A Glimpse into the Special Theory of Relativity

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1 Why relativity?

Relativity is often seen as an intricate theory that is necessary only when dealing with really high speeds or ultra-precise measurements. However, there are some quite often-encountered topics that are paradoxical if treated non-relativistically. These are also some of the main sources of Olympiad problems on relativity.

Think for a moment about two charged initially stationary particles. They "feel" only the electrostatic force from each another. But in another, moving reference frame there is also the magnetic force, in general, in a different direction! How could force depend on the choice of inertial reference frame? What principles forbid the particles from colliding in one frame and departing in another? Electromagnetism needs relativity for an explanation.

Photons (thus, much of optics) are always relativistic and other particles often are. Anything where the speed of light matters – for example, the GPS measuring the time for a radio signal to travel from satellites – uses relativity.

Particle physics needs relativity in several aspects. Particles cannot be controlled in a modern accelerator without taking into account their relativistic dynamics. The only successful quantum theory predicting the outcomes of particle collisions, quantum field theory, is relativistic. Muons in cosmic rays would decay long before reaching the ground, but we still detect them thanks to relativistic time dilation.

Relativistic theory of gravity – general relativity – allows to formulate the physics independently of whether the reference frame is inertial or not, thus unifying time and space even more tightly. It is necessary for astrophysics (precession of planets' orbits, gravitational lensing, black holes), and cosmology (history and future of large-scale structures).

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In the following, we shall derive the most important results of the special theory of relativity, starting from the fundamental postulates. Most steps of the derivation are given as problems, which are also good examples of what one can ask in relativity and exercise the reader's ability to use the theory.

The most important general technique for problem-solving is rotation of Minkowski spacetime in complex coordinates, this is described in section 4. Section 3 shows the way from postulates to the useful techniques, its problems may be skipped if concentrating purely on Olympiad preparation. Section 4 is mostly on kinematics, the following ones develop dynamics, optics and (briefly) electromagnetism from it. Finally, some problems for practising are given.

2 Postulates of special relativity

1. **The laws of all physics are the same in every inertial reference frame.**

A reference frame is inertial if and only if objects onto which no force acts move in a straight line with constant velocity.

2. **The speed of light in vacuum (c) is the same in every inertial reference frame.**

In SI, after defining the second,[*] the metre is defined through fixing (exactly!) c = 299,792,458 m/s.

3 Basic thought experiments

3.1 Time dilation

Problem 1. Consider a "light clock" that works as follows. A photon is emitted towards a mirror at a known distance l and reflected back. It is detected (almost) at the emitter again. The time from the emission to the detection (a "tick") is measured to be t. Now we

* One second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom at rest at a temperature of 0 K.

look at the clock from a reference frame where the whole apparatus is moving with velocity v *perpendicularly* to the light beam. Assume that the lengths perpendicular to the motion do not change. How long is the tick for us? (Hint: the light beam follows a zig-zag path.)

The answer is given by the following fact.

Fact 1. If the time interval between to events happening at a stationary point is t, then in a reference frame where the speed of the point is v the time interval is γt, where the *Lorentz factor*

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Another useful quantity in relativistic calculations is $\beta = v/c$. As $\gamma > 1$, we see everything in a moving vehicle take *longer* than in a stationary one – time is *dilated* (stretched) in a moving reference frame.

3.2 Length contraction

Problem 2. Now consider the same "light clock" as in Problem 1., but moving *in parallel* to the light beam, with velocity v. What is the distance to the mirror in our reference frame, if the distance in the stationary frame is l?

The answer is in the following fact.

Fact 2. If the length of a stationary rod is l, then its length in a reference frame moving in parallel to the rod with speed v is l/γ.

Lengths are *contracted* (compressed) in the direction of motion.

3.3 Proper time

Problem 3. A spaceship flies freely from (t_1, x_1, y_1, z_1) (event 1) to (t_2, x_2, y_2, z_2) (event 2). What is the *proper time* τ – time measured by a passenger on the spaceship – between these events? [Answer:
$$c^2\tau^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$
]

4 Lorentz transformations

4.1 Spacetime interval

In ordinary, *Galilean* relativity, lengths and time intervals are absolute. As we have now seen, the postulates of *Einsteinian* relativity imply that neither is so, once speeds become comparable to c. However, *proper* time – time in a comoving frame – must clearly be independent of *our* reference frame. Therefore we can define a new invariant quantity with the dimension of length.

Fact 3. The *spacetime interval*
$$s=\sqrt{c^2(\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2}$$
is independent of the choice of reference frame.

If s is a real number, the interval is called *time-like*; if s is imaginary, the interval is *space-like*. If s is zero, the interval is *light-like*.

Fact 4. The interval between two events on the same light-ray (in vacuum) is zero – thus, light-like.

4.2 Minkowski spacetime, Poincaré transformations

We can say that spacetime points are represented by *position four-vectors** $x^{\mu}=(ct, x, y, z)$ and the interval calculates the length of the *displacement four-vector* Δx^{μ}. However, this law of calculating the length has important minus signs in it, so these four-vectors form a *Minkowski spacetime*, not the usual *Euclidean space*, where lengths would be calculated using the usual Pythagoras' law.

We can reuse our familiar laws of geometry if we introduce complex numbers. Namely, the invariant quantity
$$is=\sqrt{(ic\Delta t)^2+(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$$
is now expressed just like Pythagorean theorem. So, the Euclidean distance between two events in the spacetime of (ict, x, y, z) is independent of reference frame.

What transformations of Euclidean space leave lengths invariant? Rotations and translations and combinations thereof!

Fact 5. Changes of inertial reference frames correspond to rotations and shifts in the spacetime coordinates *ict*, *x*, *y* and *z*.

In general, such transformations are called the *Poincaré transformations** and, if we only rotate and do not shift the coordinates, the *Lorentz transformations*.

4.3 Rapidity

By what angle should we rotate the axes? Clearly, as one axis has imaginary numbers on it, the angle must also be complex.

Luckily this poses no problems in drawing the angle, as long as we consider only one-dimensional motion: it turns out to be a purely imaginary angle, so its cosine

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* Four-vectors are customarily labelled by Greek indices written as superscripts; subscripts have a meaning in more advanced theory.

* The Poincaré transformations are also known as the *inhomogeneous Lorentz transformations*.

$\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2}$ (a projection of the unit direction vector) is real (can be drawn on the real *x*-axis) and its sine $\sin\alpha = \frac{e^{i\alpha}-e^{-i\alpha}}{2i}$ is purely imaginary (can be drawn on the imaginary *ict*-axis).

Problem 4. Take two coordinate systems, *O* and *O′*, with the spatial axes parallel and the (spatial) origin* of *O′* moving in the *x*-direction with velocity *v*. Calculate the angle α between the *x*- and *x′*-axis. (Hint: make a diagram with *ict* on one axis and *x* on another. Add the *ict′*- and *x′*-axes. Calculate the *x*- and *ict*-coordinate of one arbitrary point the spatial origin of *O′* passes through. The ratio of these coordinates is $\tan\alpha$.)

Such a Lorentz transformation involving only time and one spatial coordinate is called the *Lorentz boost* in the *x*-direction. The answer to the problem is the following useful fact.

Fact 6. A Lorentz boost in the *x*-direction from standstill to velocity *v* corresponds to rotation of *x*- and *ict*-axis by an angle of

$$\alpha = \arctan\frac{v}{ic} = \arctan\frac{\beta}{i}\ .$$

* In the spatial origin, *x* = *y* = *z* = 0, but *ict* changes.

Problem 5. Calculate $\cos\alpha$ and $\sin\alpha$.

Fact 7. $\cos\alpha = \gamma$, $\sin\alpha = \beta\gamma/i$.

The quantity $\varphi = \alpha/i$ is a real dimensionless number and is called the *rapidity*.

4.4 Hyperbolic trigonometry

Some imaginary units *i* and some minuses can be eliminated by using hyperbolic trigonometry. Employing the formulae $\sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2}$, $\cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2}$, $\tanh\alpha = \frac{\sinh\alpha}{\cosh\alpha}$ and $\cosh^2\alpha - \sinh^2\alpha = 1$ you can prove the following.

Problem 6. Prove for the rapidity φ that $\tanh\varphi = \beta$, $\cosh\varphi = \gamma$ and $\sinh\varphi = \beta\gamma$.

Consequently, using the inverse function of hyperbolic tangent, $\alpha = i\varphi = i\,\mathrm{artanh}\,\beta$.

4.5 Length contraction, time dilation and velocity addition

Problem 7. Prove again the length contraction formula of Fact 2. Here use rotation of Minkowski spacetime.

Problem 8. Prove similarly the time dilation formula of Fact 1.

Fact 8. If an object moves with respect to reference frame *O′* with velocity *u* and *O′* moves

with respect to frame *O* with velocity *v* in the same direction, then the velocity of the object in *O* is

$$w=\frac{u+v}{1+\frac{uv}{c^2}}.$$

Problem 9. Prove the velocity addition formula in the last fact. (Hint: $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$ and $\tanh(\alpha+\beta)=\frac{\tanh\alpha+\tanh\beta}{1+\tanh\alpha\tanh\beta}$.)

Problem 10. Show that the velocity addition formula implies the postulate that the speed of light is universal. (Hint: $u=\pm c$.)

Problem 11. Prove that if *u* and *v* in the velocity addition formula are both between $-c$ and c, then so is *w*. (Hint: show that $\frac{dw}{du}>0$ – hence *w* is monotonous – and use the result of the last problem that $u=\pm c$ corresponds to $w=\pm c$.)

Fact 9. If there exists a reference frame where an object moves slower than light, then it does so in every reference frame.

4.6 Light-cones, simultaneity and causality

The trajectory of a particle in the space-time is called its *world-line*. The world-line of a photon cuts a very special wedge from the diagram: the inside of the wedge can be influenced event at the tip of the cone; the outside cannot. The region where an event can have influence in is called the *light-cone* of the event.

Fact 10. If the spacetime diagram is scaled so that *i* metres (on the *ict*-axis) is at the same distance from the origin as 1 metre (on the *x*-axis), then the world-line of a photon is at 45° from either axis.

Fact 11. Simultaneity is relative.

Problem 12. In reference frame *O*, two events take place at the same time $t=0$, but with spatial separation Δx. What is the time $\Delta t'$ between them in reference frame *O′*, which is moving in the *x*-direction with velocity *v*? [Answer: $\Delta t'=-\gamma v\Delta x/c^2$]

Fact 12. The order of two events with time-like or light-like separation is absolute. For space-like separation, the order depends on the reference frame.

This means that only time-like or light-like separation allows one event to be the cause of another. Demanding that the causality should hold and, thus, no information may be sent to the past, we get the following fact.

Fact 13. Information cannot propagate faster than light in vacuum.

This means, among many other implications, that everything must be somewhat deformable: if we push one end of a long rod, then the push will propagate to the other end slower than c (probably much slower).

4.7 Lorentz transformations algebraically

Fact 14. When going to a reference frame moving in the *x*-direction with velocity *v*, the time and space coordinates of an event transform under Lorentz transformations as follows.

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$
$$x' = \gamma(x - vt)$$
$$y' = y$$
$$z' = z$$

Problem 13. Prove the last fact.

Problem 14. Show algebraically that if boosting in both *x*- and *y*-directions, the order of boosts matters.

Intuitively, as boosts are rotations, their order should matter just like the order of ordinary spatial rotations matters: try turning a book over around two different axes, remember the result and then repeat, switching the axes. The result of two successive boosts in different directions is actually not just a boost in a third direction, but adds some rotation that depends on the order of the boosts.

5 Dynamics

5.1 Four-velocity and four-acceleration

Generalising from the position four-vector $x^\mu = (ct, x, y, z)$ introduced in section 4.2, we now in generality define a *four-vector* as a collection of four numbers $q^\mu = (q^t, q^x, q^y, q^z)$ that transforms under Lorentz transformations. The spatial components $(q^x, q^y, q^z) \equiv \vec{q}$ rotate just like a usual vector. The time- and space-components are mixed by Lorentz boosts that act as rotations in the four-space of $(iq^t, \vec{q})$. A boost in the *x*-direction is given just as in Fact 14.

$$q^{t\,\prime} = \gamma(q^t - \beta q^x)$$
$$q^{x\,\prime} = \gamma(q^x - \beta q^t)$$

The Lorentz-invariant length of the four-vector is

$$|q^\mu| = \sqrt{(q^t)^2 - (q^x)^2 - (q^y)^2 - (q^z)^2}\ .$$

We already know that periods of proper time $d\tau$ are Lorentz-invariant. Thus, the following derivatives can be formed.

Fact 15. ***Four-velocity*** $v^\mu = \frac{dx^\mu}{d\tau}$ **and** ***four-acceleration*** $a^\mu = \frac{dv^\mu}{d\tau}$ **of a particle are four-vectors.**

Problem 15. Show that the four-velocity of a particle moving with speed v in the x-direction is $(\gamma c, \gamma v, 0,0)$.

Problem 16. What Lorentz-invariant quantity is the length of the four-velocity from the last problem? [Answer: c]

As the x-direction was arbitrary, we can generalize the answer as follows.

Fact 16. The length of any four-velocity is *c*.

Problem 17. Show that the four-acceleration of a particle moving and accelerating in the x-direction with a three-acceleration of magnitude $a = dv/dt$ is $(\beta\gamma^4 a, \gamma^4 a, 0, 0)$ with invariant length a.

5.2 Mass, momentum and energy

Some texts about relativity distinguish the *rest mass* or *invariant mass* m from the *relativistic mass* γm, but this would be misleading for discussing motion in several dimensions. Therefore, in this studying material, we refer to m as just the *mass*. This mass is an intrinsic property of any object and does not depend on the reference frame.

Fact 17. The *four-momentum* of a particle with mass *m* is the four-vector $p^\mu = m v^\mu$.

Fact 18. $p^\mu = (E/c, \vec{p})$ **where the *total energy* $E = \gamma m c^2$ and the *relativistic momentum* $\vec{p} = \gamma m \vec{v}$.**

Note that here $\vec{v}$ is the usual three-velocity and not the spatial part of the four-velocity that has an additional γ in it.

Fact 19. The length of the four-momentum is *mc*, whatever the velocity is. Therefore, $E^2 = (pc)^2 + (mc^2)^2$.

For massless particles (such as photons), $E = pc$.

Fact 20. In interactions, four-momentum is conserved.

This encompasses both the conservation of energy and the conservation of momentum.

Fact 21. The total energy can be separated into the *rest energy* $E_{\text{rest}} = mc^2$ and the *kinetic energy* $E_k = (\gamma - 1) m c^2$.

Problem 18. Show that for low speeds, $E_k \approx \frac{mv^2}{2}$.

Note that if an object has any internal structure and, thus, internal energy, then it must be taken into account in its rest energy and, thus, its (rest) mass.

On the other hand, for any *ultrarelativistic* object moving almost with a speed of *c*, the rest energy and the rest mass can be neglected; thus, $E \approx pc$.

Since the speed of light, *c*, corresponds to $\gamma = \infty$, we can deduce the following.

Fact 22. It takes infinite energy to accelerate a massive object to *c*. Massless particles move only with a speed of *c*.

5.3 Force

Fact 23. $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(\gamma m v)}{dt}$.

Fact 24. *Four-force*

$$F^{\mu} = m a^{\mu} = \frac{d p^{\mu}}{d\tau}.$$

Problem 19. Show that if all the motion is in *x*-direction, then $F^{\mu} = (\beta \gamma F, \gamma F, 0, 0)$.

In general, $F^{\mu} = (\gamma \vec{v} \cdot \vec{F} / c, \gamma \vec{F})$ where $\vec{v} \cdot \vec{F} = dE/dt$ is the *power*.

6 Optical effects

Problem 20. What is the apparent length of a rod with rest length *l* moving with velocity *v* in parallel to the rod, if you take into account the finite travel times of photons from its ends to our eyes?

Fact 25. Doppler shift of the frequency of light: $\nu' = \nu_0 \sqrt{\frac{c - v}{c + v}}$.

Problem 21. Prove the formula, considering the world-lines of two wave-crests.

Problem 22. Reprove the formula using $E = h\nu$.

At least two important relativistic optical effects have been left out of this studying material, but are still worthwhile to think about:

- Measuring the Astronomical Unit through aberration
- Compton scattering

7 Electromagnetism

The *Lorentz force* acting on a particle with charge *q* moving in an electromagnetic field is $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$. If we separate the fields into components parallel and perpendicular to $\vec{v}$, it can be shown that the electric and magnetic fields transform into each other upon Lorentz transformations:

Fact 26. $\vec{E}_{\parallel}{}'=\vec{E}_{\parallel}$, $\vec{B}_{\parallel}{}'=\vec{B}_{\parallel}$,
$$\vec{E}_{\perp}{}'=\gamma(\vec{E}_{\perp}+\vec{v}\times\vec{B}_{\perp}),$$
$$\vec{B}_{\perp}{}'=\gamma(\vec{B}_{\perp}-\vec{v}\times\vec{E}_{\perp}/c^2).$$

8 Additional problems

The following problems have been translated from an Estonian book.*

Problem 23. A rod with rest length l_0 is moving translationally with speed v in such a way that the line connecting its endpoints at an instant forms an angle φ with the direction of motion. Find its length. [Answer: $\dfrac{l_0}{\gamma\sqrt{1-\beta^2\sin^2\varphi}}$] (PK200)

Problem 24. A body is moving uniformly in a circle, an orbit takes t = 3 h. A clock inside the body sees it to take $\tau=30\,\text{min}$. Find the radius R of the orbit. [Answer: $c\sqrt{t^2-\tau^2}/(2\pi)$] (PK201)

Problem 25. The characteristic lifetime of a muon at rest is $\tau=2.2\cdot10^{-6}$ s. How long a path s can it travel since its creation, if its speed is v = 0.999c? [Answer: $\gamma v\tau=14.7$ km] (PK202)

Problem 26. A pion at rest decays into a muon and a neutrino. Find the total energy E and the kinetic energy T of the muon, if the rest masses of the pion and the muon are, respectively, m_π and m_μ ; the rest mass of the neutrino is zero. [Answers: $E=\dfrac{(m_\pi^2+m_\mu^2)c^2}{2m_\pi}$, $T=\dfrac{(m_\pi-m_\mu)^2c^2}{2m_\pi}$] (PK234)

Problem 27. A muon at rest decays into an electron and two neutrinos. The rest mass of the muon is μ , the mass of the electron is m, the mass of the neutrino is zero. Find the maximum possible energy E_{max} of the electron. [Answer: $mc^2\dfrac{1+(\mu/m)^2}{2(\mu/m)}$] (PK235)

Problem 28. At least how big must be the energy E of a pion, if its collision with a nucleon at rest produces a nucleon-antinucleon pair and the pion is absorbed? The rest masses of the nucleon and the pion are, re-

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* Paul Kard, "Elektrodünaamika ja spetsiaalse relatiivsusteooria ülesannete kogu" ("A collection of problems on electrodynamics and special relativity"), Tartu State University 1961. Here we cite it as "PK", followed by the problem number.

spectively, M and m. [Answer: $(8M^2 - m^2)c^2/(2M)$] (PK244)

Problem 29. At least how big should be the energy E of a nucleon, if its collision with a nucleon at rest produces a nucleon-antinucleon pair and the original nucleons are both intact? The rest mass of the nucleon is M. [Answer: $7Mc^2$] (PK245)

Problem 30. An atom with rest mass m, at rest, radiates a photon with frequency ν. What is the rest mass m_0 of the atom after the process? [Answer: $m\sqrt{1-2h\nu/(mc^2)}$] (PK248)

Problem 31. The difference between an excited energy level and the ground level of an atom is ΔE. What should the speed v of the excited atom be, if we want a photon, that is radiated in the direction of motion, have a frequency of $\Delta E/h$? The rest mass of the atom in its ground state is m. [Answer: $c(\gamma+3\gamma^2/4)(2+3\gamma+5\gamma^2/4)^{-1}$]
(Part of PK249)

9 Olympiad problems

See the following pages for the original texts of the problems.

- Cuba 1991 (Relativistic Square)
- Iceland 1998 (Faster than Light?)
- Taiwan 2003 (Neutrino Decay)
- China 1994 (Relativistic Particle)
- Australia 1995 (Gravitational Red-shift)
- Physics Cup 2012 (Electron-Positron annihilation)

10 Further reading

- Ta-Pei Cheng, "Relativity, gravitation, and cosmology: a basic introduction", Oxford University Press, 2005, 2006

THEORETICAL PROBLEMS

Problem 1

The figure 1.1 shows a solid, homogeneous ball radius *R*. Before falling to the floor its center of mass is at rest, but the ball is spinning with angular velocity ω_0 about a horizontal axis through its center. The lowest point of the ball is at a height *h* above the floor.

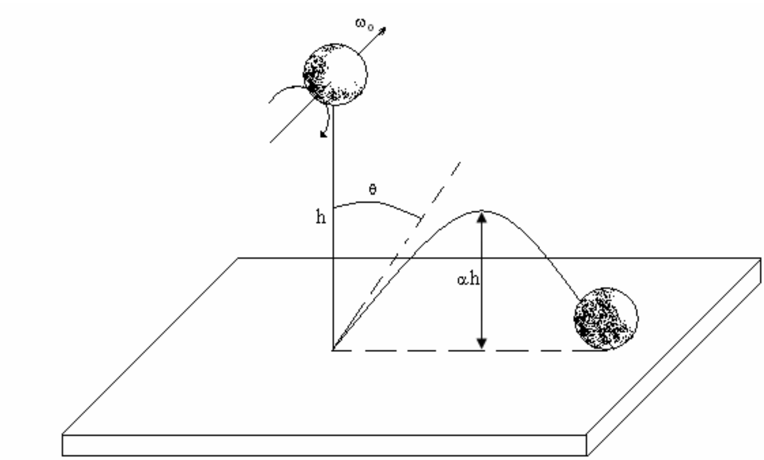


Figure 1.1

When released, the ball falls under gravity, and rebounds to a new height such that its lowest point is now *ah* above the floor. The deformation of the ball and the floor on impact may be considered negligible. Ignore the presence of the air. The impact time, although, is finite.

The mass of the ball is *m*, the acceleration due the gravity is *g*, the dynamic coefficient of friction between the ball and the floor is μ_k, and the moment of inertia of the ball about the given axis is:

$$I = \frac{2mR^2}{5}$$

You are required to consider two situations, in the first, the ball slips during the entire impact time, and in the second the slipping stops before the end of the impact time.

Situation I: slipping throughout the impact.

Find:

a) tan θ , where θ is the rebound angle indicated in the diagram;

b) the horizontal distance traveled in flight between the first and second impacts;

c) the minimum value of ω_0 for this situations.

Situation II: slipping for part of the impacts.

Find, again:

a) tan θ;

b) the horizontal distance traveled in flight between the first and second impacts.

Taking both of the above situations into account, sketch the variation of tan θ with ω_0.

Problem 2

In a square loop with a side length L, a large number of balls of negligible radius and each with a charge q are moving at a speed *u* with a constant separation *a* between them, as seen from a frame of reference that is fixed with respect to the loop. The balls are arranged on the loop like the beads on a necklace, L being much greater than *a*, as indicated in the figure 2.1. The no conducting wire forming the loop has a homogeneous charge density per unit length in the in the frame of the loop. Its total charge is equal and opposite to the total charge of the balls in that frame.

Consider the situation in which the loop moves with velocity v parallel to its side AB (fig. 2.1) through a homogeneous electric field of strength E which is perpendicular to the loop velocity and makes an angle θ with the plane of the loop.

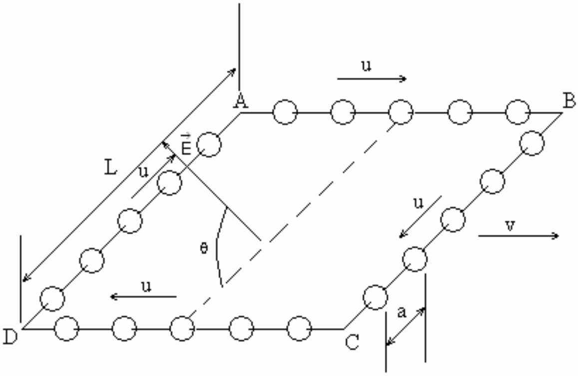


Figure 2.1

Taking into account relativistic effects, calculate the following magnitudes in the frame of reference of an observer who sees the loop moving with velocity v:

a) The spacing between the balls on each of the side of the loop, a_{AB} , a_{BC} , a_{CD} , y a_{DA}.

b) The value of the net charge of the loop plus balls on each of the side of the loop: Q_{AB}, Q_{BC}, Q_{CD} y, Q_{DA}

c) The modulus M of the electrically produced torque tending to rotate the system of the loop and the balls.

d) The energy W due to the interaction of the system, consisting of the loop and the balls with the electric field.

All the answers should be given in terms of quantities specified in the problem.

Note. The electric charge of an isolated object is independent of the frame of reference in which the measurements takes place. Any electromagnetic radiation effects should be ignored.

Some formulae of special relativity

Consider a reference frame S' moving with velocity V with reference to another reference frame S. The axes of the frames are parallel, and their origins coincide a $t = 0$. V is directed along the positive direction of the x axis.

Relativistic sum of velocities

If a particle is moving with velocity u' in the x' direction , as measured in S', the velocity of the particle measured in S is given by:

$$u = \frac{u' + V}{1 + \frac{u'V}{c^2}}$$

Relativistic Contraction

If an object at rest in frame S has length L_0 in the x-direction, an observer in frame S' (moving at velocity V in the x-direction} will measure its length to be:

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$

3 Faster than light?[7]

3.1 Problem text

In this problem we analyze and interpret measurements made in 1994 on radio wave emission from a compound source within our galaxy.

The receiver was tuned to a broad band of radio waves of wavelengths of several centimeters. Figure 3.1 shows a series of images recorded at different times. The contours indicate constant radiation strength in much the same way as altitude contours on a geographical map. In the figure the two maxima are interpreted as showing two objects moving away from a common center shown by crosses in the images. (The center, which is assumed to be fixed in space, is also a strong radiation emitter but mainly at other wavelengths). The measurements conducted on the various dates were made at the same time of day.

The scale of the figure is given by a line segment showing one arc second (as). (1 as = 1/3600 of a degree). The distance to the celestial body at the center of the figure, indicated by crosses, is estimated to be $R = 12.5$ kpc. A kiloparsec (kpc) equals $3.09 \cdot 10^{19}$ m. The speed of light is c = $3.00 \cdot 10^{8}$ m/s. Error calculations are not required in the solution.

a) (2 points) We denote the angular positions of the two ejected radio emitters, relative to the common center, by $\theta_1(t)$ and $\theta_2(t)$, where the subscripts 1 and 2 refer to the left and right hand ones, respectively, and t is the time of observation. The angular speeds, as seen from the Earth, are ω_1 and ω_2. The corresponding apparent transverse linear speeds of the two sources are denoted by $v'_{1,\perp}$ and $v'_{2,\perp}$.

Using Figure 3.1, make a graph to find the numerical values of ω_1 and ω_2 in milli-arc-seconds per day (mas/d). Also determine the numerical values of $v'_{1,\perp}$ and $v'_{2,\perp}$, and write all answers on the answer sheet. (You may be puzzled by some of the results).

b) (3 points) In order to resolve the puzzle arising in part (a), consider a light-source moving with velocity $\vec{v}$ at an angle ϕ ($0 \leq \phi \leq \pi$) to the direction towards a distant observer O (Figure 3.2). The speed may be written as $v = \beta c$, where c is the speed of light. The distance to the source, as measured by the observer, is R. The angular speed of the source, as seen from the observer, is ω, and the apparent linear speed perpendicular to the line of sight is $v'_{\perp}$.

Find ω and $v'_{\perp}$ in terms of β, R and ϕ and write your answer on the answer sheet.

c) (1 point) We assume that the two ejected objects, described in the introduction and in part (a), are moving in opposite directions with equal speeds $v = \beta c$. Then the results of part (b) make it possible to calculate β and ϕ from the angular speeds ω_1 and ω_2 and the distance R. Here ϕ is the angle defined in part (b), for the left hand object, corresponding to subscript 1 in part (a).

Derive formulas for β and ϕ in terms of known quantities and determine their numerical values from the data in part (a). Write your answers in the designated fields on the answer sheet.

d) (2 points) In the one-body situation of part (b), find the condition for the apparent perpendicular speed $v'_{\perp}$ to be larger than the speed of light c.

[7] Authors: Einar Gudmundsson, Knútur Árnason and Thorsteinn Vilhjálmsson

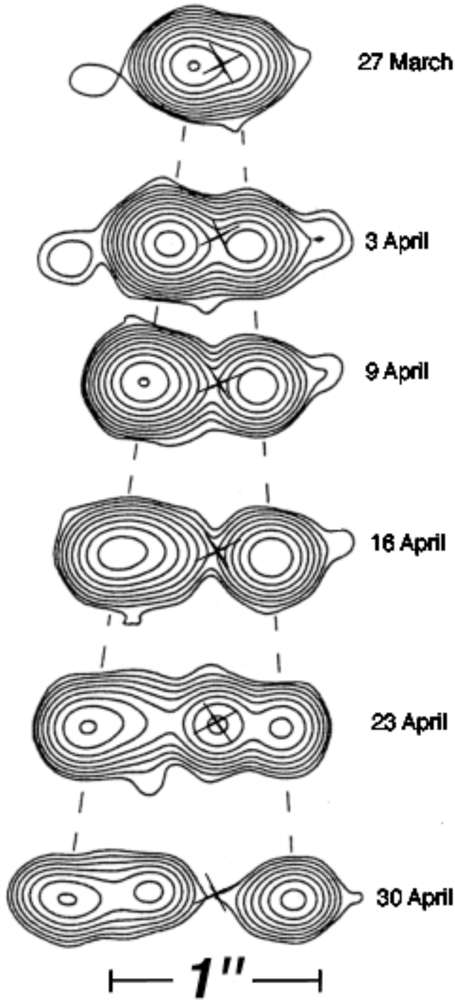


Figure 3.1: *Radio emission from a source in our galaxy.*

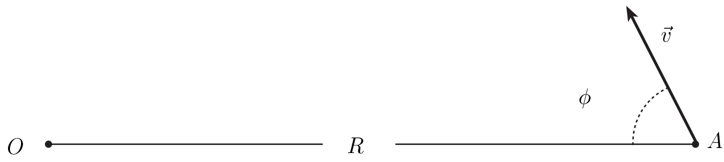


Figure 3.2: *The observer is at O and the original position of the light source is at A. The velocity vector is $\vec{v}$.*

Write the condition in the form $\beta > f(\phi)$ and provide an analytic expression for the function f on the answer sheet.

Draw on the graph answer sheet the physically relevant region of the (β, ϕ)-plane. Show by shading in which part of this region the condition $v'_\perp > c$ holds.

e) (1 point) Still in the one-body situation of part (b), find an expression for the maximum value $(v'_\perp)_{max}$ of the apparent perpendicular speed $v'_\perp$ for a given β and write it in the designated field on the answer sheet. Note that this speed increases without limit when $\beta \to 1$.

f) (1 point) The estimate for R given in the introduction is not very reliable. Scientists have therefore started speculating on a better and more direct method for determining R. One idea for this goes as follows. Assume that we can identify and measure the Doppler shifted wavelengths λ_1 and λ_2 of radiation from the two ejected objects, corresponding to the same known original wavelength λ_0 in the rest frames of the objects.

Starting from the equations for the relativistic Doppler shift,
$\lambda = \lambda_0(1 - \beta\cos\phi)(1-\beta^2)^{-1/2}$, and assuming, as before, that both objects have the same speed, v, show that the unknown $\beta = v/c$ can be expressed in terms of λ_0, λ_1, and λ_2 as

$$\beta = \sqrt{1 - \frac{\alpha\ \lambda_0^2}{(\lambda_1 + \lambda_2)^2}}\ . \tag{3.1}$$

Write the numerical value of the coefficient α in the designated field on the answer sheet.

You may note that this means that the suggested wavelength measurements will in practice provide a new estimate of the distance.

3.2 Solution

a) On Figure 3.1 we mark the centers of the sources as neatly as we can. Let $\theta_1(t)$ be the angular distance of the left center from the cross as a function of time and $\theta_2(t)$ the angular distance of the right center. We measure these quantities on the figure at the given times by a ruler and convert to arcseconds according to the given scale. This results in the following numerical data:

Theoretical Question 3

Part A

Neutrino Mass and Neutron Decay

A free neutron of mass m_n decays at rest in the laboratory frame of reference into three non-interacting particles: a proton, an electron, and an anti-neutrino. The rest mass of the proton is m_p, while the rest mass of the anti-neutrino m_ν is assumed to be nonzero and much smaller than the rest mass of the electron m_e. Denote the speed of light in vacuum by c. The measured values of mass are as follows:

$$m_n = 939.56563\ \text{MeV}/c^2,\ m_p = 938.27231\ \text{MeV}/c^2,\ m_e = 0.5109907\ \text{MeV}/c^2$$

In the following, all energies and velocities are referred to the laboratory frame. Let E be the total energy of the electron coming out of the decay.

(a) Find the maximum possible value $E_{\max}$ of E and the speed v_m of the anti-neutrino when $E = E_{\max}$. Both answers must be expressed in terms of the rest masses of the particles and the speed of light. Given that $m_\nu < 7.3\ \text{eV}/c^2$, compute $E_{\max}$ and the ratio v_m/c to 3 significant digits. [4.0 points]

Theoretical Problem 1

RELATIVISTIC PARTICLE

In the theory of special relativity the relation between energy E and momentum P or a free particle with rest mass m_0 is

$$E=\sqrt{p^2c^2+m_0^2c^4}=mc^2$$

When such a particle is subject to a conservative force, the total energy of the particle, which is the sum of $\sqrt{p^2c^2+m_0^2c^4}$ and the potential energy, is conserved. If the energy of the particle is very high, the rest energy of the particle can be ignored (such a particle is called an ultra relativistic particle).

1) consider the one dimensional motion of a very high energy particle (in which rest energy can be neglected) subject to an attractive central force of constant magnitude f. Suppose the particle is located at the centre of force with initial momentum p_0 at time t=0. Describe the motion of the particle by separately plotting, for at least one period of the motion: x against time t, and momentum p against space coordinate x. Specify the coordinates of the "turning points" in terms of given parameters p_0 and f. Indicate, with arrows, the direction of the progress of the mothon in the (p, x) diagram. There may be short intervals of time during which the particle is not ultrarelativistic. However, these should be neglected.

   Use Answer Sheet 1.

2) A meson is a particle made up of two quarks. The rest mass M of the meson is equal to the total energy of the two-quark system divided by c^2.

   Consider a one--dimensional model for a meson at rest, in which the two quarks are assumed to move along the x-axis and attract each other with a force of constant magnitude f It is assumed they can pass through each other freely. For analysis of the high energy motion of the quarks the rest mass of the quarks can be neglected. At time t=0 the two quarks are both at x=0. Show separately the motion of the two quarks graphically by a (x, t) diagram and a (p, x) diagram, specify the coordinates of the "turning points" in terms of M and f, indicate the direction of the process in your (p, x) diagram, and determine the maximum distance between the two quarks.

   Use Answer Sheet 2.

3) The reference frame used in part 2 will be referred to as frame S, the Lab frame, referred to as S, moves in the negative x-direction with a constant velocity v=0.6c. the coordinates in the two reference frames are so chosen that the point

x=0 in S coincides with the point $x'=0$ in S'' at time $t=t'=0$. Plot the motion of the two quarks graphically in a (x', t') diagram. Specify the coordinates of the turning points in terms of M, f and c, and determine the maximum distance between the two quarks observed in Lab frame S'.

Use Answer Sheet 3.

The coordinates of particle observed in reference frames S and S'' are related by the Lorentz transformation

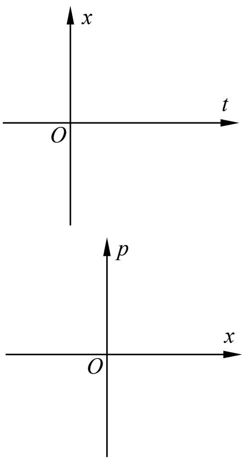
$$\begin{cases} x' = \gamma(x + \beta ct) \\ t' = \gamma(t + \beta \dfrac{x}{c}) \end{cases}$$

where $\beta = v/c$, $\gamma = 1/\sqrt{1-\beta^2}$ and v is the velocity of frame S moving relative to the frame S''.

4) For a meson with rest energy Mc^2=140 MeV and velocity 0.60c relative to the Lab frame S'', determine its energy E' in the Lab Frame S''.

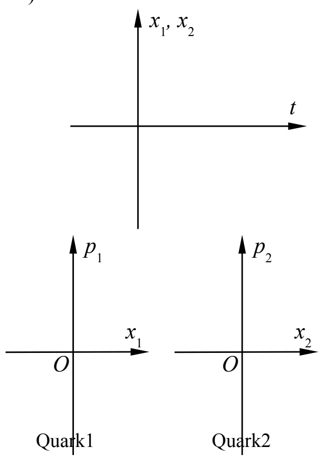
ANSWER SHEET 1

1)



ANSWER SHEET 2

2)



The maximum distance between the two quarks is d=

Theoretical Question 1

Gravitational Red Shift and the Measurement of Stellar Mass

(a) (3 marks)
A photon of frequency f possesses an effective inertial mass m determined by its energy. Assume that it has a gravitational mass equal to this inertial mass. Accordingly, a photon emitted at the surface of a star will lose energy when it escapes from the star's gravitational field. Show that the frequency shift Δf of the photon when it escapes from the surface of the star to infinity is given by

$$\frac{\Delta f}{f} \simeq -\frac{GM}{Rc^2}$$

for $\Delta f \ll f$ where:

- $G =$ gravitational constant
- $R =$ radius of the star
- $c =$ velocity of light
- $M =$ mass of the star.

Thus, the red-shift of a known spectral line measured a long way from the star can be used to measure the ratio M/R. Knowledge of R will allow the mass of the star to be determined.

(b) (12 marks)
An unmanned spacecraft is launched in an experiment to measure both the mass M and radius R of a star in our galaxy. Photons are emitted from He^+ ions on the surface of the star. These photons can be monitored through resonant absorption by He^+ ions contained in a test chamber in the spacecraft. Resonant absorption accors only if the He^+ ions are given a velocity towards the star to allow exactly for the red shifts.

As the spacecraft approaches the star radially, the velocity relative to the star ($v = \beta c$) of the He^+ ions in the test chamber at absorption resonance is measured as a function of the distance d from the (nearest) surface of the star. The experimental data are displayed in the accompanying table.

Fully utilize the data to determine graphically the mass M and radius R of the star. There is no need to estimate the uncertainties in your answer.

Data for Resonance Condition

Velocity parameter	$\beta = v/c$ $(\times 10^{-5})$	3.352	3.279	3.195	3.077	2.955
Distance from surface of star	d $(\times 10^8 \text{m})$	38.90	19.98	13.32	8.99	6.67

(c) (5 marks)
In order to determine R and M in such an experiment, it is usual to consider the frequency correction due to the recoil of the emitting atom. [Thermal motion causes emission lines to be broadened without displacing emission maxima, and we may therefore assume that all thermal effects have been taken into account.]

(i) (4 marks)
Assume that the atom decays at rest, producing a photon and a recoiling atom. Obtain the relativistic expression for the energy hf of a photon emitted in terms of ΔE (the difference in rest energy between the two atomic levels) and the initial rest mass m_0 of the atom.

(ii) (1 mark)
Hence make a numerical estimate of the relativistic frequency shift $\left(\frac{\Delta f}{f}\right)_{\text{recoil}}$ for the case of He^+ ions.

Your answer should turn out to be much smaller than the gravitational red shift obtained in part (b).

Data:

Velocity of light	c	$=$	$3.0 \times 10^8 \mathrm{ms}^{-1}$
Rest energy of He	m_0c^2	$=$	$4 \times 938(\mathrm{MeV})$
Bohr energy	E_n	$=$	$-\dfrac{13.6Z^2}{n^2}(\mathrm{eV})$
Gravitational constant	G	$=$	$6.7 \times 10^{-11}\mathrm{Nm}^2\mathrm{kg}^{-2}$

Problem No 9

Electron, initially at rest, is accelerated with a voltage $U = km_0c^2/e$, where m_0 is the electron's rest mass, e – the elementary charge, c – the speed of light, and k – a dimensionless number. The electron hits a motionless positron and annihilates creating two photons. The direction of one emitted photon defines the direction of the other one. Find the smallest possible value $\alpha_{\min}$ of the angle α between the directions of the two emitted photons (express it in terms of k and provide a numrical value for $k = 1$).