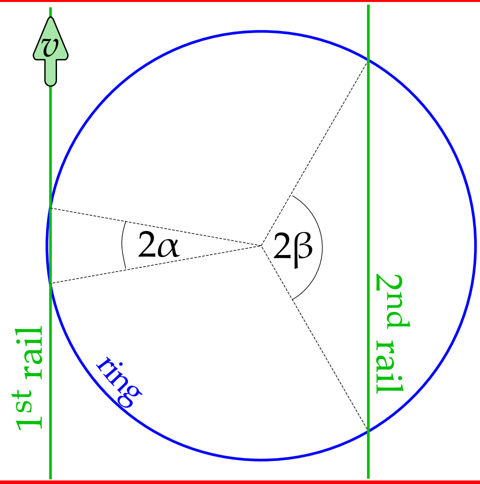
Problem 7

A homogeneous ring lays horizontally on two identical parallel rails. The first rail moves parallel to itself, with a constant speed v; the second rail is at rest. The angular distance between the ring-rail contact points, as seen from the centre of the ring, is 2α for the first rail, and 2β for the second rail, see figure. Assuming that $\alpha \ll 1$ and $\beta = \pi/3$, find the speed of the centre of the ring.



Hints after 1st week: This problem can be solved by using a brute force approach, i.e. writing down two equations for two unknown angles. However, the solution can be significantly simplified once a useful geometrical fact is noticed: then, it is enough to write down only one equation for one unknown quantity.

Hints after 2nd week: Typically in the case of static's problems, it is convenient to start with a torque balance, because the origin for the balance equation can be chosen in such a way that arms of at least two forces become equal to zero; also, you are free to choose, which forces you want to disappear from your torque balance (which are the least desirable). In particular, if there are only three forces applied to a rigid body at an equilibrium, the lines along which these forces are applied intersect always in a single point. Here, in order to derive the "geometrical fact" mentioned in the previous hint, study the torque balance with respect to the intersection point of the lines along which two forces (e.g. the friction forces due to the 1st rail) are applied.

Hints after 3rd week: Here is the "to-do-list" for the Problem 7. Let the x-axis be along the rails, y-axis — the other horizontal axis, and z — the vertical one. Do not put $\alpha = 0$; instead neglect terms which are much smaller than α^2 (e.g. using $\cos\alpha \approx 1 - \alpha^2/2$). Find the magnitudes of the friction forces using the torque balance in y-z-plane. Study the torque balance of two friction forces in x-y-plane (e.g. those due to the 1st rail) with respect to the intersection point P of the lines defined by the remaining two friction forces. As a result, you should be able to notice that the position of the point P defines the directions of all the four friction forces. Write the x-directional force balance equation of the friction forces using the distance L of the point P from the 1st rail as a single unknown quantity; solve the equation using the approximation $\alpha \ll 1$ (be careful: L has the same order of smallness as α^2!). Once you know L, the ring's speed can be easily found by finding first the distances between the instantaneous rotations centres of the ring in the reference frames of the 1st and of the 2nd rail, and the centre of the ring. Finally note that numerically approximate answer can be found using purely geometrical constructions, e.g. with the help of GeoGebra applet (still, analytical exact result is much more preferred).

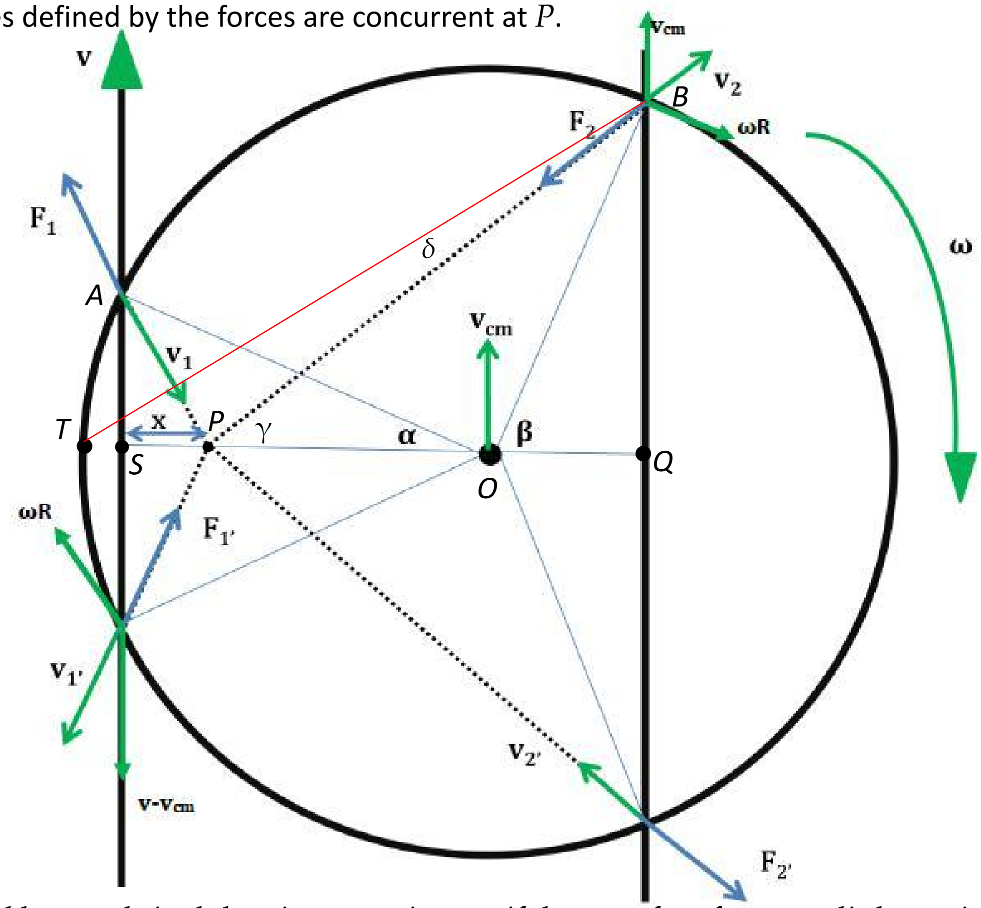
Results of the 7th problem. Correct solutions have been submitted by

(ordered according to the arrival time):

1	Szabo Attila (Hungary)	3.619859205	*(submitted during the 1st week)*
2	Oliver Edtmair (Austria)	3.674123988	*(submitted during the 2nd week)*
3	Dan-Cristian Andronic (Romania)	1.697722338	*(submitted during the 4th week)*
4	Cyuan-Han Chang (Taiwan)	1.543383943	
5	Kurenkov Mikhail (Russia)	1.9487171	
6	Lars Dehlwes (Germany)	1.61051	
7	Nguyen Ho Nam (Vietnam)	0.8433216	
8	Madhivanan Elango (United Kingdom)	1.4641	
9	Jordan Jordanov (Bulgaria)	1.21	
10	Vũ Việt Linh (Vietnam)	0.88	
11	Efim Mazhnik (Russia)	1	
12	Qu Xinyi (Singapore)	0.64	
13	Daumantas Kavolis (Lithuania)	1	
14	Jakub Mrożek (Poland)	1	
15	Jakub Safin (Slovakia)	0.9	
16	Jaemo Lim (Korea)	0.576	
17	Ismael Salvador Mendoza Serrano (Mexico)	0.9	
18	Samuel Bosch (Croatia)	0.9	
19	David Stein (Germany)	0.9	

This time, the best-solution-awards go to the two first solutions - not because these were really the best, but because all the other solutions arrived after very detailed hints, at which point solving became much easier than before. Meanwhile, the first hint was so subtle that for the second week, the decrease of difficulty was negligible. The solution of **Szabo Attila** is a brute-force one, and the solution of **Oliver Edtmair** is based on geometric observations; hence the award is divided in proportions 1:2 in favour of Oliver.
We start with the introductory paragraphs of the solution of **Madhivanan Elango**, which provides the simplest proof that all the four forces point towards the same point.

I first show the diagram above and introduce the new quantities as labelled. Notice that the velocity vectors shown are from the frame of reference of the rail, on which the points are situated on and that the decomposition of vectors v_1 and v_2 are also shown on the diagram. Additionally, label the radius of the ring as R. Label the x axis parallel and in the direction of the velocity of the moving rail, the z axes comes straight out of the page and then label the y axis to follow a right hand coordinate system. Since the ring is in equilibrium all forces and all torques about any point must sum to 0. The frictional forces by definition are in opposite directions to the velocity vectors and are proportional to the normal reaction force at that point. Let the normal reactions on the moving rail be N_1 and the reactions on the stationary rails be N_2. By considering torques in the y-z plane about the center of the ring, it is clear to see that $N_1 R\cos\alpha = N_2 R\sin\alpha$, that is $N_1\cos\alpha = N_2\cos\beta$. This means that $F_1\cos\alpha = F_2\cos\beta$. Now consider the torques about point P, the intersection of F_1 and F_2: we know that they must sum to 0. Notice that the two forces from the first rail are just reflections of each other in the line defined by the rail. Due this symmetry, the two torques about P will clearly have the same sign, so the only way that these two torques can have a total sum of 0 is if each is 0: they must therefore both go through the point P so all four lines defined by the forces are concurrent at P.



At this point, it should be empahsized that in a generic case, if there are four forces applied to a rigid body at equilibrium, these forces do not necessarily need to be pointed towards a single point (unlike what is valid for three forces).
There are two more steps left to do: finding the distance x in the figure above, and based on that result, finding the speed of the ring. Before we continue with the solutions of contestants, let us have a look, how these steps can be done in the simplest way.

Due to $F_1\cos\alpha = F_2\cos\beta$, with appropriately chosen units of force we can put F_1=1, F_2=2cosα (here we substituted $\cos\beta = ½$), and R=1. Note that $|ST| \equiv y = 0.5\alpha^2$, and $\gamma = \delta + \pi/6$, where $\delta \approx \sin(\pi/6)(x+y)/|BT| = 0.5\tan(\pi/6)(x + 0.5\alpha^2)$. Now we can express the vertical components of the forces as follows:

$$F_{1x} = |AS|/|AP| \approx \alpha/\sqrt{\alpha^2+x^2} = (1+x^2/\alpha^2)^{-1/2} \approx 1 - x^2/2\alpha^2$$

and $F_{2x} = (2 - \alpha^2)\sin\gamma \approx (2 - \alpha^2)[0.5 + \cos(\pi/6)\,\delta] \approx (2 - \alpha^2)[0.5 + 0.25(x + 0.5\alpha^2)] \approx 1 - 0.25\alpha^2 + 0.5x$.

If we denote $z = x/\alpha^2$, the condition $F_{1x} = F_{2x}$ can be written as

$2z^2 + 2z - 1 = 0$, hence (excluding the negative solution) $z = 0.5(\sqrt{3} - 1)$. From here $x = 0.5(\sqrt{3} - 1)\alpha^2$.

Now we can turn to the final step, finding the speed. To that end, we introduce the instantaneous rotation centres C_1 and C_2 (in the 1st rail's and laboratory reference frames, respectively); see also the solution of **Cristian Andronic** below (he was the only one to use the instantaneous rotation centres). From the similar right triangles C_1AP and ASP we obtain $|C_1P| \approx |C_1S| = |AS|^2/x = 1/z = \sqrt{3} + 1$. It is easy to see that C_2 is very close to the edge of the ring, so $|C_2O| \approx 1$. In the 1st rail's frame, the speed of the ring's centre is $v_1 = -\omega|C_1O|$ (down), and in the laboratory frame — $u = \omega|C_2O|$ (up); here, ω is the ring's rotation speed. The difference of the two velocities gives the 1st frame speed, $v = \omega(|C_1O| + |C_2O|)$, and hence, the lab-frame-speed of the ring's centre $u = v|C_2O|/(|C_1O| + |C_2O|) \approx v/(3+\sqrt{3})$.

Below, after the solution of *(a)* Cristian Andronic, the following solutions can be found. *(b)* **Cyuan-Han Chang**, who uses another way to show that all the friction forces are directed towards the same point; also, he obtains a correction term to the answer which describes slow dependance on α (this was also done in few other solutions); *(c)* **Kurenkov Mikhail**: the only one who cares to show that assuming equality of the dynamic and static coefficients of friction, the ring cannot stay at rest — if it were at rest, the friction forces due to the 2nd ring could be smaller; the "best" solutions of *(d)* Szabo Attila and *(e)* Oliver Edtmair.