max} is reached at $\boxed{Q=0}$.

We will make an approximation that the disk was moving with an average acceleration $\boxed{\bar{a} = \frac{Q_0}{2}} \Rightarrow \boxed{\frac{t_0 Q_0}{2} = V_{max}}$.

Then we can assume that the change of V was constant and:

$$\boxed{V_d = t \cdot \frac{Q_0}{2}}$$

Now we have $\boxed{\frac{dT_1}{dt} = -\frac{1}{4}\,\frac{T_1 \rho S(\bar{V} - V_d)}{M}}$

$$\Rightarrow \boxed{\int_{T_1}^{T_1'} -\frac{dT_1}{T_1} = \int_0^{t_0} \frac{\rho S}{4M}\cdot\left(\bar{V} - \frac{Q_0 t}{2}\right)\cdot dt} \Rightarrow \boxed{\frac{T_1'}{T_1} = e^{-\frac{\rho S}{4M}\cdot\left(\bar{V} t_0 - \frac{Q_0 t_0^2}{4}\right)}}$$

Now $\boxed{T_1' \approx T_1\left(1 - \frac{\rho S}{4M}\left(\bar{V} t_0 - \frac{Q_0 t_0^2}{4}\right)\right)}$

From $\boxed{Q=0}$ we have $\boxed{(\bar{V} + \bar{V}_1 - V_{max})(\bar{V} - V_{max}) = 8V_{max}^2 + 2(\bar{V} + V_{max})^2}$

From $\boxed{t_0 = \frac{2V_{max}}{Q_0}} \Rightarrow \boxed{T_1' = T_1\left(1 - \frac{\rho S}{4M}\left(\frac{2\bar{V} V_{max}}{Q_0} - \frac{V_{max}^2}{Q_0}\right)\right)}$

$$\boxed{\sqrt{\frac{T_1'}{T_1}} = \sqrt{\frac{T_1 - \Delta T_1}{T_1}} \approx 1 - \frac{\Delta T_1}{2T_1}} \Rightarrow \boxed{\sqrt{T_1'} = \sqrt{T_1}\left(1 - \frac{\rho S}{8M}\left(\frac{2\bar{V} V_{max}}{Q_0} - \frac{V_{max}^2}{Q_0}\right)\right)}$$

And $\boxed{\bar{V}_1 = \sqrt{\frac{3k_B T_1'}{m}} = \sqrt{\frac{3k_B T_1}{m}}\cdot\sqrt{\frac{T_1'}{T_1}} = \sqrt{\frac{3k_B T_1}{m}}\left(1 - \frac{\rho S}{8M}\left(\frac{2\bar{V} V_{max}}{Q_0} - \frac{V_{max}^2}{Q_0}\right)\right)}$

We get equation of V_{max} from third degree but it can be simplified with approximations.

Lets $\boxed{\frac{\rho S}{8M}\left(\frac{2\bar{V} V_{max}}{Q_0} - \frac{V_{max}^2}{Q_0}\right) \ll 1}$ because $\boxed{\frac{8MQ_0}{\rho S} \gg \bar{V}^2}$ and $\bar{V}$ is the same order as V_{max}.