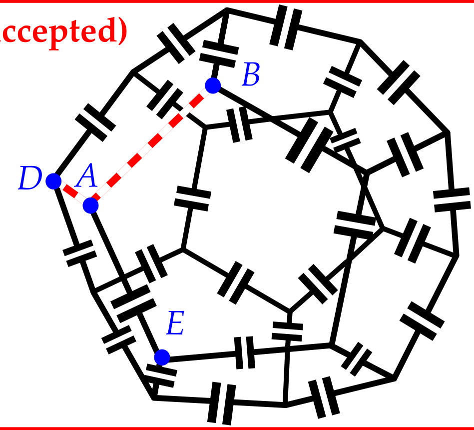
Problem 6 (deadline has passed, solutions will no longer be accepted)

Edges of a dodecahedron are made of wire of negligible electrical resistance; each wire includes a capacitor of capacitance C, see figure. Let us mark a vertex A and its three neighbours B, D and E. The wire segments AB and AD are removed. What is the capacitance between the vertices B and E?



Hints after 1st week: This problem has also **a short solution which does not use brute force**. How to be sure that you have found the short solution: using the method of that short solution, it is possible to solve also a modified problem, where the dodecahedron is replaced by an infinite honeycomb lattice (two wires are cut off in the same way as for this dodecahedron).

Hints after the 2nd week: As a first step, find the resistance between B and E when the segments DA and AB (together with the respective capacitors) are still present. This can be found in the same way as the resistance r between two neighbouring nodes P and Q of an infinite square lattice of resistors R: consider the superposition of two current distributions. (i) current I is driven into the node P and driven out symmetrically at infinity; (ii) current is driven into the lattice at infinity, and out from the node Q. Due to symmetry, in both cases there is a current $I/4$ in the wire directly connecting P and Q. For the superposition, current I enters the circuit at P, and leaves from Q, and there is a current $I/4 + I/4 = I/2$ in the wire connecting P and Q, i.e. $r = R \cdot (I/2) / I = R/2$.

Hints after the 3rd week: In the case of a dodecahedron, current I, if driven into a node P, cannot be driven out at infinity because the circuit is finite. However, there is still a way to drive it out from the nodes of the dodecahedron so that (i) the current distribution remains symmetric; (ii) for a superposition of two such current distributions of opposite polarities (when adding a current distribution with I being driven out from a node Q), the external currents driven to and from all the nodes other than P and Q cancel out. Now, suppose you know the resistance r between the nodes B and E for a uncut dodecahedron: between B and E, the whole uncut dodecahedron is equivalent to a single resistor r. Next, notice that cutting out a resistor R is mathematically equivalent to adding a parallel resistance $-R$ (a mathematician doesn't care that there are no negative resistances).

Hints after the 4th week: Notice that any calculations for a certain configuration of resistances can be carried over to the equivalent configuration of capacitors. Indeed, the overall impedance $Z = 1/iC\omega$ of a system of capacitors can be found according to the rules for resistors, with resistances being substituted by the impedances $Z_k = 1/iC_k\omega$. As for the method to drive out the current which is driven into the node P, you need to drive out equal amount from every node other than P.

Points	Results of the 6th problem.	Correct solutions have been submitted by
2,567805035	1. Lars Dehlwes (Germany)*	(ordered according to the arrival time):
2,59374246	2. Hrishikesh Menon (India)	
2,14358881	3. Ly Nguyen (Vietnam) (short solution was submitted later, order nubmber 14a)	
1,9487171	4. Dan-Cristian Andronic (Romania)	
1,771561	5. Szabo Attila (Hungary)	
1,61051	6. Jan Ondras (Slovakia)	
1,4641	7. Ng Fei Chong (Malaysia)	
1,185921	8. Tudor Ciobanu (Romania)** (short solution was submitted later, order number 13a, i.e. between Kongas and Schmidt)	
1,21	9. Vũ Việt Linh (Vietnam) (short solution was submitted later, order number 14b)	
1,1	10. Madhivanan Elango (United Kingdom)	Second week begins here
1,1	11. Nguyen Ho Nam (Vietnam) (short solution was submitted later, order nubmber 15a)	
1,491824698	12. Jaemo Lim (Korea)***	
1,349858808	13. Kristjan Kongas (Estonia)***	
1,221402758	14. David Schmidt (Germany)***	
1,105170918	15. Efim Mazhnik (Russia)***	Third week begins here
0,72	16. Colibaba Nicoleta (Moldova)**	Fourth week begins here
0,9	17. Áron Dániel Kovács (Hungary)* ***	
1	18. Jakub Safin (Slovakia)***	
1	19. Qu Xinyi (Singapore)***	Fifth week begins here
1	20. Áron Dániel Kovács (Hungary)***	
1	21. Kurenkov Mikhail (Russia)	
1	22. Jakub Mrożek (Poland)***	
1	23. Oliver Edtmair (Austria)***	
1	24. Péter Juhász (Hungary)	
1	25. Sean Seet (Singapore)***	
1	26. Ivan Tadeu Ferreira Antunes Filho (Brazil)***	
1	27. Fruzsina Agócs (Hungary)***	
1	28. Maksim Velikanov (Russia)***	
0,81	29. Shirsha Sengupta (India)**	
0,81	30. Selimović Nudžeim (Bosnia and Herzegovina)	
0,81	31.Amer Ajanovic (Bosnia and Herzegovina)	

* Solution includes a typo at the very last line **Correct version submitted at the second attempt *** Short solutions

Best solutions

There were many good solutions, but none really stood above the others. Therefore, the best solution awards were given to those short solutions which were sent before the hints of the second week were made available. The bonus coefficients were distributed by taking into account the arrival time of the solution (by that time, all the speed bonus points were already distributed, so that the speed was NOT credited twice): $e^{0.4}$, $e^{0.3}$,$e^{0.2}$, and $e^{0.1}$, respectively. Tudor Ciobanu and Vũ Việt Linh would have been also qualified, but their score would have been lower than their original speed bonus.

1. Jaemo Lim - a very typical short solution

WoPhO Selection Round-Jaemo Lim

Problem 6. Capacitance of Dodecahedron

1.

We can change the capacitance to a resistance to calculate the total capacitance.

We should change $C \leftrightarrow \frac{1}{R}$.

For series connection

capacitance: $C=\frac{1}{\frac{1}{C_1}+\frac{1}{C_2}}$, $R=R_1+R_2$. Then, if we change $\frac{1}{C}\leftrightarrow R$, $\frac{1}{R}=\frac{1}{R_1+R_2}$, $R=R_1+R_2$.

For parallel connection

capacitance: $C=C_1+C_2$, $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. Then, if we change $\frac{1}{C}\leftrightarrow R$, $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$.

Therefore, if we change C into $\frac{1}{C}$ and consider it as a resistance and calculate the total resistance R_T, we can get the total capacitance $\frac{1}{R_T}$.

2.

Each wire includes resistance R.

AE can be ignored since one end(A) has no connection.

Then, if we connect AE and AB, the total circuit becomes a parllel connection of E-A-B and the original circuit.

Let the total resistance of the original circuit R_T.

If we connect AE, AB, the total resistance becomes $R_T'=\frac{1}{\frac{1}{R_T}+\frac{1}{2R}}$.(E-A-B:R+R=2R)

Because of symmetry, D and A become equipotential. Connecting DA does not change the total resistance. Therefore, R_T' is the resistance between E and B without removing any wires from the dodecahedron.

3.

Consider a circuit that inputs current I in point B, and outputs current I/19 in other 19 points. Because of dodecahedron's symmetry, the current are like figure 1.

$\left(\frac{\frac{1}{3}-\frac{1}{19}}{2}=\frac{8}{57}\right)$

Next, consider a circuit that outputs current I in point E, and inputs current I/19 in other 19 points, including B. Because of dodecahedron's symmetry, the current are like figure 2.

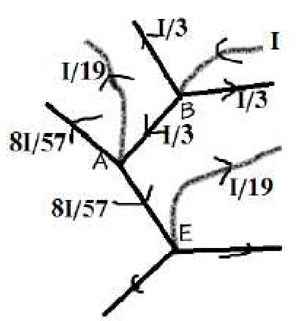


Figure 1. the "crayon lines" means the input/output current. Straight lines are wires.

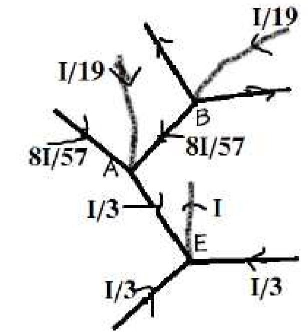


Figure 2. the "crayon lines" means the input/output current. Straight lines are wires.

If we superposition the two figures(add the currents) we get figure 3.(only considering AB and AE and the input/output currents.

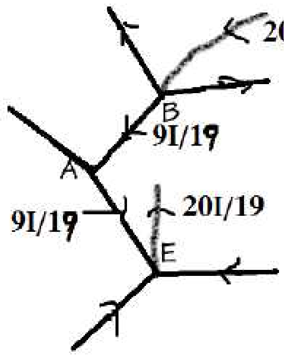


Figure 3. Superposition of Figure 1 and 2. The "crayon lines" means the input/output current. Straight lines are wires.

Input and output currents are only in point B and E. The potential difference between B and E is $\frac{9IR}{19}+\frac{9IR}{19}=\frac{18IR}{19}$.

Therefore, $\frac{20I}{19}R_T'=\Delta V_{BE}=\frac{18IR}{19}$, $R_T'=\frac{9}{10}R$.

Since $R_T'=\frac{1}{\frac{1}{R_T}+\frac{1}{2R}}$, $\frac{1}{R_T}=\frac{1}{R_T'}-\frac{1}{2R}=\frac{10}{9R}-\frac{1}{2R}=\frac{11}{18R}$, $R_T=\frac{18}{11}R$

$\frac{1}{C_T}=\frac{18}{11}\frac{1}{C}$, $C_T=\frac{11}{18}C$

$$\therefore C_T=\frac{11}{18}C$$

2. Kristjan Kongas - the same as above, but without turning to resistances or impedances

First, consider the dodecahedron before segments AB and AD were removed. Let's give vertex A (call it a Special Point) charge $\frac{19}{20}q$ and every other vertex on dodecahedron charge $-q/20$, so that the total charge of the device is 0.

From symmetry, it is clear that each of the capacitors AB, AD and AE have charge $q_1 = \frac{19}{20}q/3 = \frac{19}{60}q$. Charge conservation and again symmetry tell us that the charge q_2 of the six capacitors connecting B, D and E but not A must satisfy equation $-q/20 = -q_1 + 2q_2 \Rightarrow q_2 = (\frac{19}{60}q - \frac{1}{20}q)/2 = \frac{2}{15}q$. From the definition of capacitance, one can easily derive that voltage between A and all verteces separated by two capacitors is $V_2 = q_1/C + q_2/C = (\frac{19}{60}q + \frac{2}{15}q)/C = \frac{9}{20}q/C$.

Consider two solutions for electrical equilibrum: one with Special Point chosen to be B and with $q = Q$, other with Special Point E and $q = -Q$. Superposing these solutions gives us a new solution with every vertex other than B and E having no charge. However, the charge of B and E are respectively $\frac{19}{20}Q + \frac{1}{20}Q = Q$ and $-\frac{19}{20}Q - \frac{1}{20}Q = -Q$, while voltage between them is $2V_2 = \frac{9}{10}Q/C$. Thus, the capacitance between B and E without removing any segments from the dodecahedron is $C_{full} = Q/2V_2 = Q/(\frac{9}{10}Q/C) = \frac{10}{9}C$.

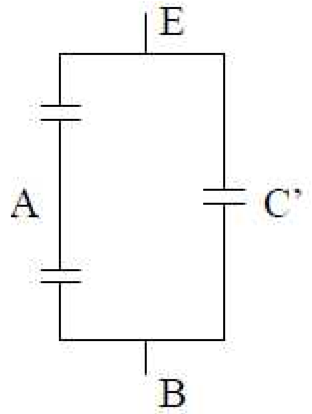


Figure 1: The capacitance between B and E on figure represents capacitance on dodecahedron's respective points, left side of it segments BA and AE, right side the dodecahedron whose segments BA and AD are removed

Our friend symmetry tells us again that voltage on capacitor AD is $0 \Rightarrow$ segment AD can be removed without changing anything else. The full dodecahedron scheme can thus be viewed as in figure. The capacitance of the left side is $C_{left} = 1/(1/C+1/C) = C/2$. Thus capacitance mentioned in the problem $C' = C_{full} - C_{left} = \frac{10}{9}C - \frac{1}{2}C = \frac{11}{18}C$. Removing segment AB gives us the scheme originally mentioned in problem and obviously it's capacitance is C'.

Answer: $\frac{11}{18}C$

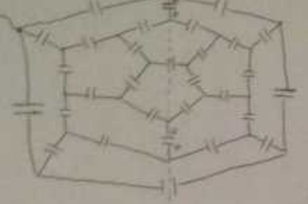
3. David Schmidt - a solution which is quite similar to the previous one

For this problem to solve, we first imagine the dodecahedron without a removed wire segment. Now let enter a charge Q_{enter} at the vertex E and a charge of $Q_{exit} = -\frac{Q_{enter}}{19}$ at every other vertex of the polyhedron. This is of course a symmetrically situation, so we conclude that the capacity of the wire segment EA contains a charge of $\frac{Q_{enter}}{3}$ and the capacity of the wire segment AB contains a charge of $\frac{\frac{Q_{enter}}{3}+Q_{exit}}{2}$. According to the Helmholtz principle (principle of superposition) we can now also take a look at the same situation and with an entering charge of $Q'_{enter} = -Q_{enter}$ at B and an exiting charge $Q'_{exit} = -\frac{Q'_{enter}}{19}$ at every vertex and add those charge distributions to the ones of the first case. In total there is at E an entering charge of $Q_{total} = Q_{enter} + Q'_{exit} = Q_{enter} - \frac{Q'_{enter}}{19} = Q_{enter} + \frac{Q_{enter}}{19} = \frac{20}{19}Q_{enter}$ and in B there is charge of $-\frac{20}{19}Q_{enter}$ entering or in other terms a charge of $\frac{20}{19}Q_{enter}$ leaving. At every other vertex there is a total of $Q_{exit} + Q'_{exit} = Q_{exit} - Q_{exit} = 0$, so there is no charge exiting or entering. And the capacities at the wire segments EA and AB both contain a charge of $\frac{Q_{enter}}{3} + \frac{\frac{Q_{enter}}{3}+Q_{exit}}{2} = \frac{Q_{enter}}{3} + \frac{\frac{Q_{enter}}{3}-\frac{Q_{enter}}{19}}{2} = Q_{enter}\cdot(\frac{1}{3} + \frac{\frac{1}{3}-\frac{1}{19}}{2}) = Q_{enter}\cdot\frac{2+1-\frac{3}{19}}{6} = Q_{enter}\cdot\frac{9}{19}$. That is why there has to be a voltage drop of $U = \frac{Q}{C} = \frac{9}{19}\cdot\frac{Q_{enter}}{C}$ over each capacity so between the vertices E and B there is a potential drop of $U_{total} = \frac{18}{19}\frac{Q_{enter}}{C}$. So we can determine the capacity of the whole dodecahedron between the vertices E and B to $C_{dodecahedron} = \frac{Q_{total}}{U_{total}} = \frac{\frac{20}{19}Q_{enter}}{\frac{18}{19}\frac{Q_{enter}}{C}} = \frac{20}{18}C$. Now, this whole capacity can be calculated from the equivalent capacity over the wire segments EA and AB, which is $\frac{1}{C_{equiv}} = \frac{1}{C} + \frac{1}{C} \Leftrightarrow C_{equiv} = \frac{C}{2}$, and the capacity $C_{solution}$ which we are looking for. Because these two capacities are parallel (and because of symmetry there is no drop of voltage over the wire segment AD) they have to fulfill the equation $C_{equiv} + C_{solution} = C_{dodecahedron} \Leftrightarrow C_{solution} = C_{dodecahedron} - C_{equiv} = (\frac{20}{18} - \frac{1}{2})C = \frac{11}{18}C$. So we determined the capacity which we were looking for.

4. Efim Mazhnik - a solution, similar to the solutions 2 and 3

... superposition.

It's more convenient to work with a whole dodecahedron, so we can notice that our scheme is a part of the parallel connection:



Thus, if C' is a capacity of a whole dodecahedron, than a capacity of our scheme should be

$C_0 = C' - \frac{C}{2}$.

Consider the situation, where $+q$ is the charge of one of the vertices and $-\frac{q}{19}$ of the other 19.

In this case, due to the symmetry, charge of the first three capacitors (from q vertice) is $\frac{q}{3}$ and charge of the next six is $\frac{1}{2}\left(\frac{q}{3} - \frac{q}{19}\right) = \frac{8q}{57}$.

Consider another situation, where $-q$ is the charge of one of the vertices and $+\frac{q}{19}$ of the other 19.

In the same way, we can get $-\frac{q}{3}$ and $-\frac{8q}{57}$.

Consider the superposition of these cases with the required vertices.

We obtain, that charges of our vertices are $+\frac{20}{19}q$ and $-\frac{20}{19}q$ and a voltage difference is

$\frac{2\left(\frac{q}{3} + \frac{8q}{57}\right)}{C} = \frac{18}{19}\frac{q}{C}$

Using the definition of capacity: $\frac{20}{19}q = C' \cdot \frac{18}{19}\frac{q}{C}$

and $C' = \frac{20}{18}C$

$C_0 = C' - \frac{C}{2} = C\left(\frac{20}{18} - \frac{9}{18}\right) = \boxed{\frac{11}{18}C}$

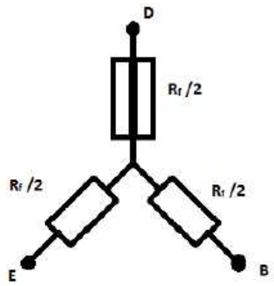
5. Tudor Ciobanu - a short solution which is in fact more general than the previous ones: uncut dodecahedron is substituted by an equivalent Y-connection. By doing so, it becomes also possible to calculate the circuit with additional (possibly different) capacitors included between *B-D*, *B-E* and/or *D-E*. This solution was a pretendent for the best-solution-award, but it was a later submitted revised solution of Tudor, and without his original speed-bonus, the final score would have been lower.

For simplicity, we will solve this problem by using resistors in place of capacitators, and then inverting the result.

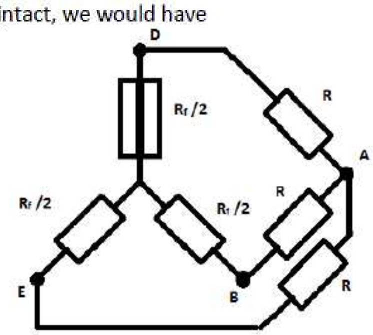
Thus, if $R_f = x\,R$, where x is the factor we seek, then $C_f = \frac{1}{x}\,C$.

The first part of the reasoning is a general one and coresponds to considering the intact figure and finding an equation that links R_{in} , the resistance between two adjacent points of the intact structure, and R_f.

If the line AE, AD and AB were missing, the points E,D,B could be thought of as a triangle formation linked like this:



If we were to put the lines AE, AD and AB back, our dodecahedron would be intact, we would have resistance R_f between A and E, D or B , and our circuit would look like this:



The resistance between A and B will be formed of a resistance R in parallel with two identical sides of resistance $R + R_f$, both of whom are in series with a resistance $\frac{R_f}{2}$.

Thus: $$R_{in} = \frac{R}{3}\,\frac{2R+3R_f}{2R+R_f} \qquad (1)$$

This result is compatible with any circuit formed of nodes at the intersection of three wires, with one such node eliminated, including the infinite honeycomb lattice. Once we find out R_{in} for such a system, we can find R_f immediately.

In order to find R_{in} for the intact dodecahedron, we shall imagine what happens if we introduce the current I in node A, while substracting $I/19$ from each of the remaining 19 nodes. Due to the symmetry of the sistem, the wire AD, AB and AE will each carry the current $I/3$.

Now, we imagine that we are taking the current I out of the node B, while again substracting $I/19$ from the remaining nodes and carrying the current $I/3$ through the wire AB.

By superimposing those two situations, we arrive at a system in which the current ${}^{20}/_{19}\ I$ goes in and out only from nodes A and B, while the wire AB is traversed by a current $2I/3$.

By using the meaning of R_{in} and thinking of the potential applied between points A and B to achieve this current: $V = \frac{20}{19}\,I\,R_{in} = \frac{2I}{3}R$ We find out: $R_{in} = \frac{19}{30}\,R$

Again, this method of finding R_{in} is commonly used in such lattices.

By introducing the value of R_{in} in equation (1) , we arrive at the result : $R_f = \frac{18}{11}\,R$

Since we have used resistors in place of capacitators, we have to invert the result: $C_f = \frac{11}{18}\,C$

On next pages , the following solutions can be found (each of the authors will receive an additional 1.1-factor):

6. Lars Dehlwes - solves a full system of 28 equations for the currents;

7. Hrishikesh Menon - makes use of Y-Δ-transform (this involves typically less calculations than the method of foop currents or that of the node potentials);

8. **Nguyen Ho Nam** - a brute force approach, but with the method of node potnetials, the No of unknowns is smaller (20 nodes vs 30 edges), and using the mirror symmetry, the No of unknowns goes down to 7.