Quasilinear Hyperbolic Systems, Nonlinear Superposition and Solitons

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ABSTRACT

The study of wave propagation described by linear hyperbolic equations and systems is, in the main, dominated by the linear superposition principle. Mainly through the influence of gas dynamics, the study of wave propagation has shifted away from linear systems to quasilinear hyperbolic systems and hyperbolic conservation laws, with the early contribution by Peter Lax [1] playing a prominent role. At around the same time, important contributions were being made by various Russian authors, largely motivated by questions from gas dynamics, of whom mention must be made of B.L. Rozhdestvenskii and N.N. Yanenko [2]. Despite the significant progress that has been made in the intervening years, various questions remain unresolved, and one objective of this paper is to draw attention to some of these questions that have important implications in the study of nonlinear wave propagation.

An early attempt was made by Zabusky and Kruskall to examine the propagation of a wave through a nonlinear elastic crystal lattice, with a view to examining some anomalous nonlinear conduction behaviour first identified by Fermi, Pasta and Ulam. Their approach was based on a nonlinear hyperbolic equation, but they discovered the propagating wave became unbounded after a finite time, showing that a nonlinear hyperbolic equation could not describe the physical phenomenon that had been observed. This work stimulated further interest in the breakdown of solutions of nonlinear quasilinear hyperbolic equations, but also interest of a different kind to discover the correct equation that describes wave propagation through a nonlinear crystal lattice. The latter question was answered by Zabusky and Kruskall numerically, who went to the first higher approximation above their nonlinear hyperbolic equation, as a result of which they arrived at the KdV equation and observed numerically and for the first time the behaviour of solitons. Another objective of this paper will be to trace these ideas to the point where weak solutions of quasilinear hyperbolic equations need to be introduced, along with so-called entropy principles to select a physically realistic solution from amongst the many possible purely mathematical weak solutions.

Finally, some observations will be presented concerning nonlinear superposition principles, where it will be shown that they are not as special as they might seem at first sight.

REFERENCES

[1] P.D. Lax. Hyperbolic systems of conservation laws II, *Comm. Pure and Appl. Math.* **10,** 537-556, 1951.

[2] B.L.Rozhdestvenskii and N.N. Yanenko. *Systems of Quasilinear Equations and Applications to Gas Dynamics,* Translations of Mathematical Monographs, **55,** American Mathematical Society, 1980.