COUNTRY : ____________________

XXIV INTERNATIONAL PHYSICS OLYMPIAD

WILLIAMSBURG, VIRGINIA, U.S.A.

THEORETICAL COMPETITION

July 12, 1993

Time available: 5 hours

READ THIS FIRST!

INSTRUCTIONS:

1. Use only the pen provided.
2. Use only the marked side of the paper.
3. Begin each problem on a separate sheet.
4. Write at the top of each and every page:

   - The number of the problem
   - The number of the page of your solution in each problem
   - The total number of pages in your solution to the problem.

   Example (for Problem 1): 1 1/4; 1 2/4; 1 3/4; 1 4/4.

General Tabulated Information

Quantity	Symbol	Value
Earth's mean radius	R_E	6.4×10^{6} m.
acceleration due to gravity	g	9.8 m s^{-2}.
Newtonian gravitational constant	G	6.67×10^{-11} N m^2 kg^{-2}.
permittivity of vacuum	ε_o	8.85×10^{-12} C^2 N^{-1} m^{-2}.
permeability of vacuum	μ_0	8.85×10^{-7} N A^{-2}.
speed of light in vacuum (or air)	c	3.00×10^{8} m s^{-1}.
elementary charge	e	1.60×10^{-19} C.
mass of electron	m_e	9.11×10^{-31} kg.
mass of proton	m_p	1.67×10^{-27} kg
Planck constant	h	6.63×10^{-34} J s.
Avogadro constant	N_A	$6{,}02 \times 10^{23}$ mol^{-1}.
Boltzmann constant	k	1.38×10^{-23} J K^{-1}.
molar gas constant	R	8.31 J mol^{-1} K^{-1}.

ATMOSPHERIC ELECTRICITY

From the standpoint of electrostatics, the surface of the Earth can be considered to be a good conductor. It carries a certain total charge Q_o and an average surface charge density σ_o.

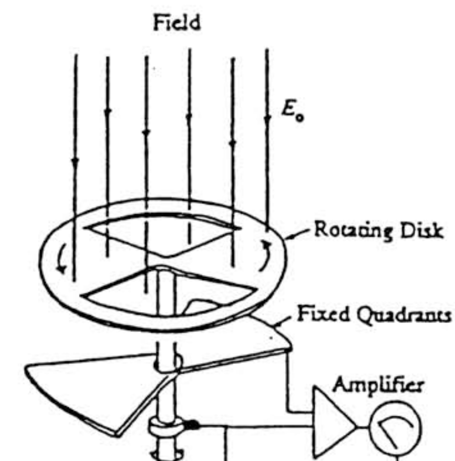
1) Under fair-weather conditions, there is a downward electric field, E_o, at the Earth's surface equal to about 150 V/m. Deduce the magnitude of the Earth's surface charge density and the total charge carried on the Earth's surface.

2) The magnitude of the downward electric field decreases with height, and is about 100 V/m at a height of 100 m. Calculate the average amount of net charge per m^3 of the atmosphere between the Earth's surface and 100 m altitude.

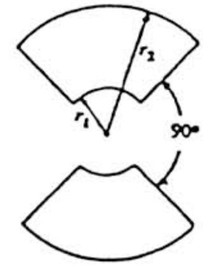
3) The net charge density you have calculated in (2) is actually the result of having almost equal numbers of positive and negative singly-charged ions per unit volume (n_+ and n_-). Near the Earth's surface, under fair-weather conditions, $n_+ \approx n_- \approx 6 \times 10^8\ m^{-3}$. These ions move under the action of the vertical electric field. Their speed is proportional to the field strength:

$$v \approx 1.5 \times 10^{-4} \cdot E,$$

where v is in m/s and E in V/m. How long would it take for the motion of the atmospheric ions to neutralize half of the Earth's surface charge, if no other processes (e.g. lightning) occurred to maintain it?



4) One way of measuring the atmospheric electric field, and hence σ_o, is with the system shown in the diagram. A pair of metal quadrants, insulated from ground but connected to each other, are mounted just underneath a grounded uniformly rotating disk with two quadrant-shaped holes cut in it. (In the diagram, the spacing has been exaggerated in order to show the arrangement.) Twice in each revolution the insulated quadrants are completely exposed to the field, and then (1/4 of a period later) are completely shielded from it. Let T be the period of revolution, and let the inner and outer radii of the insulated quadrants be r_1 and r_2 as shown.



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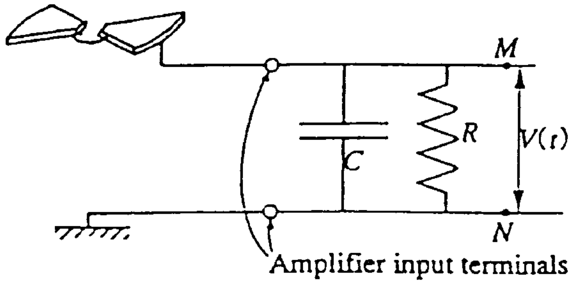
Take $t = 0$ to be an instant when the insulated quadrants are completely shielded.

Obtain expressions that give the total charge $q(t)$ induced on the upper surface of the insulated quadrants as a function of time between $t = 0$ and $t = T/2$, and sketch a graph of this variation.

[The effects of the atmospheric ion current can be ignored in this situation.]

(5) The system described in (4) is connected to an amplifier whose input circuit is equivalent to a capacitor C and a resistor R in parallel. (You can assume that the capacitance of the quadrant system is negligible compared to C.) Sketch graphs of the form of the voltage difference V between the points M and N as a function of t during one revolution of the disk, just after it has been set into rotation with period of revolution T, if:



a) $T = T_a \ll CR$;
b) $T = T_b \gg CR$.

[Assume that C and R have fixed values; only T changes between situations (a) and (b).] Obtain an expression for the approximate ratio, V_a/V_b, of the largest values of $V(t)$ in cases (a) and (b).

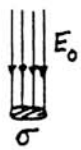
6) Assume that $E_0 = 150$ V/m, $r_1 = 1$ cm, $r_2 = 7$ cm, $C = 0.01$ μF, $R = 20$ MΩ, and suppose that the disk is set into rotation at 50 revolutions per second.

Approximately, what is the largest value of V during one revolution in this case?

Theoretical Problem 1 -- Solution

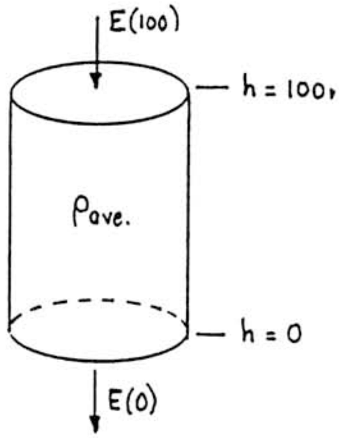
1) By Gauss' law, $\sigma = \varepsilon_0 E_0$.

$$\therefore \ \sigma = -8.85\ 10^{-12} \times 150$$
$$\approx -1.3 \times 10^{-9}\ \mathrm{C/m^2}.$$



$$Q = 4\pi R^2 \sigma = 4\pi \times (6.4 \cdot 10^6)^2 \times 1.3 \cdot 10^{-9} = -6.7 \cdot 10^5\ \mathrm{C}.$$

2) Consider a cylinder of cross-section A with faces at heights of 0 and 100 m.



By Gauss' law, $E(0)\,A - E(100)\,A = q_{enclosed}/\varepsilon_0$

$$= \rho_{ave.} \times (100A)/\varepsilon_0.$$

$$\therefore \ \rho_{ave.} = \frac{\varepsilon_0[E(0) - E(100)]}{100}$$

$$= \frac{8.85\ 10^{-12} \times 50}{100} \approx 4.4 \times 10^{-12}\ \mathrm{C/m^3}.$$

3) If a conductor contains n charges per unit volume, each with charge q and travelling with speed v, the current per unit area (j) is given by:

$$j = nqv.$$

Here, we have both positive and negative charges ($\pm e$). Clearly, with a downward electric field, the positive charges move downward and the negative charges move upward.
In the situation as described, only the positive charges can contribute to neutralization of the Earth's surface charge. Hence we have (taking downward as the positive direction for this purpose):

$$j = n_+ e\, v$$
$$\approx (6 \cdot 10^8) \times (1.6 \cdot 10^{-19}) \times (1.5 \cdot 10^{-4}\, E)$$
$$= 1.44 \times 10^{-14}\, E.$$

Now j is the rate of change ($d\sigma/dt$) of the surface charge density σ, and E (if defined as positive downward) is equal to $-\sigma/\varepsilon_0$. Thus the above equation can be written:

$$\frac{d\sigma}{dt} = -1.44 \cdot 10^{-14} \frac{\sigma}{\varepsilon_0} = -\frac{1.44 \cdot 10^{-14}}{8.85 \cdot 10^{-12}}\sigma = -1.63 \cdot 10^{-3}\sigma \approx -\frac{1}{600}\sigma.$$

This is just like the equation of radioactive decay. Its solution is an exponential decrease of σ with time:

$$\sigma(t) = \sigma_0\, e^{-t/\tau}, \quad \text{with } \tau \approx 600 \text{ sec.}$$

Putting $\sigma(t) = \sigma_0/2$ then gives $t = \tau \ln 2 = 0.693 \times 600 \approx 415$ sec ≈ 7 min.

[A simpler approximate solution is to assume that j remains constant at its initial value j_0:

$$j_0 = 1.44 \cdot 10^{-14} E_0 = 1.44 \cdot 10^{-14} \times 150 \approx 2.15 \times 10^{-12}\ \text{A/m}^2.$$

With $|\sigma_0| = 1.3\ 10^{-9}$ C/m^2 from part 1, we would then put:

$$|\sigma_0/2| = j_0 \times t\text{, giving } t = (0.65 \cdot 10^{-9})/(2.15 \cdot 10^{-12}) \approx 300 \text{ s} = 5 \text{ min.}]$$

4) If $t = 0$ is an instant at which the insulated quadrants are completely shielded, we have the following relations:

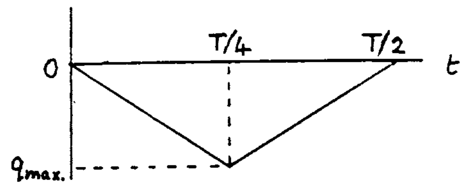
$$\text{For } 0 \le t \le \frac{T}{4},\ q(t) = -2\pi(r_2^2 - r_1^2)\varepsilon_0 E_0 \frac{t}{T}.$$

$$\text{For } \frac{T}{4} < t \le \frac{T}{2},\ q(t) = -\pi(r_2^2 - r_1^2)\varepsilon_0 E_0\left(1 - \frac{2t}{T}\right).$$

Corresponding variations occur during all the succeeding pairs of quarter-cycles.

The maximum (negative) induced charge is given by:

$$q_{max.} = -\frac{\pi}{2}(r_2^2 - r_1^2)\varepsilon_0 E_0.$$

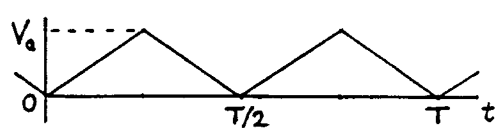


5) This question can be discussed without making a full circuit analysis. One only needs to realize that the rate of flow of charge into the amplifier is divided into a rate of charging of the capacitor, $C\ dV/dt$, and a conduction current, V/R, through the resistor. There are then two extreme situations, depending on whether the amount of charge lost by leakage during one quarter-period is small or large compared to CV.

(a) If $CV \gg (V/R) \times (T/4)$ -- i.e., $T = T_a \ll CR$ -- very little of the charge is carried away through R during the time $T/4$. Thus, when the insulated quadrants are charged negatively through induction, an almost equal *positive* charge is given to C. Thus $V(t)$ rises almost linearly with t between $t = 0$ and $t = T/4$, and then decreases almost linearly by an equal amount between $t = T/4$ and $t = T/2$. In this case,

$$V_{max.} = V_a \approx \frac{|q_{max.}|}{C},$$

where $q_{max.}$ has the value obtained in part 4.*



(b) If, however, $T = T_b \gg CR$ -- i.e., $CR \ll T_b$ -- most of the charge is quickly carried away through R. A constant positive current flows through R when the magnitude of q is increasing, and an equal negative current when the magnitude of q is decreasing. The size

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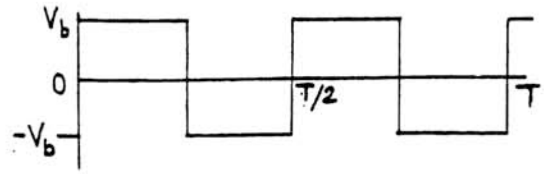
*Note: Ultimately (unless CR is infinite) the form of V_a will become a sawtooth varying symmetrically between $\pm\, q_{max.}/2C$. The statement of the problem avoids this complication by specifying that V is measured *just after* the rotation has begun.

of this current is approximately equal to $|q_{max.}|/(T_b/4)$. The resulting voltage across R is approximately constant during each quarter-period, and is alternately positive and negative. In this case,

$$V_{max.} = V_b \approx \frac{4\, q_{max.} R}{T_b}.$$



Putting these results together, we see that:

$$\frac{V_a}{V_b} \approx \frac{T_b}{4CR}.$$

6) We have $CR = 10^{-8} \times 2\ 10^{7} = 0.2$ s, and $T = 1/50 = 0.02$ s.
Thus $CR = 10 \times T$, which satisfies the criterion $CR \gg T$.
Therefore we can use the solution 5(a) above.
We have $A_{max.} = \frac{\pi}{2}(7^2 - 1^2) = 75\ \text{cm}^2 = 7.5 \times 10^{-3}\ \text{m}^2$.
$E_o = 150$ V/m $\rightarrow \sigma = \varepsilon_o E_o \approx 1.33 \times 10^{-9}$ C/m^2 (as in part 1).
$\therefore\ q_{max.} = 1.33 \cdot 10^{-9} \times 7.5\ 10^{-3} \approx 1.0 \times 10^{-11}$ C,

and so $V_{max.} = \dfrac{q_{max.}}{C} = \dfrac{1.0 \times 10^{-11}}{1.0 \times 10^{-8}} = 10^{-3}\ \text{V} = 1\ \text{mV}.$

Theoretical Problem 1: Grading Scheme

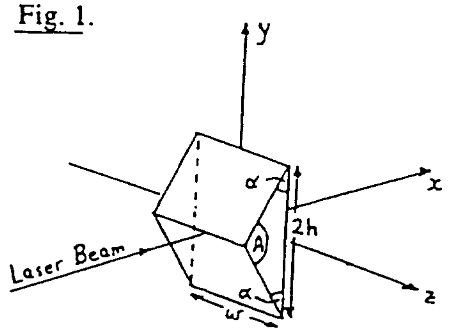
Part 1.	1 point	(1/2 point for σ_0, 1/2 point for Q)
Part 2.	1 point	
Part 3.	2 points	(1/2 point for recognizing $j = nev$; 1/2 point for recognizing $j = d\sigma/dt$; 1/2 point for getting $\sigma(t) = \sigma_0\, e^{-t/\tau}$; 1/2 point for final numerical answer.) [1 point maximum for using $t = \sigma_0/2j_0$.]
Part 4.	1-1/2 points	(1/2 point for each equation; 1/2 point for graph.)
Part 5.	3-1/2 points	(1 point for correct graphical form of (a); 1 point for correct graphical form of (b); 1-1/2 points for correct evaluation of V_d/V_b.)
Part 6.	1 point	(1/2 point for recognizing that $T \ll CR$; 1/2 point for final answer)

LASER FORCES ON A TRANSPARENT PRISM

By means of refraction a strong laser beam can exert appreciable forces on small transparent objects. To see that this is so, consider a small glass triangular prism with an apex angle $A = \pi - 2\alpha$, a base of length $2h$ and a width w. The prism has an index of refraction n and a mass density ρ.

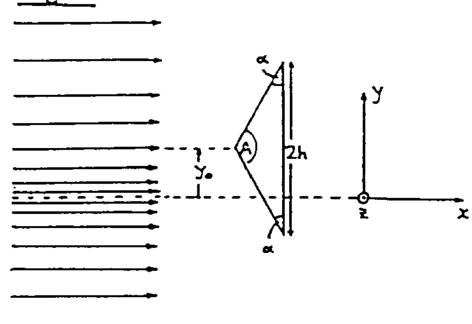
Suppose that this prism is placed in a laser beam travelling horizontally in the x direction. (Throughout this problem assume that the prism does not rotate, i.e., its apex always points opposite to the direction of the laser beam, its triangular faces are parallel to the xy plane, and its base is parallel to the yz plane, as shown in Fig. 1.) Take the index of refraction of the surrounding air to be $n_{air} = 1$. Assume that the faces of the prism are coated with an anti-reflection coating so that no reflection occurs.

Fig. 1.



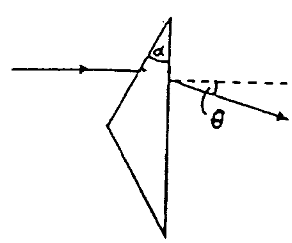
The laser beam has an intensity that is uniform across its width in the z direction but falls off linearly with distance y from the x axis such that it has a maximum value of I_0 at $y = 0$ and falls to zero at $y = \pm 4h$ (Fig. 2). [Intensity is power per unit area, e.g. expressed in W m^{-2}.]

Fig. 2.



1) Write equations from which the angle θ (see Fig. 3) may be determined (in terms of α and n) in the case when laser light strikes the upper face of the prism.

Fig. 3.



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2) Express, in terms of I_0, θ, h, w and y_0, the x and y components of the net force exerted on the prism by the laser light when the apex of the prism is displaced a distance y_0 from the x axis where $|y_0| \leq 3h$.

Plot graphs of the values of the horizontal and vertical components of force as functions of vertical displacement y_0.

3) Suppose that the laser beam is 1 mm wide in the z direction and 80 μm thick (in the y direction). The prism has $\alpha = 30°$, $h = 10$ μm, $n = 1.5$, $w = 1$ mm and $\rho = 2.5$ g cm^{-3}. How many watts of laser power would be required to balance this prism against the pull of gravity (in the $-y$ direction) when the apex of the prism is at a distance $y_0 = -h/2$ (= -5 μm) below the axis of the laser beam?

4) Suppose that this experiment is done in the absence of gravity with the same prism and a laser beam with the same dimensions as in (3), but with $I_0 = 10^8$ W m^{-2}.

What would be the period of oscillations that occur when the prism is displaced and released a distance $y = h/20$ from the center line of the laser beam?

Theoretical Problem 2 -- Solution

1. This is a simple problem in geometry and Snell's Law

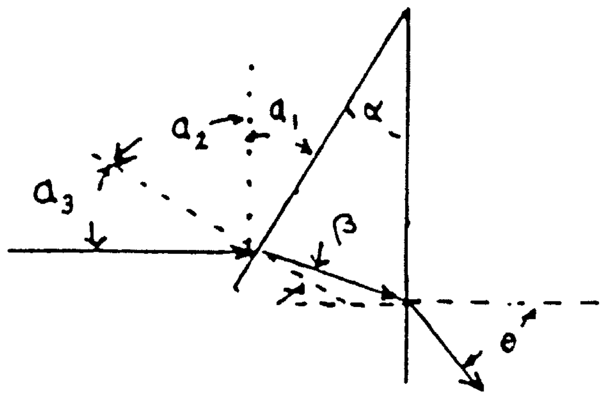


Figure 1: Refraction through a wedge.

The angle of incidence $a_3 = \alpha$ because $a_1 = \alpha$ and $a_1 + a_2 = a_2 + a_3 = 90°$. The angle β is found from Snell's law $\sin\alpha = n\sin\beta$. The angle of incidence on the base is

$$\frac{\pi}{2} - (\pi - \alpha - (\frac{\pi}{2} - \beta)) = \alpha - \beta$$

from which it follows that

$$\sin\theta = n\sin(\alpha - \beta)$$

implying that

$$\theta = \sin^{-1}\left[n \sin\left(\alpha - \sin^{-1}\left(\frac{\sin\alpha}{n}\right)\right)\right]$$

2. The force on the prism is equal and opposite to the rate of change of momentum of the laser light passing through it. To analyze this, consider the momentum changes of the laser light incident on the upper half of the prism.

   Think of the laser beam as delivering to the upper half of the prism r_u photons per second parallel to the x axis. If the energy of a photon is E, then its momentum is $\vec{p}_i = \frac{E}{c}\,\hat{\imath}$, and a photon leaving the prism at an angle θ to the x axis will differ in momentum from the incident photon by

$$\delta\vec{p} = \frac{E}{c}(\cos\theta - 1)\,\hat{\imath} - \frac{E}{c}\sin\theta\,\hat{\jmath}.$$

   The rate of charge of momentum of these photons will then be

$$\vec{F}_{\text{up}} = r_u\delta\vec{p} = \frac{r_u E}{c}\left[(\cos\theta - 1)\,\hat{\imath} - \sin\theta\ \hat{\jmath}.\right]$$

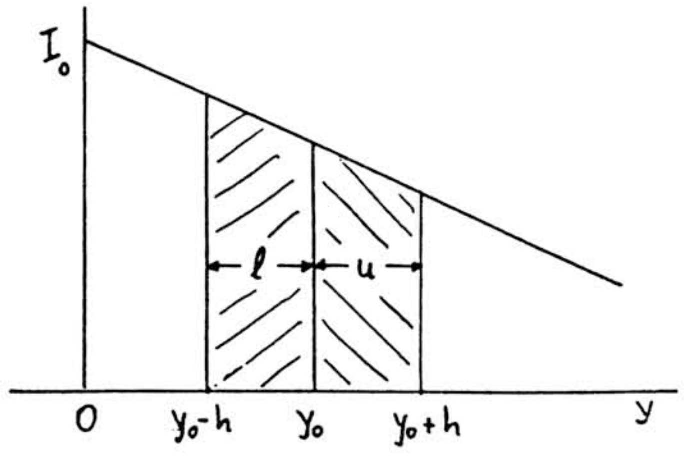


Figure 2: $\bar{I}_u$ and $\bar{I}_\ell$ when $y_0 \geq h$

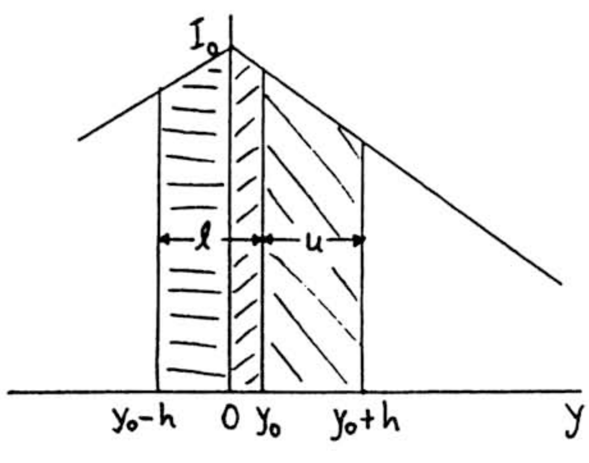


Figure 3: $\bar{I}_u$ and $\bar{I}_\ell$ when $0 < y_0 < h$

The quantity $r_u E$ is the power P_u delivered to the upper face, and the recoil force $\vec{F}_u$ produced by light refracting through the upper half of the prism will be

$$\vec{F}_u = \frac{P_u}{c}\left[(1-\cos\theta)\,\hat{\imath} + \sin\theta\,\hat{\jmath}\right].$$

A similar argument gives the force on the lower half as

$$\vec{F}_\ell = \frac{P_\ell}{c}\left[(1-\cos\theta)\,\hat{\imath} - \sin\theta\,\hat{\jmath}\right].$$

From these two results we see that the net force on the prism will be

$$\vec{F} = \frac{1}{c}\left[(P_u + P_\ell)(1-\cos\theta)\right]\,\hat{\imath} + \frac{1}{c}\left[(P_u - P_\ell)\,\sin\theta\right]\,\hat{\jmath}.$$

The angle θ can be expressed in terms of α (see answer to part 1).

To find the values of P_u and P_ℓ calculate the average intensities, $\bar{I}_u$ and $\bar{I}_\ell$, incident on each half of the prism and multiply by hw, the area of each half of the prism projected perpendicular to the laser beam. Because the intensity distribution $I(y)$ is a linear function of y, the average intensities are easily determined.

The problem states that

$$\begin{aligned} I(y) &= I_0\left(1-\frac{y}{4h}\right) && \text{for } 0 < y < +4h \\ &= I_0\left(1+\frac{y}{4h}\right) && \text{for } -4h < y < 0. \end{aligned}$$

Now suppose that the prism is lifted a distance y_0 from the x axis ($y_0 > 0$). There are two distinct cases:

(a) When $h \le y_0 \le 3h$, the whole prism is entirely in the upper half of the beam. As Fig. 2 shows, for this case the average is equal to the intensity at the center of each face which is at $y_0 + h/2$ for the upper face and at $y_0 - h/2$ for the lower one. This gives

$$\bar{I}_u = I_0\left(1-\frac{y_0+h/2}{4h}\right) = I_0\left(\frac{7}{8}-\frac{y_0}{4h}\right)$$
$$\bar{I}_\ell = I_0\left(1-\frac{y_0-h/2}{4h}\right) = I_0\left(\frac{9}{8}-\frac{y_0}{4h}\right)$$

From these it follows that

$$\begin{aligned} F_x &= \frac{2hwI_0}{c}\left(1-\frac{y_0}{4h}\right)(1-\cos\theta) \\ F_y &= -\frac{hwI_0}{4c}\sin\theta. \end{aligned}$$

(b) When $0 < y_0 < h$, the lower half of the prism is partly in the lower half of the laser beam as shown in Fig. 3. Then the part of the lower half of the prism between 0 and y_0 has a fraction y_0/h of the area of the lower half of the prism and sees an average intensity

$$\bar{I}_{\ell_1} = I(y_0/2) = I_0\left(1 - \frac{y_0}{8h}\right).$$

The part between 0 and $y_0 - h$ has a fraction $1 - y_0/h$ of the area and sees an average intensity of

$$\bar{I}_{\ell_2} = I(\frac{h - y_0}{2}) = I_0\left(\frac{7}{8} + \frac{y_0}{8h}\right).$$

Putting these together we get

$$\begin{aligned} P_\ell &= hw\,\frac{y_0}{h}\bar{I}_{\ell_1} + hw\,(1 - \frac{y_0}{h})\bar{I}_{\ell_2} \\ &= hwI_0\left(\frac{7}{8} + \frac{y_0}{4h} - \frac{y_0^2}{4h^2}\right). \end{aligned}$$

The average intensity on the upper face has the same functional dependence on y_0 as in the first case. Therefore, $P_u = hwI_0\left(\frac{7}{8} - \frac{y_0}{4h}\right)$ as before.

Putting these together gives

$$\begin{aligned} P_u + P_\ell &= hwI_0\left(\frac{7}{4} - \frac{y_0^2}{4h^2}\right) \\ P_u - P_\ell &= -hwI_0\frac{y_0}{2h}\left(1 - \frac{y_0}{2h}\right) \end{aligned}$$

from which it follows that

$$\begin{aligned} F_x &= \frac{hwI_0}{c}\left(\frac{7}{4} - \frac{y_0^2}{4h^2}\right)(1 - \cos\theta) \\ F_y &= -\frac{hwI_0}{c}\frac{y_0}{2h}\left(1 - \frac{y_0}{2h}\right)\sin\theta. \end{aligned}$$

Because the intensity distribution is symmetric about the axis of the laser beam, the solutions for $y_0 < 0$ will mirror the solutions for $y_0 > 0$. Graphs of the F_x and F_y as functions of y_0 are shown in Fig. 4.

3. Both the equation and the graph of F_y show that to have $F_y > 0$ and opposite the force of gravity, y_0 must be < 0. Then to find the force necessary to support the prism against gravity, find the prism's mass, and equate the expression for the vertical component of force from the laser beam to the weight of the prism, and find I_0 for the parameters given. Use that result to find the total power in the laser beam. This

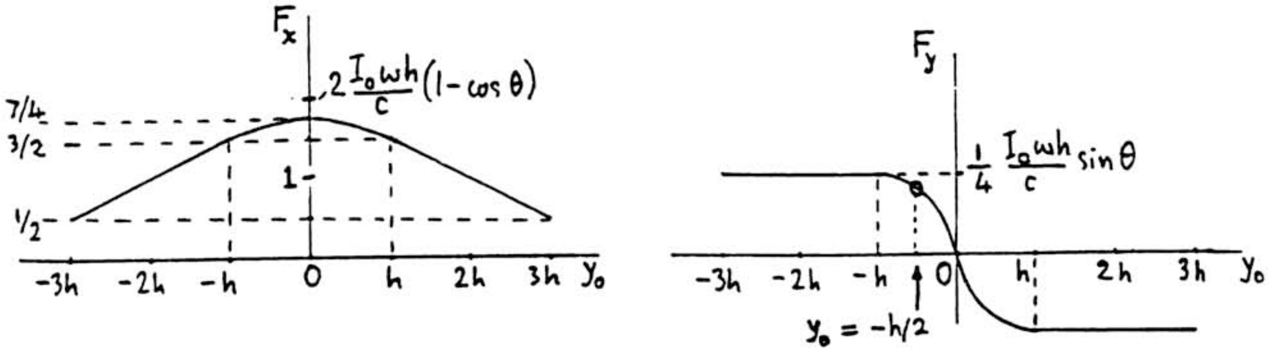


Figure 4: (a) F_x vs y_0; (b) F_y vs y_0

can be done by finding the average value $\bar{I}$ over the specified cross sectional area of the laser beam.

To find the mass of the prism first find its volume $= \tan\alpha\, h^2 w$ then

$$\begin{aligned} m &= \frac{1}{\sqrt{3}} \times (10^{-3})^2 \times .1 \times 2.5 \\ &= 1.44 \times 10^{-7}\ \text{g} \\ &= 1.44 \times 10^{-10}\ \text{kg}; \\ mg &= 1.42 \times 10^{-9}\ \text{N} \end{aligned}$$

The solution to (2) assumed a displacement in the $y > 0$ direction, but the problem is symmetric so we can use that solution. We want the value of I_0 that satisfies

$$\frac{I_0 hw}{c}\frac{y_0}{2h}\left(1 - \frac{y_0}{2h}\right)\sin\theta = mg = 1.42 \times 10^{-9}$$

when

$$\begin{aligned} \theta &= 15.9^\circ \\ y_0 &= \frac{h}{2} \\ h &= 10 \times 10^{-6}\ \text{m} \\ w &= 10^{-3}\ \text{m} \end{aligned}$$

$$I_0 = \frac{3 \times 10^8 \times 1.42 \times 10^{-9}}{10^{-5} \times 10^{-3} \times .274 \times \frac{3}{16}} = 8.30 \times 10^8\ \text{W/m}^2$$

since the power P is given by $P = \bar{I} \times$ area of laser beam where $\bar{I} = \frac{I_0}{2}$. This yields

$$P = \frac{1}{2} \times 8.30 \times 10^8 \times 10^{-3} \times 80 \times 10^{-6} = 33.2\ \text{W}.$$

4. A displacement of $h/20$ corresponds to $y_0/h = .05 << 1$ so that the vertical force component is well approximated by

$$F_y = -\frac{I_0 w \sin\theta}{2c} y.$$

This is the equation of a harmonic oscillator with angular frequency

$$\omega = \sqrt{\frac{I_0 w \sin\theta}{2mc}} = \sqrt{\frac{I_0 \sin\theta}{2c\rho h^2 \tan\alpha}}.$$

Putting numbers into this gives

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{2 \times 3 \times 10^8 \times 2.5 \times 10^3 \times 10^{-10} \times 1/\sqrt{3}}{10^8 \times .274}} = 11.2\times10^{-3} \text{ s}.$$

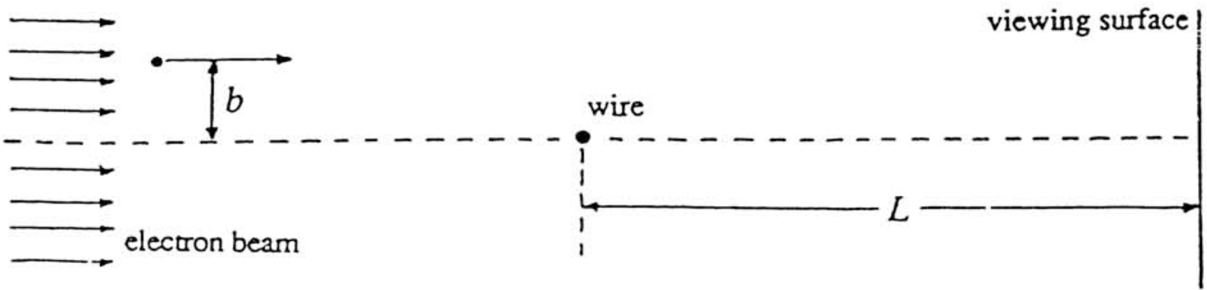
Theoretical Problem 2: Grading Scheme

Part 1.	1.5 points	
Part 2.	5 points	(2 points for obtaining expression for net force in terms of θ and powers P_u, P_l incident on upper and lower prism faces ; 1 point for finding F_x and F_y explicitly in terms of I_0, y_0 and θ for $h \le y_0 \le 3h$; 1 point for finding F_x and F_y explicitly in terms of I_0, y_0 and θ for $0 \le y_0 \le h$; 1 point for drawing appropriate graphs)
Part 3.	1.5 points	
Part 4.	2 points	

Theoretical Problem 3

ELECTRON BEAM

An accelerating voltage V_0 produces a uniform, parallel beam of energetic electrons. The electrons pass a thin, long, positively charged copper wire stretched at right angles to the original direction of the beam, as shown in the figure. The symbol b denotes the distance at which an electron would pass the wire if the wire were uncharged. The electrons then proceed to a screen (viewing surface) a distance L ($\gg b$) beyond the wire, as shown. The beam initially extends to distances $\pm b_{max}$ with respect to the axis of the wire. Both the width of the beam and the length of the wire may be considered infinite in the direction perpendicular to the paper.



The charged wire extends perpendicularly to the plane of the paper. The sketch is not to scale.

Some numerical data are provided here; you will find other numerical data in the table at the front of the examination:

radius of wire = r_o = 10^{-6} m

maximum value of b = b_{max} = 10^{-4} m

electric charge per unit length of wire = q_{linear} = 4.4×10^{-11} C m^{-1}

accelerating voltage = V_o = 2×10^4 V

length from wire to observing screen = L = 0.3 m.

Note: For parts 2 - 4, make reasonable approximations that lead to analytical and numerical solutions.

1) Calculate the electric field **E** produced by the wire. Sketch the magnitude of **E** as a function of distance from the axis of the wire.

(Continued on next page)

2) Use classical physics to calculate the angular deflection of an electron. Do this for values of the parameter b such that the electron does not strike the wire. Let θ_{final} denote the (small) angle between the initial velocity of the electron and the velocity when the electron reaches the viewing surface. Hence, calculate θ_{final}.

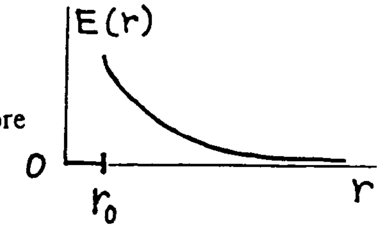
3) Calculate and sketch the pattern of impacts (i.e., the intensity distribution) on the viewing screen that classical physics predicts.

4) Quantum physics predicts a major difference in the intensity distribution (relative to what classical physics predicts). Sketch the pattern for the quantum prediction and provide quantitative detail.

Theoretical Problem 3 -- Solution

1. By symmetry, the electric field will point radially away from the wire, and its magnitude will depend only on the radius r (in cylindrical coordinates). Place an imaginary cylinder around the wire and use Gauss's law:

$$2\pi r E(r) = \frac{q_{\text{linear}}}{\varepsilon_0}$$



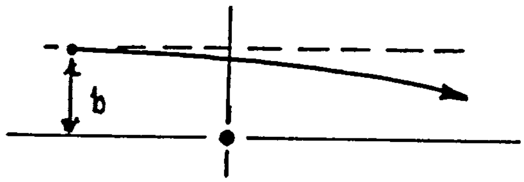
for a cylinder of radius r and unit length, provided $r \geq r_0$. Therefore

$$E(r) = \frac{q_{\text{linear}}}{2\pi r \varepsilon_0} = \frac{0.791}{r} N/C \quad \text{provided} \quad r \geq r_0.$$

When $r < r_0$, the electric field is zero (because copper is a good conductor), that is, the electric field is zero inside the wire.

2. The problem stated that the angular deflection is small. Estimate the deflection angle θ_{final} by forming a quotient: the momentum acquired transverse to the initial velocity divided by the initial momentum:

$$\theta_{\text{final}} \cong \frac{|\Delta \mathbf{p}_{\perp}|}{m v_0}.$$



A first estimate of the transverse momentum can be made as follows:

The transverse force (where it is significant) is of order $\frac{e q_{\text{linear}}}{2\pi\varepsilon_0 b}$.

The (significant) transverse force operates for a time such that the electron goes a distance of order $2b$, and hence that transverse force operates for a time of order $2b/v_0$.

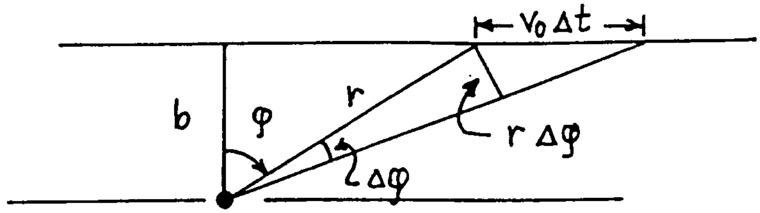
The product of force and operating time gives an estimate the transverse momentum:

$$|\Delta \mathbf{p}_{\perp}| \cong \frac{e q_{\text{linear}} 2b}{2\pi\varepsilon_0 b v_0} = \frac{e q_{\text{linear}}}{\pi\varepsilon_0 v_0},$$

and so $\theta_{\text{final}} \cong \frac{e q_{\text{linear}}}{\pi\varepsilon_0 m v_0^2} = \frac{q_{\text{linear}}}{\pi\varepsilon_0 2V_0} = 3.96 \times 10^{-5}$ radians

after one uses energy conservation to say $\frac{1}{2} m v_0^2 = eV_0$. Note that the deflection is extremely small and that the deflection is independent of the impact parameter b. Because the force between the positively charged wire and the electron is attractive, the deflection will bend the trajectory toward the wire--though only ever so slightly.

A more accurate estimate can be made by setting up an elementary integration for $|\Delta \mathbf{p}_\perp|$, as follows. For the sake of the integration, approximate the actual trajectory by a straight line that passes the wire at distance b, as shown in the sketch.



$$|\mathbf{F}_\perp| = \frac{eq_{\text{linear}}}{2\pi\varepsilon_0 r}\cos\varphi \qquad v_0\,\Delta t\cos\varphi = r\Delta\varphi \quad \text{and so} \quad \Delta t = \frac{r\Delta\varphi}{v_0\cos\varphi}$$

$$|\mathbf{F}_\perp|\,\Delta t = \frac{eq_{\text{linear}}}{2\pi\varepsilon_0 r}\cos\varphi\frac{r\Delta\varphi}{v_0\cos\varphi} = \frac{eq_{\text{linear}}}{2\pi\varepsilon_0 v_0}\Delta\varphi.$$

Adding up the increments in $\Delta\varphi$ over the range $-\pi/2$ to $\pi/2$ yields $|\Delta \mathbf{p}_\perp| = \dfrac{eq_{\text{linear}}}{2\varepsilon_0 v_0}$.

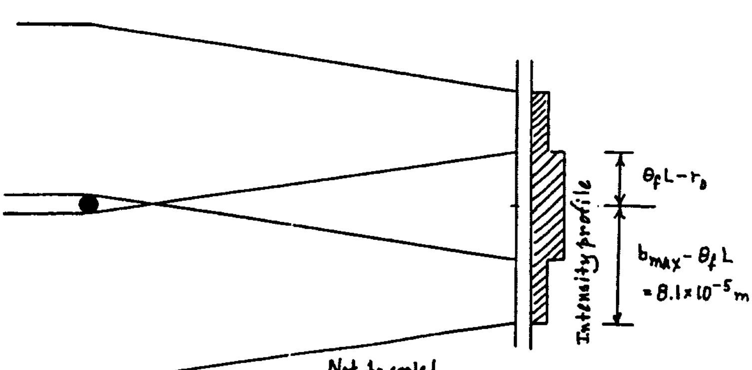
The better estimate differs from the first estimate by merely the factor $\pi/2$. The better estimate yields

$$\theta_{\text{final}} \cong \frac{eq_{\text{linear}}}{2\varepsilon_0 m v_0^2} = \frac{q_{\text{linear}}}{2\varepsilon_0 2V_0} = 6.21\times10^{-5}\ \text{radians}.$$

3. Most of the bending of the trajectory occurs within a distance from the wire of order b. On the scale of L, order b is very small indeed. Therefore we may approximate the trajectory by two straight lines with a kink near the wire. Thus, at the viewing surface, the transverse displacement of each trajectory is

$$\begin{pmatrix}\text{transverse}\\ \text{displacement}\end{pmatrix} = \theta_{\text{final}}L = 6.21\times10^{-5}\times0.3 = 1.86\times10^{-5}\ \text{meter} \cong 19\,r_0 >> r_0.$$

Thus the portions of the beam that pass on opposite sides of the wire have a region of overlap, as shown in the sketch.



The full width of the overlap region is

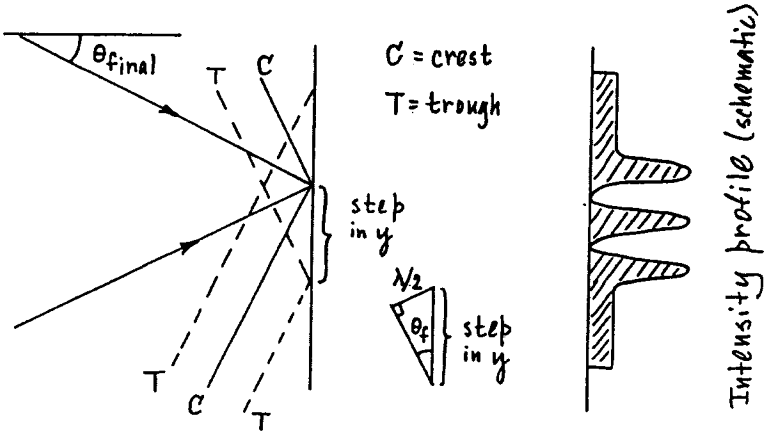
$$\begin{pmatrix}\text{full}\\ \text{width}\end{pmatrix} = 2\times(\theta_{\text{final}}L - r_0) \cong 36\, r_0 = 36\times10^{-6} \quad \text{meter}.$$

The density of impacts is constant within each region and doubled in the overlap region.

4. Associated with the electron beam is a quantum wave pattern whose de Broglie wavelength is

$$\lambda = \frac{h}{mv_0} = \frac{h}{\sqrt{2meV_0}} = 8.68\times10^{-12} \quad \text{meter}.$$

The de Broglie wavelength is so much smaller than the beam width $2b_{\text{max}}$ that one may ignore "single slit diffraction" effects. Rather, to the right of the wire, two plane waves that travel at a fixed angle relative to each other (an angle $2\theta_{\text{final}}$) overlap and interfere. In the region where, classically, the two halves of the original beam overlap, there will be interference maxima and minima.



Reference to the sketch indicates that

$$\begin{pmatrix}\text{Interval between}\\ \text{adjacent constructive}\\ \text{interference locations}\end{pmatrix} = \begin{pmatrix}\text{step}\\ \text{in } y\end{pmatrix} = \frac{\lambda/2}{\sin\theta_{\text{final}}} \cong \frac{\lambda/2}{\theta_{\text{final}}} \cong \frac{\frac{1}{2}\times8.68\times10^{-12}}{6.21\times10^{-5}} = 7.00\times10^{-8} \quad \text{meter}.$$

Because the region of overlap has a full width of $\cong 36\times10^{-6}$ meter, there will be roughly 500 interference maxima. Note that the interval between adjacent maxima does *not* depend on either b or b_{max} (unlike the situation with ordinary "double slit interference").

Historical note. This problem is based on the now-classic experiment by G. Mollenstedt and H. Duker, "Observation and Measurement of Biprism Interference with Electron Waves," Zeitschrift fur Physik, 145, pp. 377-397 (1956).

Theoretical Problem 3: Grading Scheme

Part 1. 1 point.

E(r) correct outside of wire: 1 point.

E(r) inside wire: ignore in the grading. (Some students may ignore the interior because there is no field there.)

Part 2. 5 points, distributed as follows:

θ_{final} independent of b: 1 pt.

$\theta_{\text{final}} \propto \frac{e q_{\text{linear}}}{\varepsilon_0 m v_0^2}$ or $\frac{q_{\text{linear}}}{\varepsilon_0 V_0}$ or equivalent: + 1 pt.

Numerical coefficient correct to within a factor of 4: + 2 pts.

Numerical coefficient correct to within 20 %: + 1 pt.

Part 3: 1.5 points:

Overlap region exists: 0.5 pt.

Constant densities of impacts within each region: + 0.25 pt.

Correct ratio of intensities: + 0.25 pt.

Full width of pattern correct, given student's value for θ_{final}: + 0.25 pt.

Width of overlap region correct, given student's value for θ_{final}: + 0.25 pt.

Part 4: 2.5 points:

Recognizes that "two wave" interference occurs: 0.5 pt.

Correct de Broglie wavelength : 0.5 pt.

Correct separation of maxima: + 1.5 pts.

[If separation of maxima is wrong by merely a factor of 2, then partial credit: +1 pt.]

Maxima in intensity = 4 times single-wave intensity: ignore in grading.

COUNTRY : ____________________

XXIV INTERNATIONAL PHYSICS OLYMPIAD

WILLIAMSBURG, VIRGINIA, U.S.A.

PRACTICAL COMPETITION

Experiment No. 1

July 14, 1993

Time available: 2.5 hours

READ THIS FIRST!

INSTRUCTIONS:

1. Use only the pen provided.
2. Use only the marked side of the paper.
3. Write at the top of each and every page:

   - The number of the problem
   - The number of the page of your report
   - The total number of pages in your report.

   Example (for Problem 1): 1 1/4; 1 2/4; 1 3/4; 1 4/4.

THE HEAT OF VAPORIZATION OF NITROGEN

The object of this experiment is to measure the heat of vaporization per unit mass (L) of nitrogen using two different methods. In Method #1, you will add a piece of aluminum to the sample of liquid nitrogen and measure how much liquid nitrogen evaporates as the aluminum cools. In Method #2, you will add energy in the form of heat at a known rate to the sample of liquid nitrogen and measure the rate at which the liquid nitrogen vaporizes.

The liquid nitrogen is supplied to you in the "reservoir" container. Some of it can be poured into the "sample" container, which can be placed on the mass balance. The reading of the mass balance will decrease as liquid nitrogen vaporizes. This occurs (1) because the container is not a perfect insulator, (2) because energy is being added to the liquid nitrogen in the form of heat when the aluminum cools (in Method #1), and (3) because energy is being added to the liquid nitrogen in the form of heat when current passes through a resistor placed in the liquid nitrogen (in Method #2). A multimeter, which can be used to measure voltage (V), current (I), and resistance (R), as well as a stopwatch are supplied. Instructions for using the multimeter and stopwatch are attached.

Warnings

(1) Liquid nitrogen is very cold, so do not let it, or any object which has been cooled by it, touch you or your clothing in any way.

(2) Do not drop anything in the liquid nitrogen, and wear safety goggles at all times.

(3) Place the piece of aluminum in the liquid nitrogen slowly, as it will cause the liquid nitrogen to boil rapidly until equilibrium is reached. A piece of string is supplied for this purpose.

(4) The resistor can get very hot if it is not immersed in the liquid nitrogen. Pass current through the resistor only when it is in the container and completely immersed in liquid nitrogen.

Method #1

The specific heat of aluminum (c) varies significantly between room temperature and the temperature at which liquid nitrogen vaporizes under atmospheric pressure (77 K). A graph showing the variation of c with temperature (T) is attached. Conduct an experiment to measure how much liquid nitrogen vaporizes when the aluminum block is cooled. Use this determination and the specific heat graph to determine the heat of vaporization per unit mass of nitrogen. You may assume that room temperature is 21±2°C. Be sure to provide a quantitative estimate of the accuracy of your heat of vaporization value.

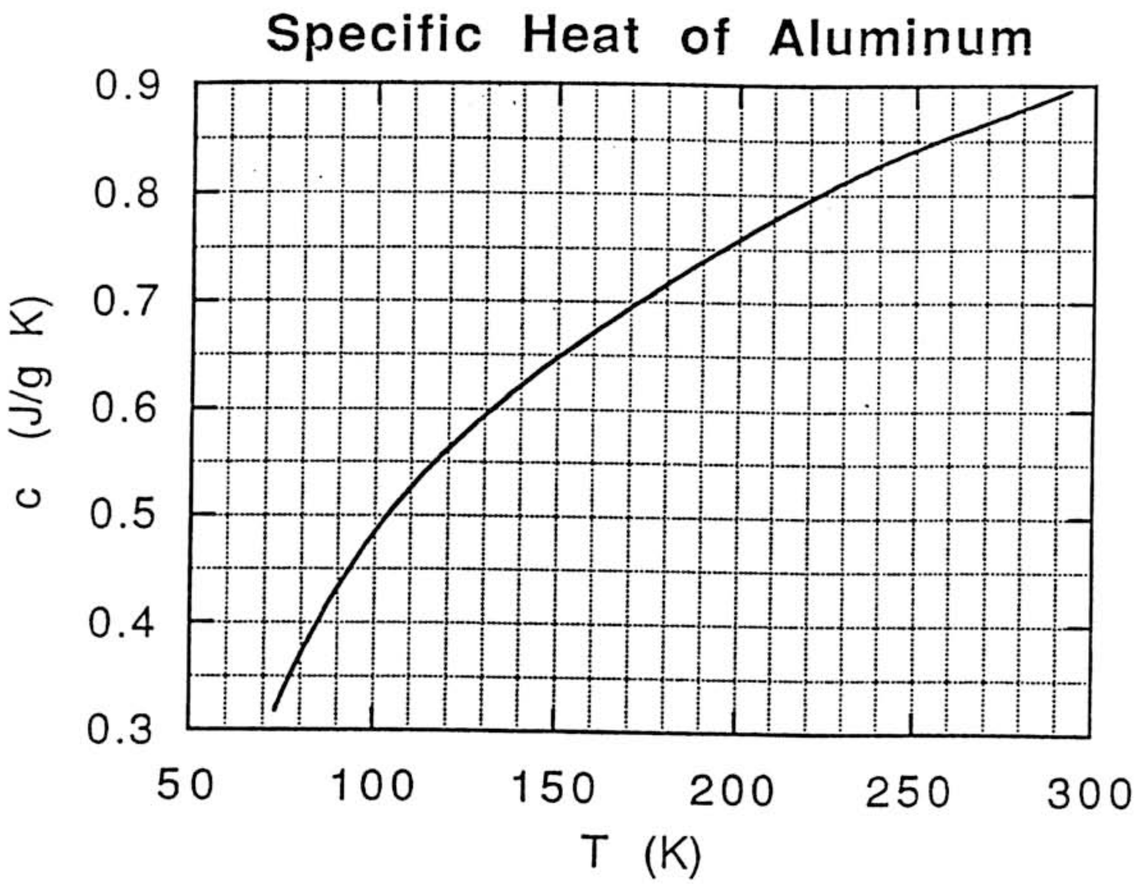
Method #2

Conduct an experiment to measure the rate at which liquid nitrogen vaporizes when current is passed through the resistor placed in the liquid nitrogen. A direct current power supply is provided; use it only with the dial in the "8" position and do not disconnect the capacitor installed across its terminals. Use this result to determine the heat of vaporization per unit mass of nitrogen. Be sure to provide a quantitative estimate of the accuracy of your result.

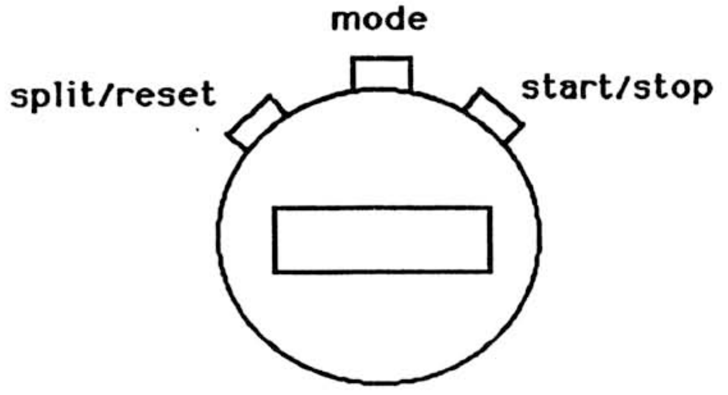
Notes:

(1) Please include sketches, schematic diagrams, properly labelled tables, numbers with the proper units, etc. so the graders can determine exactly what you did.

(2) Ask for assistance if any piece of equipment is not working properly.



Digital Stopwatch



To Perform Timing Operations

1. Press "Mode" until 0 00 00 appears
   (You may have to press "Mode" several times to get the 0 00 00 to appear)

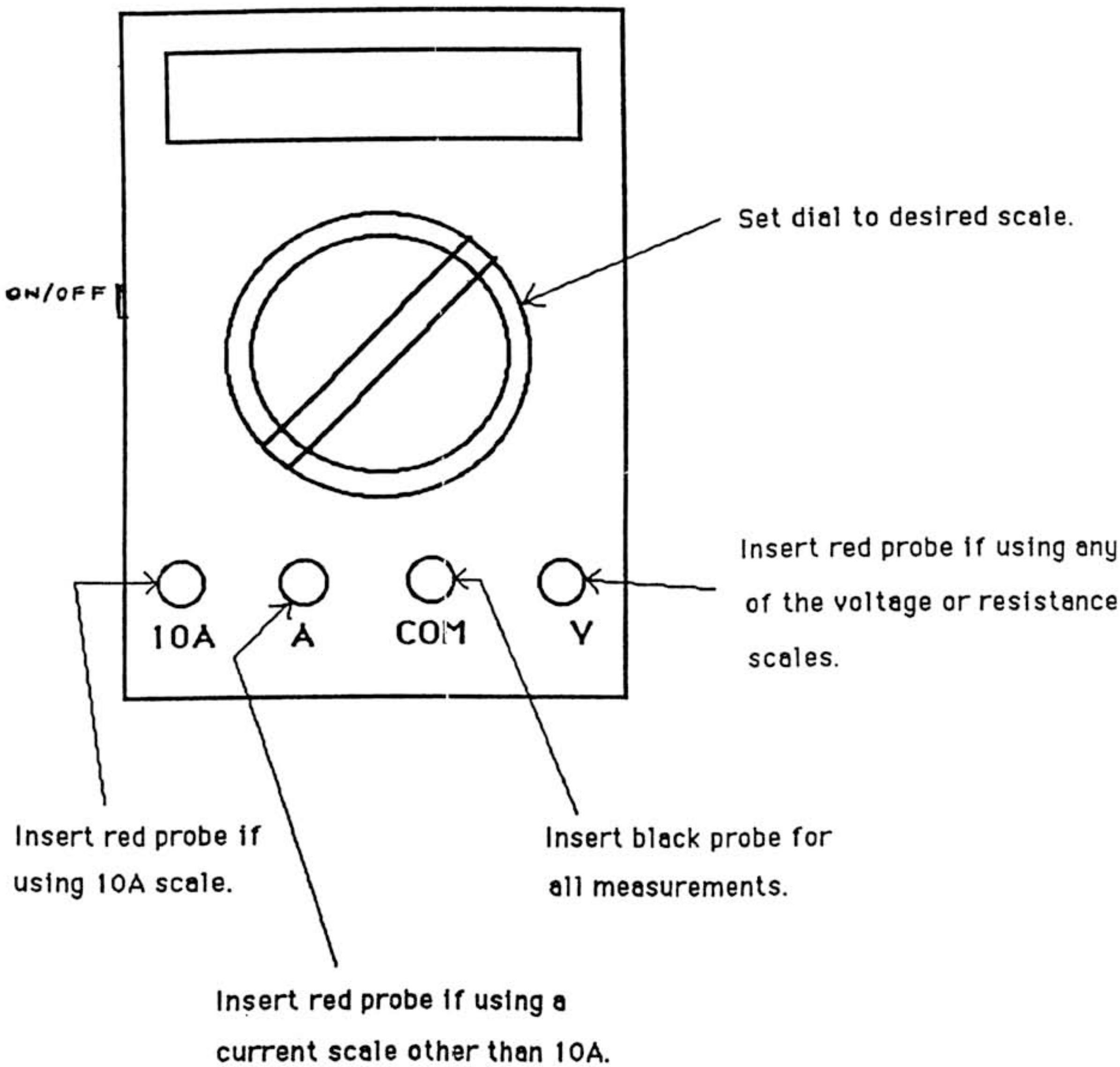
To Time a Single Interval

1. Press "Start/Stop" to start stopwatch.
2. Press "Start/Stop" to stop stopwatch.
3. Press "Split/Reset" to reset stopwatch to zero.

To Time Multiple Events Without Stopping the Stopwatch

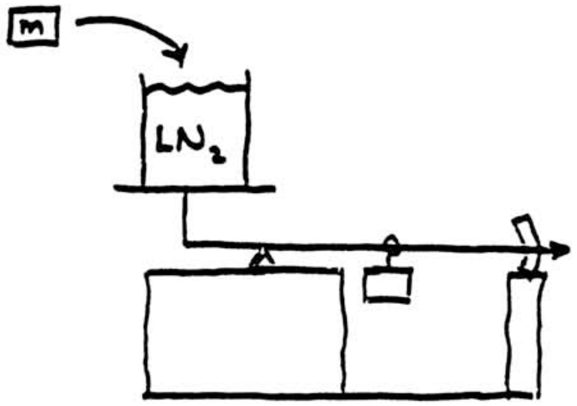
1. Press "Start/Stop" to start stopwatch.
2. Press "Split/Reset" to stop the display while stopwatch keeps running.
3. Press "Split/Reset" to reset display to actual time.
4. Press "Start/Stop" to stop stopwatch after last event.
5. Press "Split/Reset" to reset stopwatch to zero.

Multimeter


ON/OFF
10A
A
COM
V
Set dial to desired scale.
Insert red probe if using any of the voltage or resistance scales.
Insert red probe if using 10A scale.
Insert black probe for all measurements.
Insert red probe if using a current scale other than 10A.

Experimental Problem 1 -- Solutions

Method #1

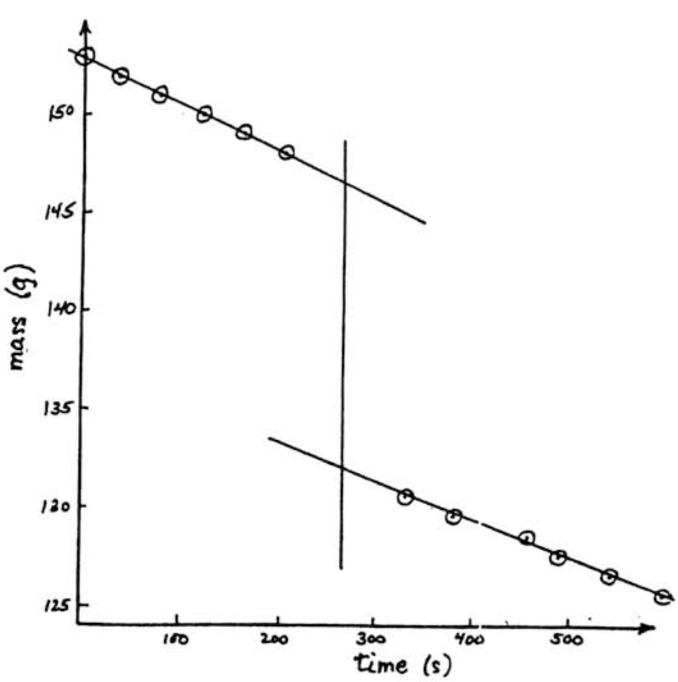


$Q = mc\Delta T = m \int c \, dT$

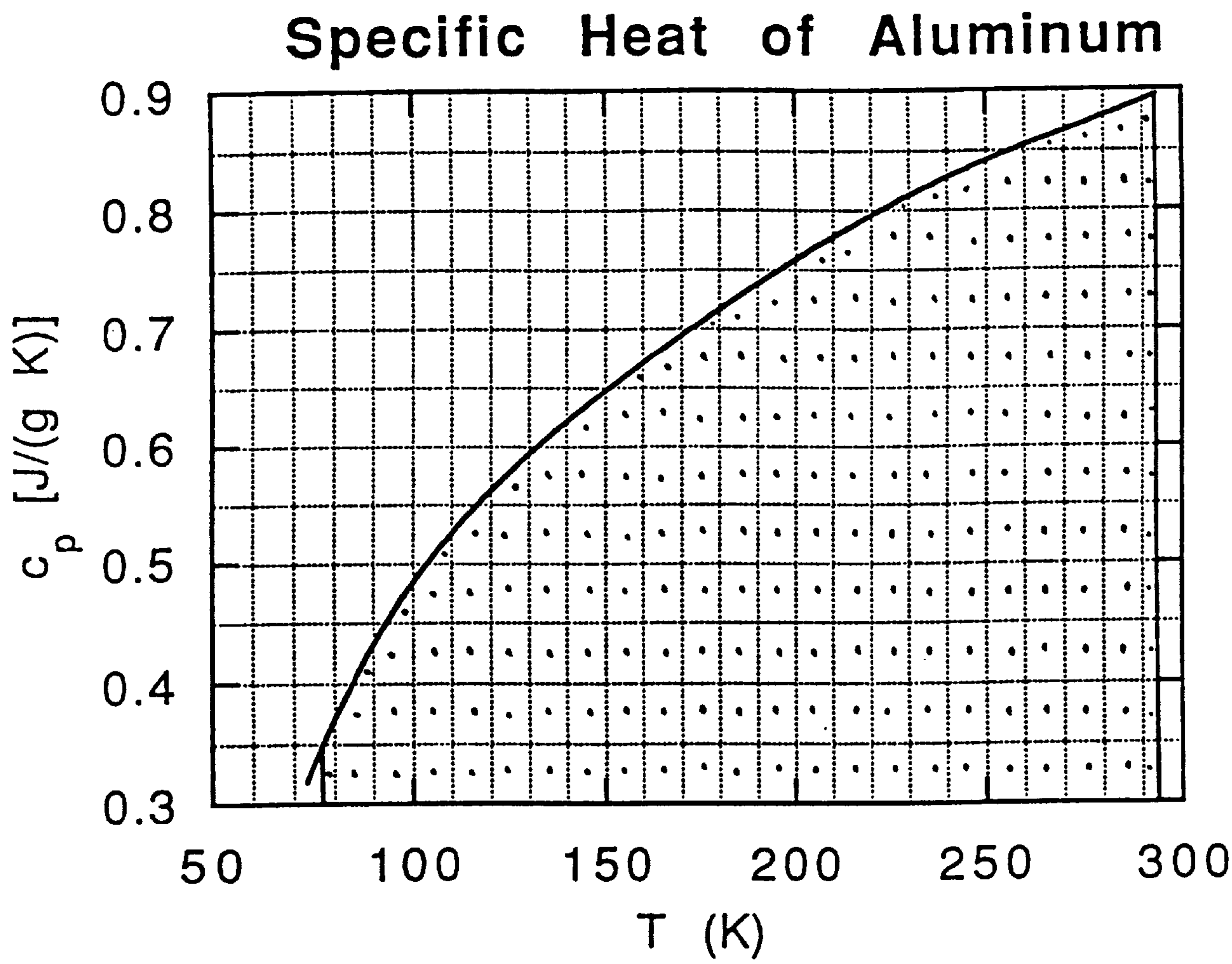
$Q = L \, \Delta M_{LN_2}$

$m = 19.4 \pm 0.1$ g

	total mass	clock time	time
	153 g	0:00.0	0
	152	0:36.8	36.8
	151	1:19.1	79.1
	150	2:00.7	120.7
	149	2:40.5	160.5
	148	3.23.1	203.1
Add Al mass			
	150 (130.6)	5:31.8	331.8
	149 (129.6)	6:21.6	381.6
	148 (128.6)	7:17.3	457.3
	147 (127.6)	8:08.6	488.6
	146 (126.6)	9:00.9	540.9
	145 (125.6)	9:54.6	594.6



$\Delta M_{LN_2} = 146.5 - 132.0$
$= 14.5 \pm 0.3$ g



$$\int_{77}^{293} c\, dT \approx (0.3)(293-77) + (173)(0.5)$$

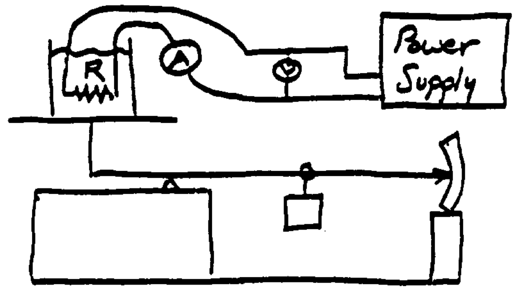
$$\approx 64.8 + 86.5 = 151 \pm 2 \text{ J/g}$$

$$Q = \int m\, c\, dT = (19.4 \pm 0.1 \text{ g})(151 \pm 2 \text{ J/g})$$

$$= 2930 \pm 42 \text{ J.}$$

$$L = \frac{Q}{\Delta M_{LN2}} = \frac{2930 \pm 42 \text{ J}}{14.5 \pm 0.3 \text{ g}} = 202 \pm 5 \text{ J/g}$$

Method #2



$P = IV = V^2/R = I^2R$

$P = \Delta Q/\Delta t$

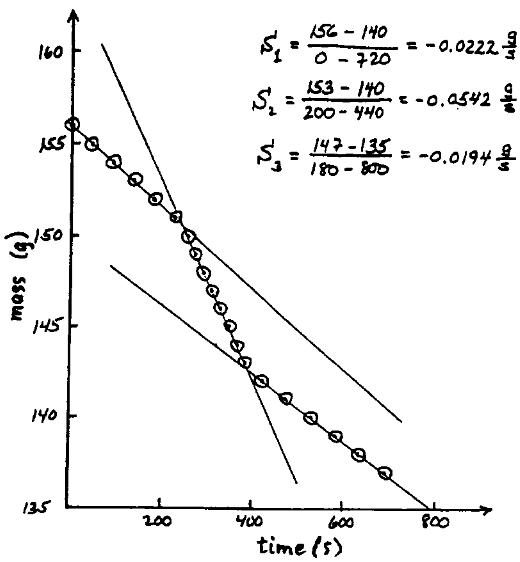
$Q = M_{LN_2}L$

$R = 23.0\ \Omega$ (in LN_2)

$V = 12.7$ V

$I = 0.56$ A

	total mass	clock time	time
	156 g	0:00.0	0 s
	155	0:45.2	45.2
	154	1:31.4	91.4
$P = 0$	153	2:16.2	136.2
	152	2:60.0	180.0
	151	3:47.2	227.2
	150	4:13.6	253.6
	149	4:32.1	272.1
	148	4:50.1	290.1
$P \neq 0$	147	5:08.9	308.9
	146	5:27.2	327.2
	145	5:45.7	345.7
	144	6:04.1	364.1
	143	6:21.9	381.9
	142	7:02.3	422.3
	141	7:58.4	478.4
$P = 0$	140	8:51.2	531.2
	139	9:43.7	583.7
	138	10:34.6	634.6
	137	11:30.7	690.7



$$S_{P\neq 0} = -\,0.054 \pm 0.001 \text{ g/s}$$

$$\langle S_{P=0}\rangle = -\,0.020 \pm 0.001 \text{ g/s}$$

$$\text{Power} = P = \left|\frac{Q}{\Delta t}\right| = L\left|\frac{\Delta M_{LN2}}{\Delta t}\right|$$

$$\left.\begin{aligned} P &= IV &= 7.11\text{ W}\\ P &= I^2R &= 7.21\text{ W}\\ P &= V^2/R &= 7.01\text{ W}\end{aligned}\right\} P = 7.1 \pm 0.1\text{ W}$$

$$|\Delta M_{LN_2}/\Delta t| = 0.054 - 0.020 = 0.034 \pm 0.0014 \text{ J/s}$$

$$L = \frac{P}{\Delta M_{LN2}/\Delta t} = \frac{7.1 \pm 0.1}{0.034 \pm 0.0014} = 209 \pm 9 \text{ J/g}$$

Experimental Problem 1: Grading Scheme

Method No. 1 (5 points maximum)

1) 0.5 Uses $Q = mc\Delta T$ or $Q = m \int c\, dT$

2) 0.5. Uses $Q = L\Delta M_{LN_2}$

3) 0.5 Measures mass of aluminum correctly

4) 0.5 Measures ΔM_{LN_2} in some way

5) 0.5 Takes into account "thermal leakage" in some way and corrects for aluminum added to container

6) 0.5 Takes into account "thermal leakage" not being constant in time

7) 0.5 Uses reasonable values for c and ΔT or does $\int c\, dT$ integral in a reasonable way

8) 0.5 No mistakes made in computing L

9) 0.5 Error estimate is reasonable for methods used

10) 0.5 Value for L is within bounds set by grading team using good procedures

Method No. 2 (5 points maximum)

1) 0.5 Uses $P = \Delta Q/\Delta t$

2) 0.5 Uses $P = IV = I^2R = V^2/R$

3) 0.5 Uses $Q = LM_{LN_2}$

4) 0.5 Measures two parameters (to get P) correctly

5) 0.5 Measures M_{LN_2} in some way

6) 0.5 Takes into account "thermal leakage" in some way

7) 0.5 Takes into account "thermal leakage" not being constant in time

8) 0.5 No mistakes made in computing L

9) 0.5 Error estimate is reasonable for methods used

10) 0.5 Value for L is within bounds set by grading team using good procedures

COUNTRY:________________

XXIV INTERNATIONAL PHYSICS OLYMPIAD
WILLIAMSBURG, VIRGINIA, U.S.A.

PRACTICAL COMPETITION

Experiment No. 2

July 14, 1993

Time available: 2.5 hours

READ THIS FIRST!

INSTRUCTIONS:

1. Use only the pen provided, and only the equipment supplied.
2. Use only the marked side of the paper.
3. Write at the top of each page:
   - The number of the problem
   - The number of the page of your report
   - The total number of pages in your report.

   Example (for problem 1): 1 1/4; 1 2/4; 1 3/4; 1 4/4

Experimental Problem 2

MAGNETIC MOMENTS AND FIELDS

This experiment has two parts:

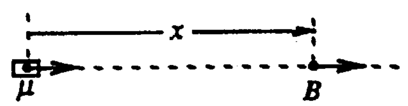
Part 1: To determine the absolute magnitude μ_X of the magnetic moment of a small cylindrical permanent magnet, contained in the envelope marked "X". (A similar magnet, also needed for the experiment, is contained in the envelope marked "A".)

Part 2: To investigate the magnetic field of a given axially symmetric distribution of magnets, contained in the envelope marked "B".

In your experiments, you should make use of the following facts:

(1) The magnetic field **B** produced by a dipole magnet at a point along its axis at distance x from its center is parallel to that axis and of strength given by:

$$B(x) = \frac{2\mu K}{x^3},$$



where **B** is in Tesla [= N/(A m)], $K = 10^{-7}$ Tesla m/A, x is in m, and μ is in A m^2.

(2) The period of small torsional (angular) oscillations of a horizontal freely suspended magnet, such as a compass needle in the Earth's magnetic field, is given by:

$$T = 2\pi\sqrt{\frac{I}{\mu B_h}},$$

where B_h is the horizontal component of the net field at the magnet, and I is the moment of inertia of the magnet about a vertical axis through its center.

Apparatus

The apparatus is illustrated in the diagrams at the end. A thin thread is suspended from the upper of two shelves on a wooden stand. A magnet ("X" or "A") can be attached to the bottom end of the thread. A copper plate can be placed on the lower shelf, just below the suspended magnet, to damp out its motion if desired. Two auxiliary wooden stands are provided. One of these serves as a holder for either "A" or "X" in Part 1; the other holds the magnet system B (used in Part 2). Distances between a suspended magnet and a magnet mounted in one of the auxiliary stands can be measured with a ruler mounted on that stand.

Warning: These magnets are extremely strong. Hold onto them tightly and be careful not to let them be pulled out of your fingers.

PART 1

The magnetic moment to be determined (μ_X) is that of the pair of magnets in envelope X, labelled at the ends with a letter-number combination. Always keep this pair together. The moment of inertia of this pair has been calculated and written on envelope X. Envelope A contains another pair of magnets with north and south poles marked respectively with black and red spots. This pair is similar to the pair from envelope X, though its magnetic moment (μ_A) cannot be assumed equal to μ_X. A given pair of magnets can be "split" and placed around the bronze disk attached to the thread, forming a "compass" whose torsional oscillation period may be measured. (The value I_X given on envelope X includes the effects of the bronze disk.)

One magnet-pair, centered in the hole in the wooden holder, can be used to influence the "compass" pair, possibly affecting its period and its angular equilibrium position. The angular position is best studied by placing the copper plate a few millimeters below the "compass" so as to provide electromagnetic damping. **Please do not mark or write on the copper plate.**

You will need to use more than one arrangement of the magnets. **Draw clearly labelled diagrams showing each experimental arrangement used. Also, write equations to show how you will combine your different observations to obtain the value of μ_X.**

Keep all magnets in the same horizontal plane. Note for the main stand that the top knob can be rotated, and the thread length adjusted. The position of each shelf can also be adjusted.

Practical Details (IMPORTANT!)

1) COMPASS ASSEMBLY AND USE: Hold one magnet from a given pair between the thumb and forefinger of one hand. Center the bronze disk over one end. Then, carefully, and without pulling on the thread, slowly bring in the second magnet. This forms the compass pair ("X" or "A"). Also, avoid pulling on the thread in taking the compass apart.
   Warning: Rapid snapping of magnets or magnet pairs together can break the thread or chip the magnets. The tiny loop can be threaded again if thread breakage occurs. (Consult the organizers if necessary.)

2) Study the torsional mode of oscillation. To prevent excitation of the "pendulum" mode, a small assembly made of copper wire is mounted on the lower shelf of the main stand. Rotate this assembly so that the horizontal piece is up against the thread at a point about 2 mm above where the thread is tied. With a slight additional rotation in the same direction, move the wire a few mm further .
   Warning: If this is not done, the two modes can "couple," causing a periodic variation in the amplitude of the torsional oscillations, and affecting their period.
   Use the nail (see diagrams at end) to start the torsional oscillations in a controlled way.

3) Keep magnetic or magnetizable objects stationary, and as far as possible from the experimental area. Consider such items as the nail, wrist watches, pens, etc. The table has some steel support parts; if you want to change the position of the apparatus, consider this fact.

Suggestions

(i) The torsion constant of the thread is quite small. It turns out that you can neglect its effect in the analysis provided the thread is reasonably long, e.g. around 15 cm.

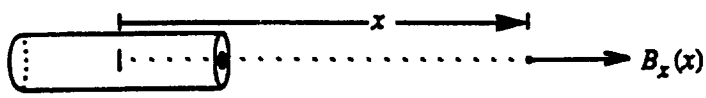
(ii) You may notice that a given magnet pair does not hang horizontally. This is because of the vertical component of the Earth's field. The effect of this on the analysis is small and should be neglected. In other words, simply pretend that the magnet is horizontal.

(iii) We suggest that you postpone the error analysis for Part 1 until after you have made the measurements needed for Part 2.

(iv) **You should not make any assumptions about the magnitude of the Earth's field.**

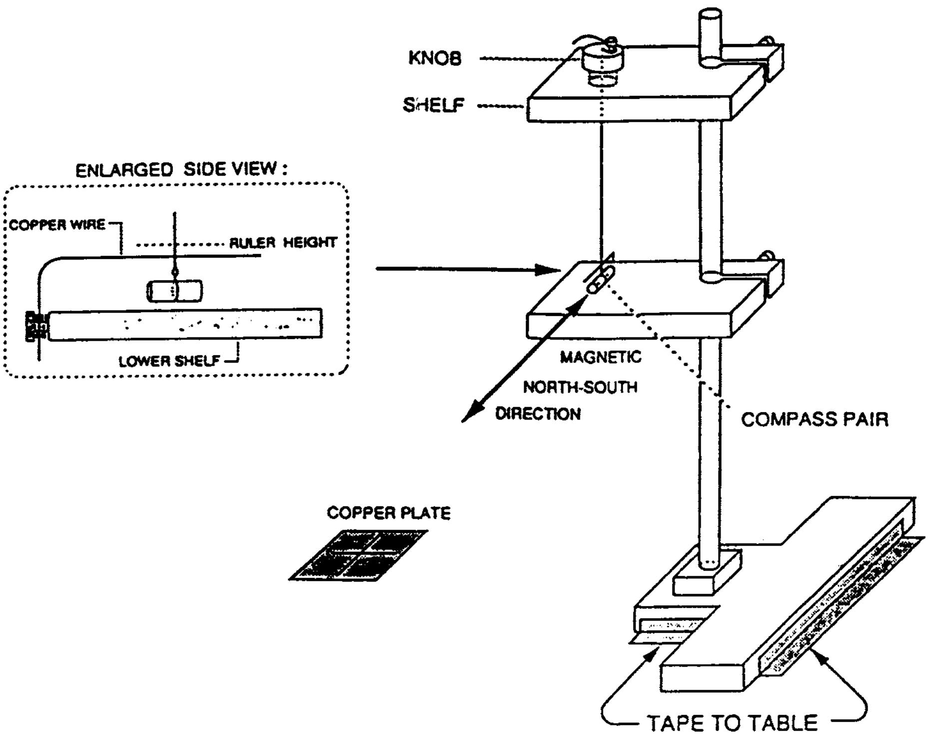
PART 2

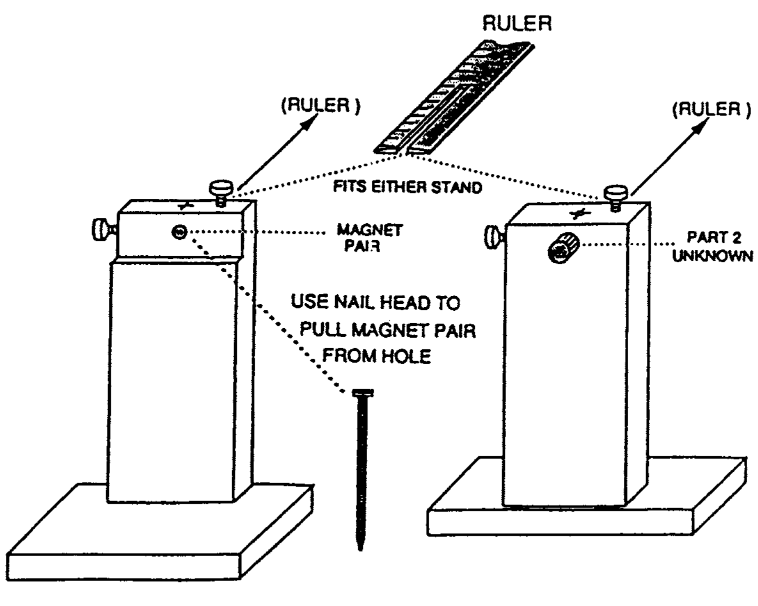
The aluminum tube (in envelope B) contains an axially symmetrical distribution of magnets. The magnetic field along the x axis, B_x, of this assembly varies as a function of distance x measured from the center of the tube according to the relation $B_x(x) = Cx^p$. Determine the exponent p, with its approximate error. As sketched below, you should study the field on the side in the direction of the end marked with a black spot.



WRITE YOUR SET-UP NUMBER ON YOUR REPORT. THIS IS THE LETTER-NUMBER COMBINATION PRINTED ON THE EQUIPMENT BOX AND ALSO ON THE MAGNET ENVELOPES LIKE THIS:

☐


KNOB
SHELF
ENLARGED SIDE VIEW :
COPPER WIRE
RULER HEIGHT
LOWER SHELF
MAGNETIC
NORTH-SOUTH
DIRECTION
COMPASS PAIR
COPPER PLATE
TAPE TO TABLE


RULER
(RULER)
(RULER)
FITS EITHER STAND
MAGNET
PAIR
PART 2
UNKNOWN
USE NAIL HEAD TO
PULL MAGNET PAIR
FROM HOLE

Experimental Problem 2 -- Solutions

PART 1 : DETERMINATION OF μ_X

Basic Insight :

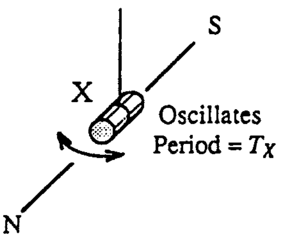
The idea which enables one to "see into" the problem is contained in the following remark: The oscillation period of a given suspended magnet depends on the product of its moment and the (horizontal component of) the Earth's field, while the extent to which that magnet can influence the direction of another magnet used as a compass depends on the ratio of those two quantities.

It follows that by making measurements of both types, both the unknown moment and the horizontal component of the Earth's field can be determined. We suspect that this idea goes historically back to Gauss.

First Solution : The "Turn-Around Method"

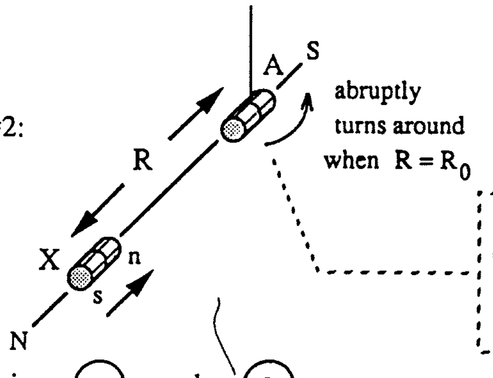
Experimental Arrangement | Equation

#1:



$$\mu_X B_h = I_X (2\pi/T_X)^2 \qquad (1)$$

#2:



$$\mu_X \frac{2K}{R_0^3} = B_h \qquad (2)$$

use copper damping plate beaneath compass

note that the μ and I values of the compass magnet do not matter

Combining (1) and (2) one easily finds:

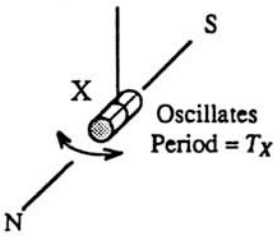
$$\mu_X = \frac{R_0^{3/2}}{(2K)^{1/2}} \frac{2\pi}{T_X} (I_X)^{1/2}$$

Second Solution : Dynamic Method with 3 Unknowns

The experience from our tests was that the "Turn-Around" method did not occur naturally to most students. They were much more comfortable with the idea of using one magnet to influence the period of another. Since the magnetic moments are not necessarily equal, it is clear that two measurements will no longer suffice. Our guess is that the following 3-measurement scheme will be the most common student choice.

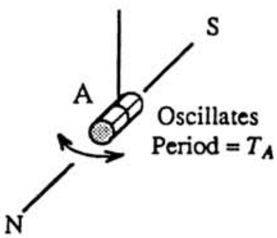
Experimental Arrangement | Equation

#1:



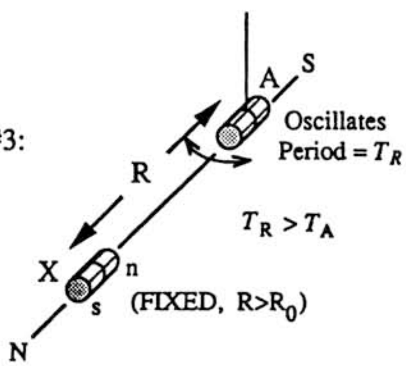
$$\mu_X B_h = I_X (2\pi/T_X)^2 \qquad (1)$$

#2:



$$\mu_A B_h = I_A (2\pi/T_A)^2 \qquad (2)$$

#3:



$$\mu_A \left[B_h - \mu_X \frac{2K}{R^3}\right] = I_A (2\pi/T_R)^2 \qquad (3)$$

Note that the X magnet (positioned at a distance R which is somewhat larger than the turn-around distance R_0) is being used here to slow the oscillations of the A magnet on the compass.

One worries at first that there are actually 4 unknowns, since the inertial moment of A need not equal that of X. Inspection of equations (2) and (3) shows, however, that the ratio μ_X/B_h can be expresed in terms of experimentally known quantities. Since (1) gives the <u>product</u> $\mu_X B_h$, the calculational strategy is clear. One easily finds:

$$\mu_X = \frac{R^{3/2}}{(2K)^{1/2}} \frac{2\pi}{T_X} (I_X)^{1/2} \left[1 - (T_A/T_R)^2\right]^{1/2} \quad (4)$$

Alternatively, by reversing its poles, one can use the X magnet to <u>speed-up</u> the oscillations of the A magnet. Then, of course we have $T_R < T_A$. In this case (which is formally equivalent to the first case, with a reversal of the sign of K), one finds:

$$\mu_X = \frac{R^{3/2}}{(2K)^{1/2}} \frac{2\pi}{T_X} (I_X)^{1/2} \left[(T_A/T_R)^2 - 1\right]^{1/2} \quad (4')$$

SAMPLE EXPERIMENT

The Dynamic Method just outlined was used (in the case where the X magnet was used to slow down the oscillations of the A magnet in Arrangement #3). In all cases 20 oscillations were timed. The distance R was (17.0±0.1) cm. The X moment of inertia was $I_X = (4.95 \pm 0.1) \times 10^{-8}$ kg m^2. Using the notation given previously, the data were as follows:

Measurements (in seconds) of $20T_X$: 10.83, 10.99, 10.91, 10.94. [Arrangement #1]

Measurements (in seconds) of $20T_A$: 10.95, 11.10, 11.01, 10.92. [Arrangement #2]

Measurements (in seconds) of $20T_R$: 21.70, 21.65, 21.78, 21.59. [Arrangement #3]

Using a pocket calculator (HP32S) to obtain the averages and statistical errors gives:

T_X = (0.546±.003) sec

T_A = (0.550±.004) sec

T_R = (1.084±.004) sec

The "statistical errors"" here are naively based on what the calculator gave for the estimated standard deviation <u>around</u> the sample mean. More carefully, one should divide this by the square root of the number of observations to give the estimated standard error <u>of</u> the sample mean. [Still more carefully, for such a <u>small</u> sample, one should apply the appropriate statistical correction factor]. For simplicity we will use the naively calculated results. This will suffice for our purposes.

Write ④ as $\mu_X = G\,F$, where

$$G = \frac{R^{3/2}}{(2K)^{1/2}} \frac{2\pi}{T_X} (I_X)^{1/2} \qquad \text{and} \qquad F = \left[1 - (T_A/T_R)^2\right]^{1/2}$$

The expression for G is identical for that for μ_X in the "turnaround method" when $R=R_0$. This must be true, since in that case T_R goes to infinity.

Numerically

$$G = \frac{[(0.170\pm0.001)\,\text{m}]^{3/2}}{[2\times10^{-7}\,\text{N}/\text{A}^2]^{1/2}} \;\frac{2\pi}{(0.546\pm.003)\,\text{sec}}\; \left[(4.95\pm0.1)\times10^{-8}\,\text{kg m}^2\right]^{1/2}$$

then standard error propagation and reduction of the units give

$$G = (0.401 \pm 0.006)\ \text{Am}^2$$

which is a 1.5% uncertainty. For F we find numerically :

$$F = \left\{1.000 - \left[\frac{(0.550\pm.004)\,\text{sec}}{(1.084\pm.004)\,\text{sec}}\right]^2\right\}^{1/2}$$

The central value here is 0.862. One can easily use a pocket calculator to see the effects of the permitted statistical variations in each of the two places above. This shows that the effect of the numerator uncertainty is essentially ± 0.0022, while that of the denominator is ± 0.0013. Combining these statistically gives an net uncertainty in F of 0.0026, so that the fractional uncertainty in F is 0.0033. [An analysis of this by calculus is straightforward, but cumbersome.] Then the fractional uncertainty in μ_X is practically that in G. We find:

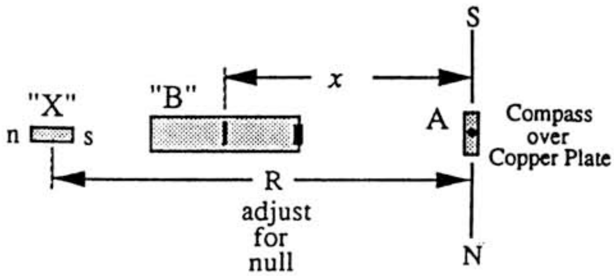
$$\mu_X = (0.862 \pm 0.0026)\,(0.401 \pm 0.006)\ \text{Am} = (0.346\pm.005)\ \text{A m}^2.$$

By way of comparison, measurement of the same magnet X using Fluxgate Magnetometry (at a distance of around 16 cm) gave $\mu_X = (0.345 \pm .003)\ \text{A m}^2$.

PART 2 : DISTANCE DEPENDENCE OF FIELD OF "B" UNKOWN

Method I (Close Distances) : Nulling of Transverse Static Deflection

Arrangement (top view)



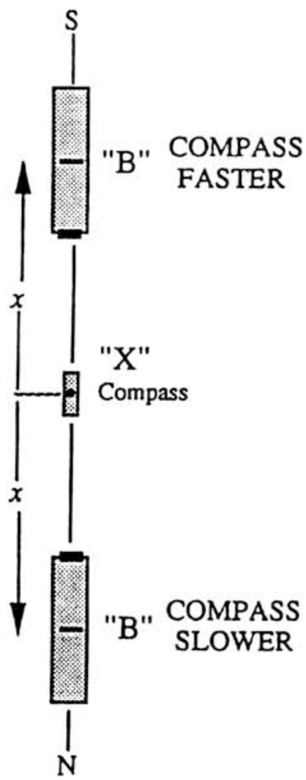
Equation

$$B_X(x) = \frac{2K\mu_X}{R^3}$$

Method II (Intermediate Distances) : Differential $1/T^2$ Technique

General Relation : $T = T_X$; B_h =local (horiz.) field $\left\{ (2\pi/T)^2 = \frac{\mu_X B_h}{I_X} \right.$

Arrangement (top view)



DEFINE:

$$\Delta(1/T^2) \equiv (1/T^2)_{faster} - (1/T^2)_{slower}$$

THEN:

$$\Delta(1/T^2) = \frac{\mu_X \Delta B_h}{4\pi^2 I_X} \quad \text{where} \quad \Delta B_h = 2B_x(x)$$

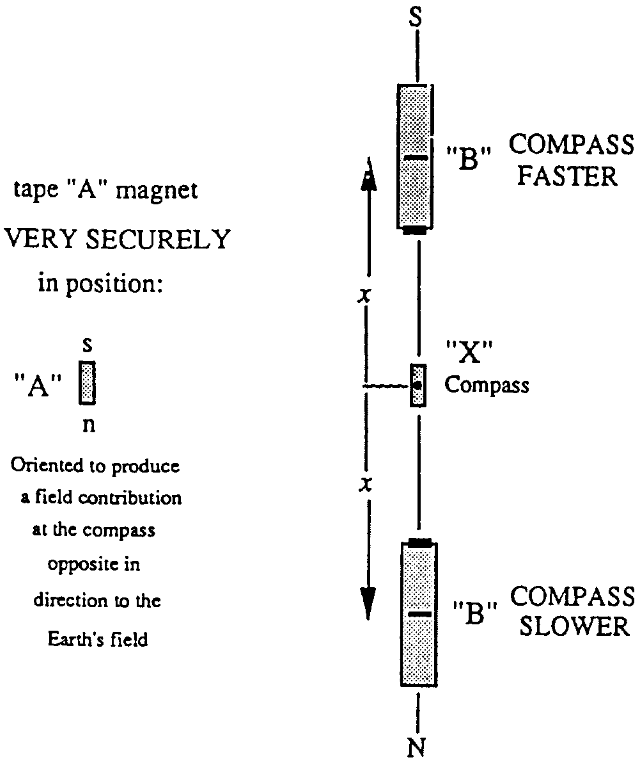
$$\boxed{B_x(x) = \frac{2\pi^2 I_X}{\mu_X}\Delta(1/T^2)}$$

"Master Equation"

Method III (Large Distances) :

Differential $1/T^2$ Technique with Partial "Bucking" of the Earth's Field

Arrangement (top view)



"Master Equation"

$$B_x(x) = \frac{2\pi^2 I_X}{\mu_X} \Delta(1/T^2)$$

remains the same

Use only partial buckout --(slow natural oscillations typically by a factor of 2)

In working at a given distance x, $\Delta(1/T^2)$ must be constant (independent of the "bucking").

$$\Delta(1/T^2) = \text{const.}$$

$$\Delta T/T^3 = \text{const.} \longrightarrow \boxed{\Delta T \propto T^3}$$

Sample Experiment

Method I $\quad B_x(x) = \frac{2K\mu_X}{R^3} = \frac{[(2\times10^{-7})\text{T m/A}][0.346\pm0.005]\text{Am}^2}{[\text{R (m)}]^3}$

DATA TABLE FOR METHOD I

measured data		calculated	standard error propagation	see below	
x (m)	R (m)	$B_x(x)$ $(10^{-7}$ T)	ΔB/B	$4\Delta x/x$	$(\Delta B/B)_{eff}$
.062±.001	.112±0.112	493.	.031	.065	.072
.0705±.0015	.133±.0015	294	.019	.085	.087
.0845±.0015	.167±.002	149	.039	.071	.081
.102±.0015	.206±.005	79	.074	.059	.095

The uncertainty in R includes the ruler reading error, together with the imprecision in locating the null position, the latter effect becoming predominant at larger x. The R uncertainty, together with the small uncertainty in μ_X define the ΔB/B values listed in the 4th column.

Of course there are also the uncertainties in the x values, which we could represent graphically by horizontal error bars. Since this is technically awkward, we choose instead to define an effective vertical uncertainty. Since it turns out that the log-log plot slope is about -4, a given fractional error in x corresponds to 4 times as much in B(x). These fractional errors have been tabulated in the 5th column. From this it is clear that we should take the effective ΔB/B as the square root of the sum of the squares of thecontributions in colums 4 and 5. These values, listed in column 6, form the basis for the error bars used. Though we would certainly not expect a student to do this, we would expect him to be aware of the horizontal uncertainties.

Method II $\quad B_x(x) = \frac{2\pi^2 I_X}{\mu_X}\,\Delta(1/T^2) = (28.2\pm .51)\times10^{-7}\text{ Tesla sec}^2 \cdot \Delta(1/T^2)$

● $x = (.120\pm.001)$m :

Data in seconds for 20 oscillations: Pocket Calculator Results:

$20\,T_{slow}$: 14.56, 14.50, 14.52, 14.58 $\quad T_{slow} = (.727\pm.0018)$sec

$20\,T_{fast}$: 11.32, 11.34, 11.31, 11.28 $\quad T_{fast} = (.5656\pm.0013)$sec

$\Delta(1/T^2) = [(3.1257\pm.0138)-(1.892\pm.0095)]\text{ sec}^{-2} = (1.23\pm.017)\text{sec}^{-2}$

⟶ $B_x(x) = (34.7\pm0.8)\times10^{-7}$ Tesla

Method III

Introduced bucking magnet in transverse position to slow oscillations in Earth's Field to about 1.2sec

Master equation is still:

$$B_x(x) = \frac{2\pi^2 I_X}{\mu_X}\,\Delta(1/T^2) = (28.2\pm .51)\times 10^{-7}\ \text{Tesla sec}^2 \cdot \Delta(1/T^2)$$

- $x = (.150\pm.001)\text{m}$:

Data in seconds for 20 oscillations: Pocket Calculator Results:

$20\,T_{slow}$: 27.90, 27.80, 27.78, 27.77 — $T_{slow} = (1.391\pm.003)\text{sec}$

$20\,T_{fast}$: 19.56 19.66, 19.50, 19.64 — $T_{fast} = (.9795\pm.0037)\text{sec}$

$\Delta(1/T^2) = [(1.0422\pm.0079)-(.5171\pm.0022)]\text{sec}^{-2} = (.525\pm.0082)\text{sec}^{-2}$

$\longrightarrow B_x(x) = (14.8\pm.35)\times 10^{-7}$ Tesla

- $x = (.170\pm.001)\text{m}$:

Data in seconds for 20 oscillations: Pocket Calculator Results:

$20\,T_{slow}$: 24.97, 24.97, 24.87 — $T_{slow} = (1.2468\pm.0029)\text{sec}$

$20\,T_{fast}$: 20.55, 20.46, 20.79, 20.65 — $T_{fast} = (1.0306\pm.00708)\text{sec}$

$\Delta(1/T^2) = [(.9415\pm.013)-(.6433\pm.0030)]\text{sec}^{-2} = (.298\pm.013)\text{sec}^{-2}$

$\longrightarrow B_x(x) = (8.4\pm0.4)\times 10^{-7}$ Tesla

- $x = (.190\pm.001)\text{m}$:

Data in seconds for 20 oscillations: Pocket Calculator Results:

$20\,T_{slow}$: 17.17, 17.15, 17.11, 17.10 — $T_{slow} = (.8566\pm.0017)\text{sec}$

$20\,T_{fast}$: 16.01, 15.93, 15.91, 15.92 — $T_{fast} = (.797\pm.0029)\text{sec}$

$\Delta(1/T^2) = [(1.574\pm.028)-(1.3628\pm.0053)]\text{sec}^{-2} = (.2112\pm.029)\text{sec}^{-2}$

$\longrightarrow B_x(x) = (6.0\pm0.8)\times 10^{-7}$ Tesla

- $x = (.220\pm.001)\text{m}$:

Data in seconds for 20 oscillations: Pocket Calculator Results:

$20\,T_{slow}$: 23.80, 23.76, 23.70 — $T_{slow} = (1.1877\pm.00252)\text{sec}$

$20\,T_{fast}$: 22.27, 21.98, 21.86, 21.94 — $T_{fast} = (1.1006\pm.0089)\text{sec}$

$\Delta(1/T^2) = [(.8255\pm.0134)-(.7089\pm.0030)]\text{sec}^{-2} = (.1166\pm.014)\text{sec}^{-2}$

$\longrightarrow B_x(x) = (3.3\pm0.4)\times 10^{-7}$ Tesla

DATA TABLE FOR METHODS II and III

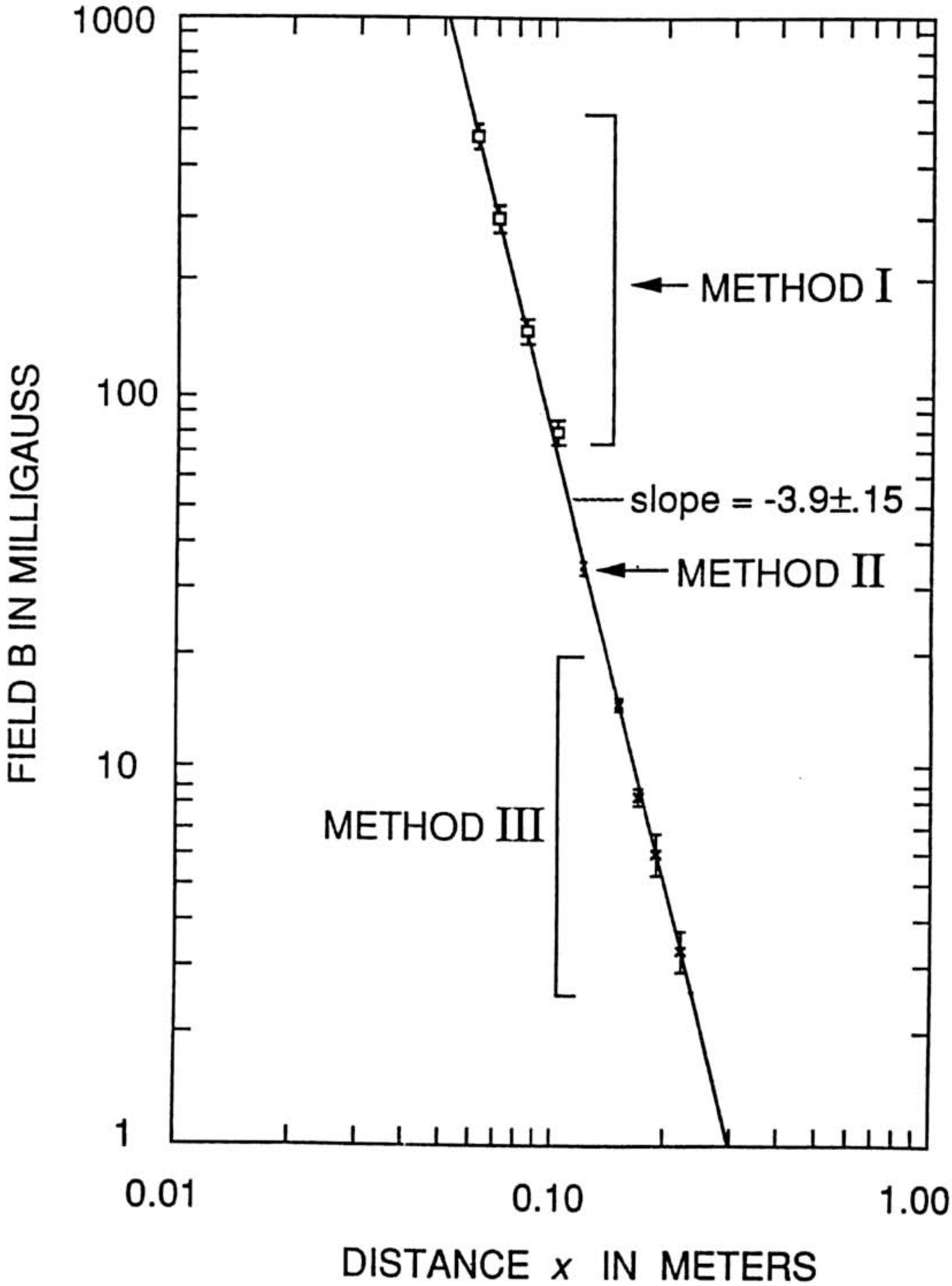
x (m)	Method	calculated $B_x(x)$ (10^{-7} T)	standard error propagation $\Delta B/B$	see above $4\Delta x/x$	$(\Delta B/B)_{eff}$
.120±.001	II	34.7	.023	.033	.040
.150±.001	III	14.8	.024	.027	.036
.170±.001	III	8.4	.05	.024	.055
.190±.001	III	6.0	.13	.021	.13
.220±.001	III	3.3	.12	.018	.12

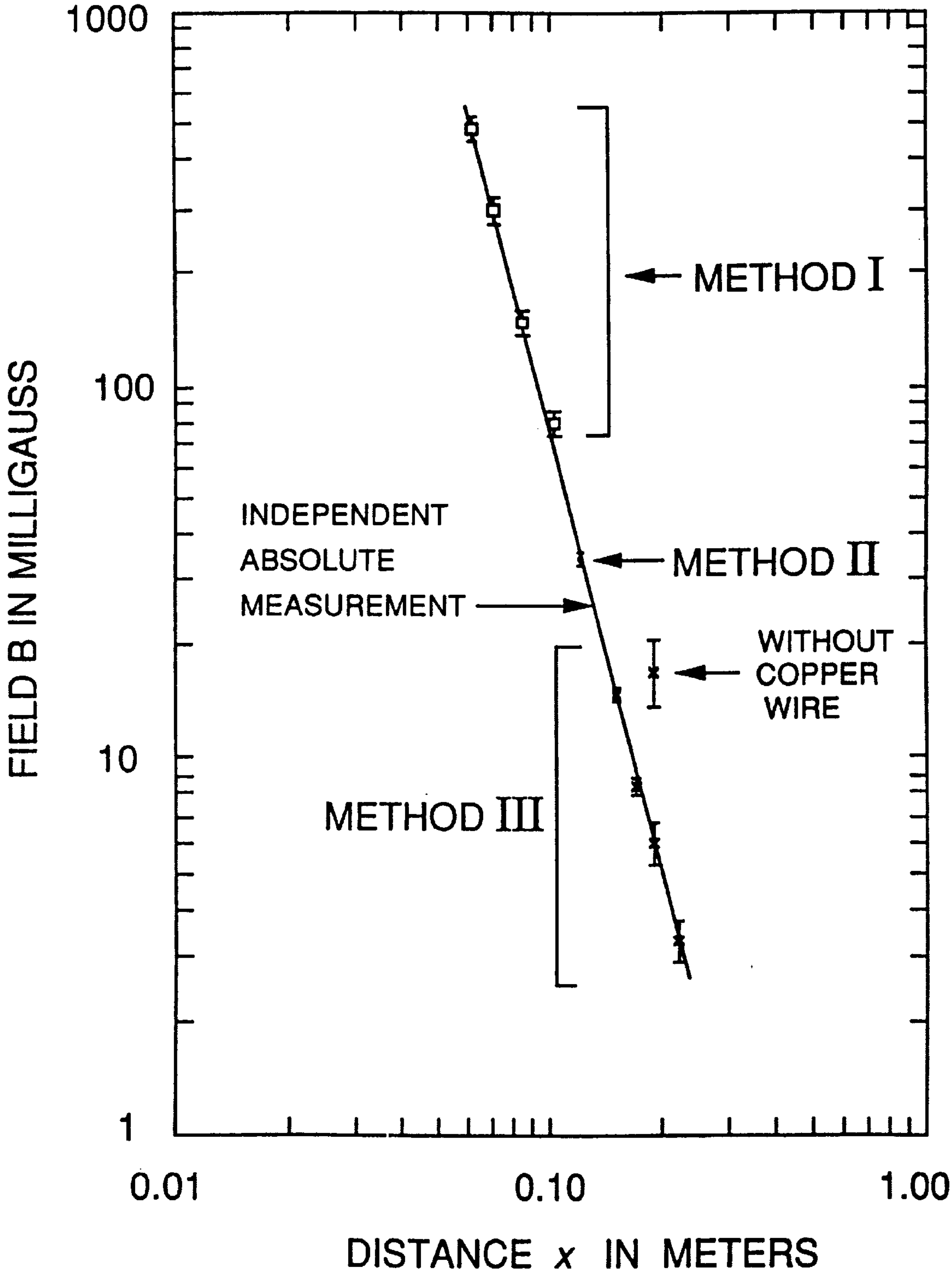
The equivalent vertical uncertainties have calculated as before and tabulated in the last column above. These give the error bars on the log-log plot shown on the next page. The three different methods are nicely consistent, and the whole data set well fits the power law indicated by the drawn line. When this is done on the regular log paper (as provided), the easiest way in this case to get the slope is to use a pocket calculator to find the ratio of the log of the vertical rise ratio to that of the horizontal run ratio for the possible lines consistent with the errors. Since the line has to drop vertically through three decades in total, this is roughly

$$\text{slope} = \frac{-3}{\log_{10}\left[\dfrac{(0.30\pm.02)}{(.051\pm.03)}\right]} = -3.9 \pm 0.15$$

For this particular unknown, the fluxgate magnetometer data gave an effective exponent of -3.92 over the range from 0.07m to 0.22m. A more detailed absolute comparison with those measurements is shown on the second graph. Here the drawn line corresponds to the actual magnetometer data. The student experiment is clearly doing an excellent job. Of particular interest is the next to the lowest point (x=0.19m). For this point, the "buckout" magnet had been moved out a little bit so that the natural compass period in the Earth's field was about 0.89 sec., which was close to the period of the "pendulum mode". This was done deliberately to test the effectiveness of the copper wire "mode-decoupler". The point at x=0.19 m which is on the line was taken using the decoupler. The point at the same x value which is almost a factor of 3 higher than the line was taken without the decoupler.

This shows that the decoupler is both effective and important. Without it, the "fast" and "slow" measurements are effected differently by the coupling to the pendulum mode. Then the small difference between them can be very poorly determined.


1000
100
10
1
FIELD B IN MILLIGAUSS
METHOD I
slope = -3.9±.15
METHOD II
METHOD III
0.01
0.10
1.00
DISTANCE x IN METERS


1000
100
10
1
0.01
0.10
1.00
FIELD B IN MILLIGAUSS
DISTANCE x IN METERS
METHOD I
METHOD II
METHOD III
INDEPENDENT
ABSOLUTE
MEASUREMENT
WITHOUT
COPPER
WIRE

Experimental Problem 2: Grading Scheme

Part 1

2.5 points	Show how μ_X is calculated, clearly labeled diagram
1.5 points	μ_X is correctly stated
0 - 1 points (sliding scale)	error analysis
0 - 1 points (sliding scale)	consistency with "correct" range

Part 2

1.0 points	A diagram of a technique that can be used
1.0 points	Correct measurements at 3 distances at least
0 - 1 points (sliding scale)	Accuracy of the result (correct value of p)
0 - 1 points (sliding scale)	Precision and error analysis