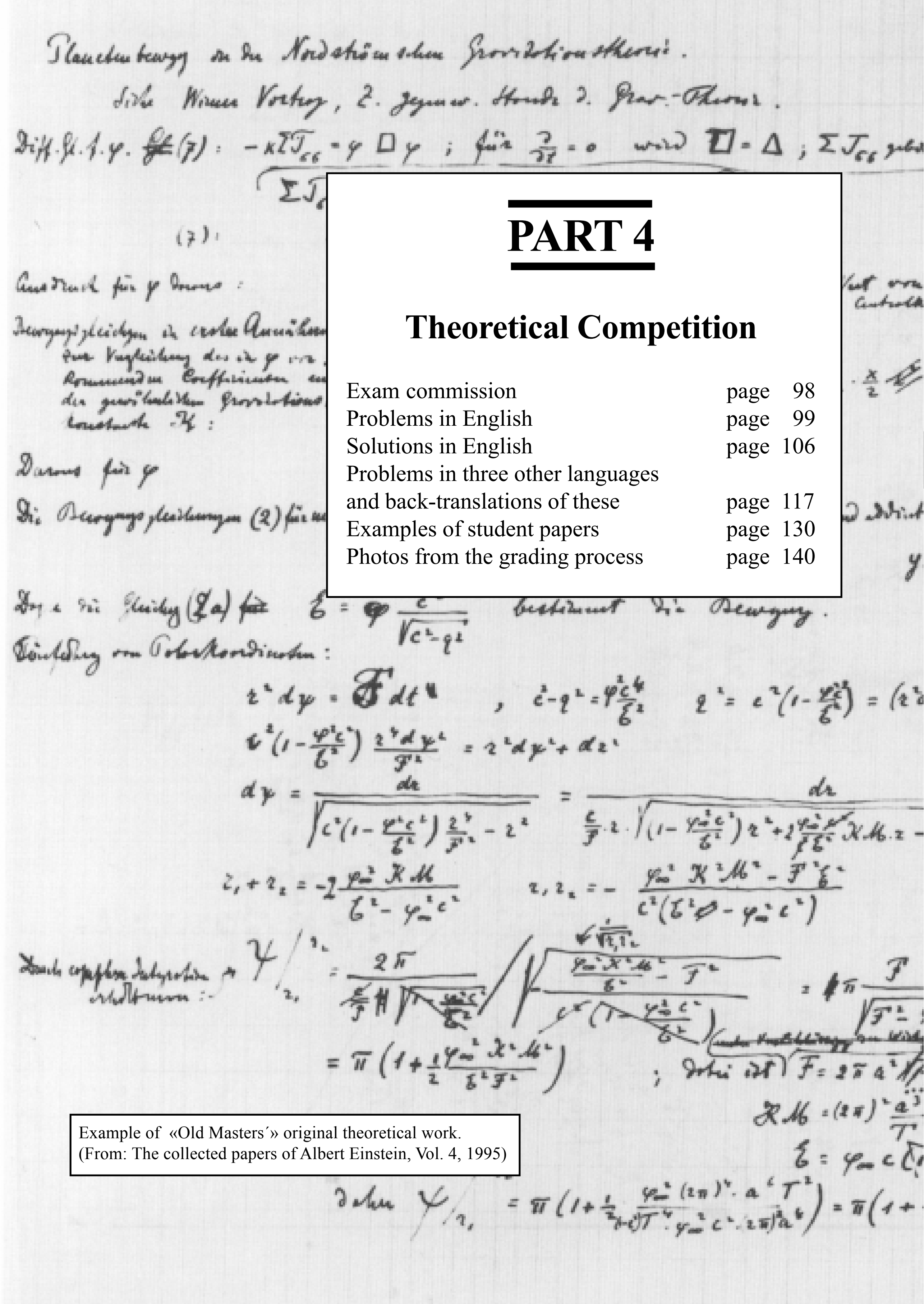

PART 4

Theoretical Competition

Example of «Old Masters´» original theoretical work.
(From: The collected papers of Albert Einstein, Vol. 4, 1995)

Per Chr. Hemmer
Chief examiner

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27th INTERNATIONAL PHYSICS OLYMPIAD
OSLO, NORWAY

THEORETICAL COMPETITION
JULY 2 1996

Time available: 5 hours

READ THIS FIRST :

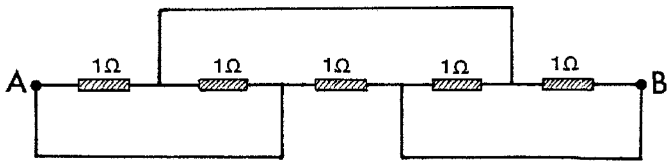
1. Use only the pen provided
2. Use only the marked side of the paper
3. Each problem should be answered on separate sheets
4. In your answers please use primarily equations and numbers, and as little *text* as possible
5. Write at the top of *every* sheet in your report:
   - Your candidate number (IPhO identification number)
   - The problem number and section identification, e.g. 2/a
   - Number each sheet consecutively
6. Write on the front page the total number of sheets in your report

This set of problems consists of 7 pages.

PROBLEM 1

(The five parts of this problem are unrelated)

a) Five 1Ω resistances are connected as shown in the figure. The resistance in the conducting wires (fully drawn lines) is negligible.



100 Determine the resulting resistance R between A and B. *(1 point)*

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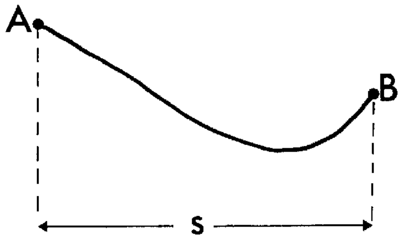
b)



A skier starts from rest at point A and slides down the hill, without turning or braking. The friction coefficient is μ. When he stops at point B, his horizontal displacement is s. What is the height difference h between points A and B? (The velocity of the skier is small so that the additional pressure on the snow due to the curvature can be neglected. Neglect also the friction of air and the dependence of μ on the velocity of the skier.) *(1.5 points)*

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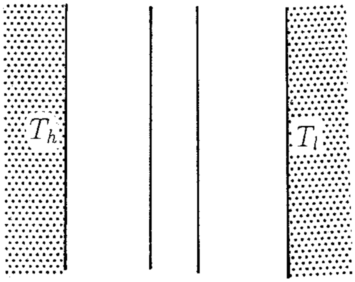
c) A thermally insulated piece of metal is heated under atmospheric pressure by an electric current so that it receives electric energy at a constant power P. This leads to an increase of the absolute temperature T of the metal with time t as follows:

$$T(t) = T_0\left[1 + a(t - t_0)\right]^{1/4}.$$

Here a, t_0 and T_0 are constants. Determine the heat capacity $C_p(T)$ of the metal (temperature dependent in the temperature range of the experiment). *(2 points)*

d) A black plane surface at a constant high temperature T_h is parallel to another black plane surface at a constant lower temperature T_l. Between the plates is vacuum.

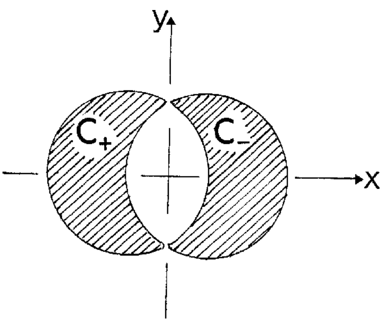
In order to reduce the heat flow due to radiation, a heat shield consisting of two thin black plates, thermally isolated from each other, is placed between the warm and the cold surfaces and parallel to these. After some time stationary conditions are obtained.



By what factor ξ is the stationary heat flow reduced due to the presence of the heat shield? Neglect end effects due to the finite size of the surfaces. *(1.5 points)*

---

e) Two straight and very long nonmagnetic conductors C_+ and C_-, insulated from each other, carry a current I in the positive and the negative z direction, respectively. The cross sections of the conductors (hatched in the figure) are limited by circles of diameter D in the x-y plane, with a distance $D/2$ between the centres. Thereby the resulting cross sections each have an area $(\frac{1}{12}\pi + \frac{1}{8}\sqrt{3})D^2$. The current in each conductor is uniformly distributed over the cross section.

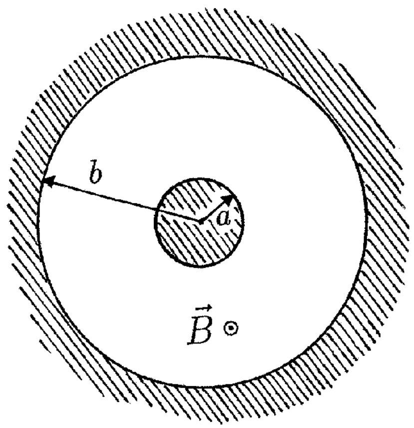


Determine the magnetic field $B(x,y)$ in the space between the conductors. *(4 points)*

PROBLEM 2

The space between a pair of coaxial cylindrical conductors is evacuated. The radius of the inner cylinder is a, and the inner radius of the outer cylinder is b, as shown in the figure below. The outer cylinder, called the anode, may be given a positive potential V relative to the inner cylinder. A static homogeneous magnetic field $\vec{B}$ parallel to the cylinder axis, directed out of the plane of the figure, is also present. Induced charges in the conductors are neglected.

We study the dynamics of electrons with rest mass m and charge $-e$. The electrons are released at the surface of the inner cylinder.



a) First the potential V is turned on, but $\vec{B} = 0$. An electron is set free with negligible velocity at the surface of the inner cylinder. Determine its speed v when it hits the anode. Give the answer both when a non-relativistic treatment is sufficient, and when it is not. *(1 point)*

For the remaining parts of this problem a non-relativistic treatment suffices.

b) Now $V = 0$, but the homogeneous magnetic field $\vec{B}$ is present. An electron starts out with an initial velocity $\vec{v}_0$ in the radial direction. For magnetic fields larger than a critical value B_c, the electron will not reach the anode. Make a sketch of the trajectory of the electron when B is slightly more than B_c. Determine B_c. *(2 points)*

From now on *both* the potential V *and* the homogeneous magnetic field $\vec{B}$ are present.

c) The magnetic field will give the electron a non-zero angular momentum L with respect to the cylinder axis. Write down an equation for the rate of change *dL/dt* of the angular momentum. Show that this equation implies that

$$L - keBr^2$$

is constant during the motion, where *k* is a definite pure number. Here *r* is the distance from the cylinder axis. Determine the value of *k*. *(3 points)*

d) Consider an electron, released from the inner cylinder with negligible velocity, that does not reach the anode, but has a maximal distance from the cylinder axis equal to r_m. Determine the speed v at the point where the radial distance is maximal, in terms of r_m. *(1 point)*

e) We are interested in using the magnetic field to regulate the electron current to the anode. For *B* larger than a critical magnetic field B_c, an electron, released with negligible velocity, will not reach the anode. Determine B_c.
(1 point)

f) If the electrons are set free by heating the inner cylinder an electron will in general have an initial nonzero velocity at the surface of the inner cylinder. The component of the initial velocity parallel to $\vec{B}$ is v_B, the components orthogonal to $\vec{B}$ are v_r (in the radial direction) and v_φ (in the azimuthal direction, i.e. orthogonal to the radial direction).

Determine for this situation the critical magnetic field B_c for reaching the anode. *(2 points)*

PROBLEM 3

In this problem we consider some gross features of the magnitude of mid-ocean tides on earth. We simplify the problem by making the following assumptions:

(i) The earth and the moon are considered to be an isolated system,
(ii) the distance between the moon and the earth is assumed to be constant,
(iii) the earth is assumed to be completely covered by an ocean,
(iv) the dynamic effects of the rotation of the earth around its axis are neglected, and
(v) the gravitational attraction of the earth can be determined as if all mass were concentrated at the centre of the earth.

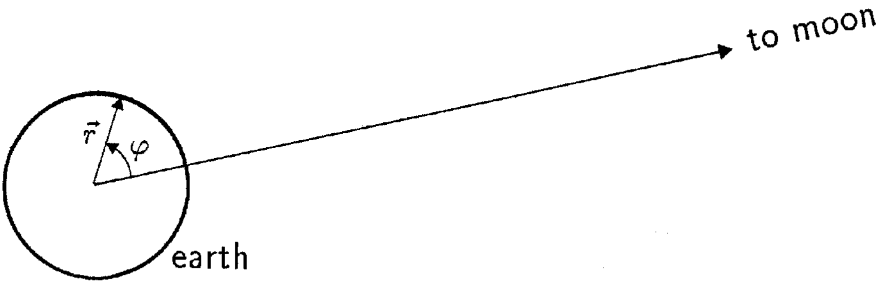
The following data are given:

Mass of the earth: $M = 5.98 \cdot 10^{24}$ kg
Mass of the moon: $M_m = 7.3 \cdot 10^{22}$ kg
Radius of the earth: $R = 6.37 \cdot 10^{6}$ m
Distance between centre of the earth and centre of the moon:
$L = 3.84 \cdot 10^{8}$ m
The gravitational constant: $G = 6.67 \cdot 10^{-11}$ m^3 kg^{-1} s^{-2}.

a) The moon and the earth rotate with angular velocity ω about their common centre of mass, *C*. How far is *C* from the centre of the earth? (Denote this distance by *l*.)

Determine the numerical value of ω. *(2 points)*

We now use a frame of reference that is co-rotating with the moon and the center of the earth around *C*. In this frame of reference the shape of the liquid surface of the earth is static.



In the plane P through C and orthogonal to the axis of rotation the position of a point mass on the liquid surface of the earth can be described by polar coordinates r, φ as shown in the figure. Here r is the distance from the centre of the earth.

We will study the shape

$$r(\varphi) = R + h(\varphi)$$

of the liquid surface of the earth in the plane P.

b) Consider a mass point (mass m) on the liquid surface of the earth (in the plane P). In our frame of reference it is acted upon by a centrifugal force and by gravitational forces from the moon and the earth. Write down an expression for the potential energy corresponding to these three forces.

Note: Any force $F(r)$, radially directed with respect to some origin, is the negative derivative of a spherically symmetric potential energy $V(r)$:
$F(r) = -V'(r)$. *(3 points)*

c) Find, in terms of the given quantities M, M_m , etc, the approximate form $h(\varphi)$ of the tidal bulge. What is the difference <u>in meters</u> between high tide and low tide in this model?

You may use the approximate expression

$$\frac{1}{\sqrt{1 + a^2 - 2a\cos\theta}} \approx 1 + a\cos\theta + \tfrac{1}{2}a^2(3\cos^2\theta - 1),$$

valid for a much less than unity.

In this analysis make simplifying approximations whenever they are reasonable. *(5 points)*

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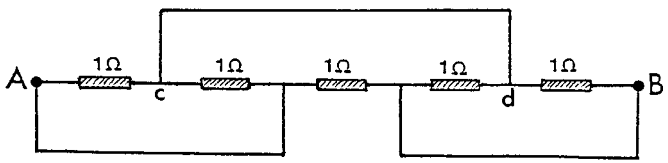
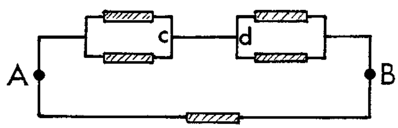
Solution Problem 1

a) The system of resistances can be redrawn as shown in the figure:





The equivalent drawing of the circuit shows that the resistance between point c and point A is 0.5Ω, and the same between point d and point B. The resistance between points A and B thus consists of two connections in parallel: the direct 1Ω connection and a connection consisting of two 0.5Ω resistances in series, in other words two parallel 1Ω connections. This yields

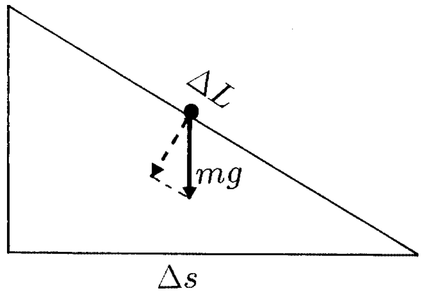
$$R = \underline{\underline{0.5\ \Omega}}\ .$$

b) For a sufficiently short horizontal displacement Δs the path can be considered straight. If the corresponding length of the path element is ΔL, the friction force is given by

$$\mu\, mg\frac{\Delta s}{\Delta L}$$

and the work done by the friction force equals force times displacement:

$$\mu\ mg\frac{\Delta s}{\Delta L}\cdot\Delta L=\mu\ mg\Delta s.$$



Adding up, we find that along the whole path the total work done by friction forces i $\mu\ \ mg\, s$. By energy conservation this must equal the decrease $mg\, h$ in potential energy of the skier. Hence

$$h=\underline{\underline{\mu s}}.$$

---

c) Let the temperature increase in a small time interval dt be dT. During this time interval the metal receives an energy $P\, dt$.

The heat capacity is the ratio between the energy supplied and the temperature increase:

$$C_p=\frac{Pdt}{dT}=\frac{P}{dT/dt}.$$

The experimental results correspond to

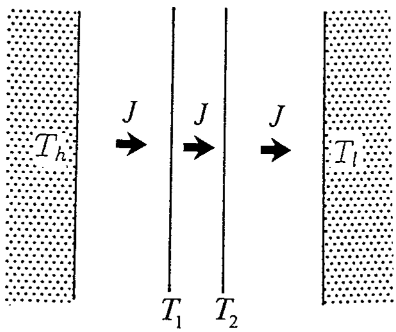
$$\frac{dT}{dt}=\frac{T_0}{4}a[1+a(t-t_0)]^{-3/4}=T_0\frac{a}{4}\left(\frac{T_0}{T}\right)^3.$$

Hence

$$C_p=\frac{P}{dT/dt}=\frac{4P}{aT_0^{\ 4}}T^3.$$

(*Comment*: At low, but not extremely low, temperatures heat capacities of metals follow such a T^3 law.)

d)



Under stationary conditions the net heat flow is the same everywhere:

$$J = \sigma(T_h^4 - T_1^4)$$
$$J = \sigma(T_1^4 - T_2^4)$$
$$J = \sigma(T_2^4 - T_l^4)$$

Adding these three equations we get

$$3J = \sigma(T_h^4 - T_l^4) = J_0,$$

where J_0 is the heat flow in the absence of the heat shield. Thus $\xi = J/J_0$ takes the value

$$\xi = \underline{\underline{1/3}}.$$

---

e) The magnetic field can be determined as the superposition of the fields of two *cylindrical* conductors, since the effects of the currents in the area of intersection cancel. Each of the cylindrical conductors must carry a larger current I', determined so that the fraction I of it is carried by the actual cross section (the moon-shaped area). The ratio between the currents I and I' equals the ratio between the cross section areas:

$$\frac{I}{I'} = \frac{(\frac{\pi}{12} + \frac{\sqrt{3}}{8})D^2}{\frac{\pi}{4}D^2} = \frac{2\pi + 3\sqrt{3}}{6\pi}.$$

Inside one cylindrical conductor carrying a current I' Ampère's law yields at a distance r from the axis an azimuthal field

$$B_\phi = \frac{\mu_0}{2\pi r}\frac{I'\pi r^2}{\frac{\pi}{4}D^2} = \frac{2\mu_0 I' r}{\pi D^2}.$$

The cartesian components of this are

$$B_x = -B_\phi \frac{y}{r} = -\frac{2\mu_0 I'y}{\pi D^2}; \qquad B_y = B_\phi \frac{x}{r} = \frac{2\mu_0 I'x}{\pi D^2}.$$

For the superposed fields, the currents are $\pm I'$ and the corresponding cylinder axes are located at $x = \mp D/4$.

The two x-components add up to zero, while the y-components yield

$$B_y = \frac{2\mu_0}{\pi D^2}[I'(x + D/4) - I'(x - D/4)] = \frac{\mu_0 I'}{\pi D} = \underline{\underline{\frac{6\mu_0 I}{(2\pi + 3\sqrt{3})D}}},$$

i.e., a *constant* field. The direction is along the positive y-axis.

Solution Problem 2

a) The potential energy gain eV is converted into kinetic energy. Thus

$$\tfrac{1}{2}mv^2 = eV \qquad \text{(non-relativistically)}$$

$$\frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 = eV \qquad \text{(relativistically).}$$

Hence

$$v = \begin{cases} \sqrt{2eV/m} & \text{(non - relativistically)} \\ c\sqrt{1 - \left(\dfrac{mc^2}{mc^2 + eV}\right)^2} & \text{(relativistically).} \end{cases} \qquad (1)$$

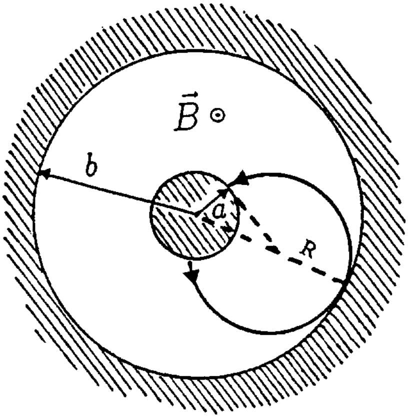
b) When $V = 0$ the electron moves in a homogeneous static magnetic field. The magnetic Lorentz force acts orthogonal to the velocity and the electron will move in a circle. The initial velocity is tangential to the circle.

The radius R of the orbit (the "cyclotron radius") is determined by equating the centripetal force and the Lorentz force:

i.e.

$$eBv_0 = \frac{mv_0^2}{R},$$

$$B = \frac{mv_0}{eR}. \tag{2}$$



From the figure we see that in the critical case the radius R of the circle satisfies

$$\sqrt{a^2 + R^2} = b - R$$

By squaring we obtain

i.e.

$$a^2 + R^2 = b^2 - 2bR + R^2,$$

$$R = (b^2 - a^2)/2b.$$

Insertion of this value for the radius into the expression (2) gives the critical field

$$B_c = \frac{mv_0}{eR} = \underline{\underline{\frac{2bmv_0}{(b^2 - a^2)e}}}.$$

c) The change in angular momentum with time is produced by a torque. Here the azimuthal component F_ϕ of the Lorentz force $\vec{F} = (-e)\vec{B}\times\vec{v}$ provides a torque $F_\phi r$. It is only the radial component $v_r = dr/dt$ of the velocity that provides an azimuthal Lorentz force. Hence

$$\underline{\underline{\frac{dL}{dt} = eBr\frac{dr}{dt}}},$$

which can be rewritten as

$$\frac{d}{dt}\left(L - \frac{eBr^2}{2}\right) = 0.$$

Hence

$$C = \underline{\underline{L - \tfrac{1}{2} eBr^2}} \qquad (3)$$

is constant during the motion. The dimensionless number k in the problem text is thus $k = \underline{1/2}$.

d) We evaluate the constant C, equation (3), at the surface of the inner cylinder and at the maximal distance r_{m}:

$$0 - \tfrac{1}{2} eBa^2 = mvr_m - \tfrac{1}{2} eBr_m^2$$

which gives

$$\underline{v = \frac{eB(r_m^2 - a^2)}{2mr_m}}. \qquad (4)$$

Alternative solution: One may first determine the electric potential $V(r)$ as function of the radial distance. In cylindrical geometry the field falls off inversely proportional to r, which requires a logarithmic potential, $V(s) = c_1 \ln r + c_2$. When the two constants are determined to yield $V(a) = 0$ and $V(b) = V$ we have

$$V(r) = V \frac{\ln(r/a)}{\ln(b/a)}.$$

The gain in potential energy, $sV(r_m)$, is converted into kinetic energy:

$$\frac{1}{2} mv^2 = eV \frac{\ln(r_m/a)}{\ln(b/a)}.$$

Thus

$$v = \sqrt{\frac{2eV}{m} \frac{\ln(r_m/a)}{\ln(b/a)}}. \qquad (5)$$

(4) and (5) seem to be different answers. This is only apparent since r_m is not an independent parameter, but determined by B and V so that the two answers are identical.

e) For the critical magnetic field the maximal distance r_m equals b, the radius of the outer cylinder, and the speed at the turning point is then

$$v = \frac{eB(b^2 - a^2)}{2mb}.$$

Since the Lorentz force does no work, the corresponding kinetic energy $\frac{1}{2}mv^2$ equals eV (question a):

$$v = \sqrt{2eV/m}$$

The last two equations are consistent when

$$\frac{eB(b^2 - a^2)}{2mb} = \sqrt{2eV/m}.$$

The critical magnetic field for current cut-off is therefore

$$\underline{\underline{B_c = \frac{2b}{b^2 - a^2}\sqrt{\frac{2mV}{e}}.}}$$

f) The Lorentz force has no component parallel to the magnetic field, and consequently the velocity component v_B is constant under the motion. The corresponding displacement parallel to the cylinder axis has no relevance for the question of reaching the anode.

Let v denote the final azimuthal speed of an electron that barely reaches the anode. Conservation of energy implies that

$$\tfrac{1}{2}m(v_B^2 + v_\phi^2 + v_r^2) + eV = \tfrac{1}{2}m(v_B^2 + v^2),$$

giving

$$v = \sqrt{v_r^2 + v_\phi^2 + 2eV/m}. \qquad (6)$$

Evaluating the constant C in (3) at both cylinder surfaces for the critical situation we have

$$mv_\phi a - \tfrac{1}{2}eB_c a^2 = mvb - \tfrac{1}{2}eB_c b^2.$$

Insertion of the value (6) for the velocity v yields the critical field

$$B_c = \frac{2m(vb - v_\phi a)}{e(b^2 - a^2)} = \frac{2mb}{e(b^2 - a^2)}\left[\sqrt{v_r^2 + v_\phi^2 + 2eV/m} - v_\phi a/b\right].$$

Solution Problem 3

a) With the centre of the earth as origin, let the centre of mass C be located at $\vec{l}$. The distance l is determined by

$$M\,l = M_m\,(L - l),$$

which gives

$$l = \frac{M_m}{M + M_m} L = \underline{4.63 \cdot 10^6\,\mathrm{m},} \tag{1}$$

less than R, and thus inside the earth.

The centrifugal force must balance the gravitational attraction between the moon and the earth: 113

$$M\omega^2 l = G\frac{MM_m}{L^2},$$

which gives

$$\omega = \sqrt{\frac{GM_m}{L^2 l}} = \underline{\underline{\sqrt{\frac{G(M + M_m)}{L^3}}}} = \underline{\underline{2.67 \cdot 10^{-6}\,\mathrm{s}^{-1}}}. \tag{2}$$

(This corresponds to a period $2\pi/\omega = 27.2$ days.) We have used (1) to eliminate l.

b) The potential energy of the mass point m consists of three contributions:

(1) Potential energy because of rotation (in the rotating frame of reference, see the problem text),

$$-\frac{1}{2} m\omega^2 r_1^2,$$

where $\vec{r}_1$ is the distance from C. This corresponds to the centrifugal force $m\omega^2 r_1$, directed outwards from C.

(2) Gravitational attraction to the earth,

$$-G\frac{mM}{r}.$$

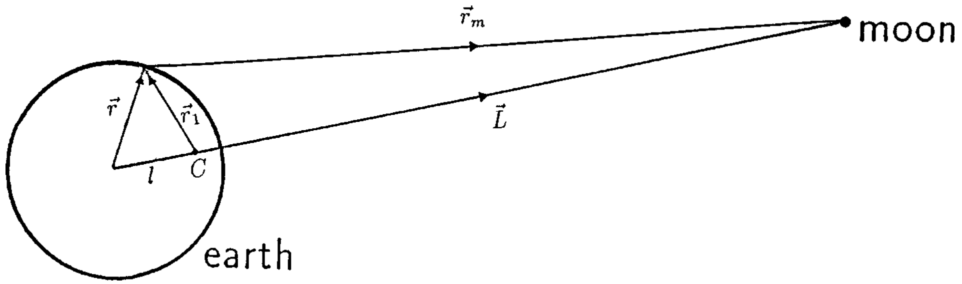
(3) Gravitational attraction to the moon,

$$-G\frac{mM_m}{|\vec{r}_m|},$$

where $\vec{r}_m$ is the distance from the moon.

Describing the position of m by polar coordinates r; ϕ in the plane orthogonal to the axis of rotation (see figure), we have

$$\vec{r}_1^{\,2} = (\vec{r}-\vec{l}\,)^2 = r^2 - 2rl\cos\phi + l^2.$$



Adding the three potential energy contributions, we obtain

$$V(\vec{r}) = -\frac{1}{2}m\omega^2(r^2 - 2rl\cos\phi + l^2) - G\frac{mM}{r} - G\frac{mM_m}{|\vec{r}_m|}. \tag{3}$$

Here l is given by (1) and

$$|\vec{r}_m| = \sqrt{(\vec{L}-\vec{r})^2} = \sqrt{L^2 - 2\vec{L}\vec{r} + r^2} = L\sqrt{1+(r/L)^2 - 2(r/L)\cos\phi}\,.$$

c) Since the ratio $r/L = a$ is very small, we may use the expansion

$$\frac{1}{\sqrt{1+a^2-2a\cos\phi}} = 1 + a\cos\phi + a^2\,\tfrac{1}{2}(3\cos^2\phi - 1).$$

Insertion into the expression (3) for the potential energy gives

$$V(r,\phi)/m = -\tfrac{1}{2}\omega^2 r^2 - \frac{GM}{r} - \frac{GM_m r^2}{2L^3}(3\cos^2\phi - 1), \tag{4}$$

apart from a constant. We have used that

$$m\omega^2 rl\cos\phi - GmM_m\frac{r}{L^2}\cos\phi = 0,$$

when the value of ω_2, equation (2), is inserted.

The form of the liquid surface is such that a mass point has the same energy V *everywhere on the surface*. (This is equivalent to requiring no net force tangential to the surface.) Putting

$$r = R + h,$$

where the tide h is much smaller than R, we have approximately

$$\frac{1}{r} = \frac{1}{R+h} = \frac{1}{R} \cdot \frac{1}{1+(h/R)} \cong \frac{1}{R}(1-\frac{h}{R}) = \frac{1}{R} - \frac{h}{R^2},$$

as well as

$$r^2 = R^2 + 2Rh + h^2 \cong R^2 + 2Rh.$$

Inserting this, and the value (2) of ω into (4), we have

$$V(r,\phi)/m = -\frac{G(M+M_m)R}{L^3}h + \frac{GM}{R^2}h - \frac{GM_m r^2}{2L^3}(3\cos^2\phi - 1), \quad (5)$$

again apart from a constant.

The magnitude of the first term on the right-hand side of (5) is a factor

$$\frac{(M+M_m)}{M}\left(\frac{R}{L}\right)^3 \cong 10^{-5}$$

smaller than the second term, thus negligible. If the remaining two terms in equation (5) compensate each other, i.e.,

$$h = \frac{M_m r^2 R^2}{2ML^3}(3\cos^2\phi - 1),$$

then the mass point m has the same energy everywhere on the surface. Here r^2 can safely be approximated by R^2, giving the tidal bulge

$$\underline{\underline{h = \frac{M_m R^4}{2ML^3}(3\cos^2\phi - 1).}}$$

The largest value $h_{\max} = M_m R^4 / ML^3$ occurs for $\phi = 0$ or π, in the direction of the moon or in the opposite direction, while the smallest value

$$h_{\min} = -M_m R^4 / 2ML^3$$

corresponds to $\phi = \pi/2$ or $3\pi/2$.

The difference between high tide and low tide is therefore

$$h_{\max} - h_{\min} = \frac{3M_m R^4}{2ML^3} = \underline{\underline{0.54\text{m.}}}$$

(The values for high and low tide are determined up to an additive constant, but the difference is of course independent of this.)

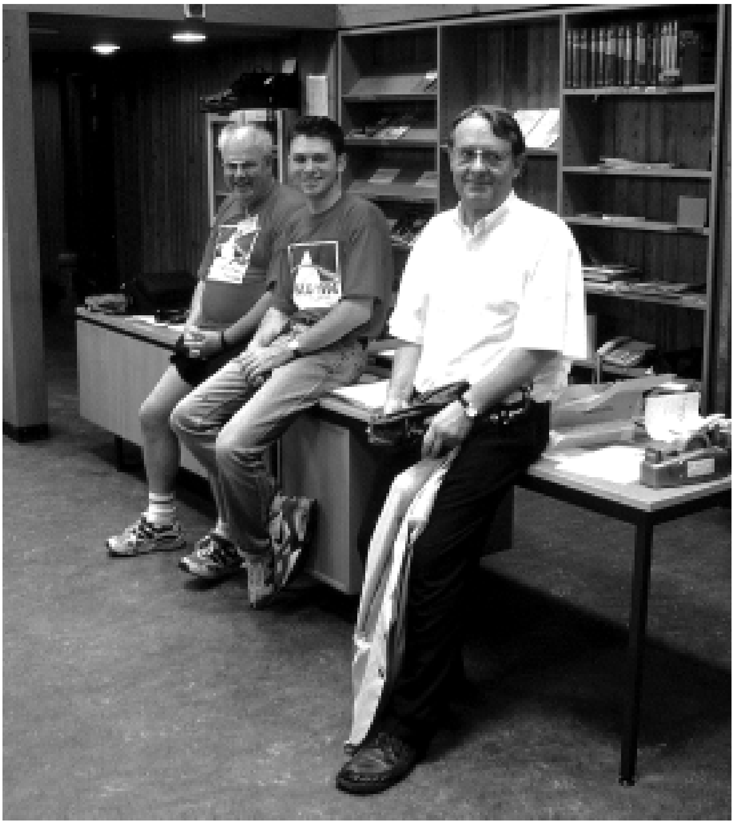


Photo: Arnt Inge Vistnes

Here we see the Exam Officer, Michael Peachey (in the middle), with his helper Rod Jory (at the left), both from Australia, as well as the Chief examiner, Per Chr. Hemmer. The picture was taken in a silent moment during the theory examination. Michael and Rod had a lot *of experience from the 1995 IPhO in Canberra, so their help was very effective and highly appreciated!*