Solitary Shock Waves and Periodic Shock Waves in a Compressible Mooney-Rivlin Elastic Rod

Hui-Hui Dai
Department of Mathematics,
City University of Hong Kong
Email: mahhdai@cityu.edu.hk

This is a joint work with Jibin Li of Kunming University of Science and Technology

- Background
- The rod equations and governing equations for traveling waves
- Equilibrium points and bifurcation analysis
- Phase planes
- Bounded Traveling Waves
- Conclusions

Background

1. Weakly nonlinear waves

Nariboli (1970) and Ostrovskii and Sutin (1977): the KdV equation

Sorensen, Chritiansen et al (1984): the improved Boussinesqu equation

Samsonov (1988): the double dispersive equations

Samsonov & Sokurinskaya (1988): experiments

Cohen and Dai (1993): two coupled nonlinear equations

Porubov et al (1993): the double dispersive equation (refined)

Dai (1998a,b) and Dai & Huo (2000): a new type of nonlinear dispersive equation which takes into account the coupling effect of material nonlinearity and the geometrical size. It is found that this equation admits peakons, compactons and other singular waves.

Dai and Huo (2002): analytical descriptions for the solitary wave with a shelf and fissions of solitons

2. Strongly nonlinear waves or finite-amplitude waves

Wright (1984, 1985) pointed out that the possibility that a variety of traveling waves can arise in incompressible hyperelastic rods, including solitary shock waves:

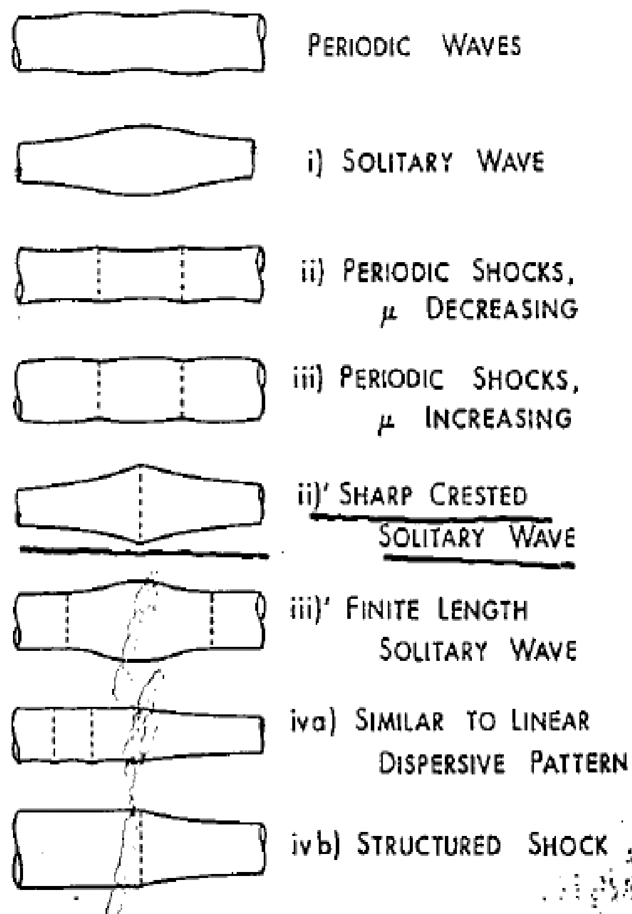


Figure 4. Typical mode shapes of steady waves. Dashed lines indicate shock waves. Roman numerals refer to cases discussed in Section 6.

Note: Actually, the last two cannot arise

- Coleman and Newman (1990) showed that only periodic and solitary waves can arise in an incompressible neo-Hookean rod and gave their explicit solutions.

- Dai (2001) showed that a number of traveling waves, including solitary shock waves, can arise in an incompressible Mooney-Rivlin rod, the existence conditions and explicit solutions were given.

Here, we shall consider **finite-amplitude waves** in a compressible Mooney-Rivlin material. A major difference is that the problem is governed by two partial differential equations

The rod equations for a compressible Mooney-Rivlin material

- The strain energy function of a compressible Mooney-Rivlin material

$$\Phi = \frac{1}{2}\mu(\frac{1}{2}+\beta)(I_1-3)+\frac{1}{2}\mu(\frac{1}{2}-\beta)(I_2-3)+\frac{1}{2}\mu k(I_3-1)-\frac{1}{2}\mu(k+\frac{3}{2}-\beta)\ln I_3,$$

Its motion, under the approximation of the Navier-Bernoulli hypothesis (the planar section remains planar and normal to the rod axis; cf. Alwis *et al.* 1994), is described by

$$z=\tilde{z}(Z,T), r=\tilde{r}(Z,T)R, \theta=\Theta, \tag{2.1}$$

By further considering the kinetic energy one can obtain the Lagrangian. Then, the variational principle yields the rod equations:

$$-\frac{\rho}{\mu} z_{TT} + \frac{1}{2}(1+2\beta) z_{ZZ} + \frac{1}{2}(2k+3-2\beta)\frac{z_{ZZ}}{z_Z^2} + (1-2\beta) r^2 z_{ZZ}$$

$$+2(1-2\beta) r r_Z z_Z + k r^4 z_{ZZ} + 4k r^3 r_Z z_Z = 0, \tag{2.6}$$

$$(1+2\beta) r + (1-2\beta) r^3 + 2k r^3 z_Z^2 + (1-2\beta) r z_Z^2 - \frac{(2k+3-2\beta)}{r}$$

$$-a^2 \frac{1}{4}(1-2\beta) r r_Z^2 - a^2 \frac{1}{4}(1-2\beta) r^2 r_{ZZ} + \frac{\rho}{\mu} a^2 \frac{1}{2} r_{TT} - a^2 \frac{1}{4}(1+2\beta) r_{ZZ} = 0, \tag{2.7}$$

There are two very complicated nonlinear PDE's. Not much can be done analytically on the initial/boundary-value problems. Next, we turn our attention on traveling waves only.

Governing equations for traveling waves

Here, we are interested in travelling waves. Denote λ as the axial stretch z_Z. For travelling waves, we have

$$\lambda = \lambda(\xi), \quad r = r(\xi), \quad \xi = Z - cT, \tag{3.1}$$

where c is the propagating wave velocity. Noting that $z_{TT} = c^2\lambda_\xi$, an integration of (2.6) with respect to ξ yields that

$$(\frac{1}{2}(1+2\beta) - \frac{c^2\rho}{\mu})\lambda - \frac{1}{2}(2k+3-2\beta)\frac{1}{\lambda} + (1-2\beta)r^2\lambda + kr^4\lambda = g, \tag{3.2}$$

The first equation becomes an algebraic one for the two unknowns!

The second equation becomes

$$\left(\frac{c^2a^2\rho}{2\mu} - \frac{a^2(1+2\beta)}{4} - \frac{a^2(1-2\beta)r^2}{4}\right)r_{\xi\xi} - \frac{a^2(1-2\beta)}{4}rr_{\xi}^2$$
$$-\frac{(2k+3-2\beta)}{r} + (1+2\beta)r + (1-2\beta)r^3 + ((1-2\beta)r + 2kr^3)\lambda^2 = 0. \qquad (3.3)$$

Using the algebraic relation, we obtain

$$(b_1 - b_2r^2)r_{\xi\xi} - b_2rr_{\xi}^2 + \Phi(r) = 0, \qquad (3.13)$$

where

$$\Phi(r) = \frac{\psi(r)}{2r\eta^2(r)}, \text{ for } r \neq r_*, \tag{3.14}$$

$$\psi(r) = 2\eta(r)[\eta(r)\zeta(r) + a_2\theta(r)] + g^2\theta(r) + g\,\theta(r)\sqrt{\Delta}. \tag{3.15}$$

$$\eta(r) = kr^4 + (1-2\beta)r^2 - a_1.$$

$$\zeta(r) = (1-2\beta)r^4 + (1+2\beta)r^2 - 2a_2,$$

$$\theta(r) = 2kr^4 + (1-2\beta)r^2.$$

We rewrite the equation for the traveling waves as a first-order system

$$\frac{dr}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{b_2 r y^2 - \Phi(r)}{b_1 - b_2 r^2}. \qquad (3.17)$$

A very important feature is that the denominator term, which implies that there is a singular line in the phase plane. In fact, it is the reason why some exotic waves can arise.

This singularity causes considerable inconvenience for direct analysis. So, we introduce a topologically equivalent system:

$$\frac{dx}{dt} = 2r(b_1 - b_2r^2)\eta^2(r)y, \quad \frac{dy}{dt} = 2b_2r^2y^2\eta^2(r) - \psi(r). \qquad (3.22)$$

Then, we shall relate these two systems.

Actually, in the second system a hetroclinic orbit connecting two saddle points together corresponding to the singular line of the first system.

Positive Equilibrium Points

Case 1 $b_1 - b_2 r^2 = 0$

$$y_s^{\pm} = \pm \frac{\sqrt{\psi(r_s)}}{r_s \eta(r_s) \sqrt{2b_2}}.$$

Case 2 $b_1 - b_2 r^2 \neq 0,$

$$[2\eta(r)(\zeta(r)\eta(r) + a_2\theta(r)) + g^2\theta(r)]^2 = -g^2\theta^2(r)\Delta.$$

To determine the types of equilibrium points, we can calculate the Jacobain Determinant:

$$\sigma = \left. \frac{\psi'(r)}{2r(b_1 - b_2 r^2)\eta^2(r)} \right|_{r=r_e}. \tag{4.9}$$

The sign of σ can be used to determine the type of an equilibrium. In fact, since (3.17) is integrable, by the theory of planar dynamical systems, when $\sigma > 0$ the equilibrium point $(r_e, 0)$ is a center; when $\sigma < 0$ it is a saddle point; if $\sigma = 0$ and $(r_e, 0)$ has index zero, then it is a cusp. Therefore, if $b_1 - b_2 r^2 > 0 (< 0)$, then when $\psi'(r_e) > 0 (< 0), (r_e, 0)$ is a center; when $\psi'(r_e) < 0 (> 0)$, it is a saddle point. The case of $\psi'(u_e) = 0$ will be discussed below.

Bifurcation analysis of equilibrium points in Case 2:

$$[2\eta(r)(\zeta(r)\eta(r) + a_2\theta(r)) + g^2\theta(r)]^2 = -g^2\theta^2(r)\Delta.$$

This equation contains four parameters: two material constants, the speed of the traveling waves and an integration constant.

We need to find out, as these four parameters vary, the number of roots of the above equations and the types of each equilibrium point, i.e., a global bifurcation analysis.

The analysis is very technical and tedious. Here, we just summarize the results.

For such a dynamical system, if no centre point, then there is no bounded traveling wave solution; see Dai (2001).

So, we only interested in the parameters domain in which there exists at least one centre point

Theorem

There exists a lower bound c_0 (shear-wave speed for linear waves) for bounded traveling waves, i.e., no traveling waves with a speed less than c_0 can arise.

The cases (in total twelve cases) in which there exists at least a centre point are given below

Proposition 5.2 For the case $a_1 = 0, g \geq 0$,

(i) if $g = 0$, then (3.17) has one saddle point at $(r_2, 0)$, where $\psi_0(r_2) = 0$;

(ii) if $0 < g < g_M$, then (3.17) has one center point at $(r_1, 0)$ and one saddle point at $(r_2, 0)$;

(iii) if $g = g_M$, then (3.17) has a cusp point at $(r_{12}, 0)$;

(iv) if $g > g_M$, then (3.17) has no equilibrium point.

Proposition 5.3 For the case $a_1 > 0, g \geq 0$ and $r > r_*$, we have

(i) if $\psi_0(\bar{u}) = 0$ and $g = 0$, then the equilibrium point $(\bar{r}, 0)$ of (3.17) is a cusp point;

(ii) if $\psi_0(\bar{u}) < 0$ and $0 < g < g_M$, then when $r_s < r_1 < r_2$ the equilibrium point $(r_1, 0)$ is a center point, $(r_2, 0)$ is a saddle point; when $r_1 < r_s < r_2$ both of $(r_1, 0)$ and $(r_2, 0)$ are saddle points; when $r_1 < r_2 < r_s$, $(r_1, 0)$ is a saddle point, $(r_2, 0)$ is a center point;

(iii) if $\psi_0(\bar{u}) < 0$ and $g = g_M$, then the equilibrium point $(r_{12}, 0)$ is a cusp point.

Proposition 6.4 Assume that the condition (6.19) holds.

(i) If $\psi_0(\bar{u}) \geq 0$, then there exists a $g = g_b \geq 0$ such that when $g = g_b$ system (3.17) has one equilibrium point at $(r_b, 0)$, which is a cusp point. If $\psi_0(\bar{u}) < 0$, then when $g = 0$ (3.17) has two equilibrium points (see (ii) below).

(ii) When $g_b < g \leq g_*$, system (3.17) has two equilibrium points at $(r_1, 0), (r_2, 0)$, satisfying $r_* \leq r_1 < r_2 < \hat{r}$; when $g_* < g \leq g_m < g_0$, system (3.17) also has two equilibrium points with $r_1 < r_* < r_2 < \hat{r}$; if $r_1 < r_2 < r_s$, then $(r_1, 0)$ is a saddle point and $(r_2, 0)$ is a center point; if $r_1 < r_s \leq r_2$, then both of $(r_1, 0)$ and $(r_2, 0)$ are saddle points; if $r_s < r_1 < r_2$, then $(r_1, 0)$ is a center point and $(r_2, 0)$ is a saddle point.

(iii) When $g > g_m$, system (3.17) has only one equilibrium point at $(r_2, 0)$ with $r_* < r_2 < \hat{r}$. If $r_2 < r_s$, then $(r_2, 0)$ is a center point; if $r_s \leq r_2$, then $(r_2, 0)$ is a saddle point.

Proposition 6.5 Assume that the condition (6.20) holds.

(i) When $g = g_*$, system (3.17) has only one equilibrium point $(r_*, 0)$, which is a cusp point.

(ii) When $g_* < g \leq g_m < g_0$, system (3.17) has two equilibrium points at $(r_1, 0), (r_2, 0)$ satisfying $r_1 < r_* < r_2 < \hat{r}$. The equilibrium point $(r_1, 0)$ is a saddle point and $(r_2, 0)$ is a center (saddle) point if $r_2 < r_s (r_2 \geq r_s)$.

(iii) When $g > g_m$, system (3.17) has only one equilibrium point $(r_2, 0)$ with $r_* < r_2 < \hat{r}$, which is a center (saddle) point if $r_2 < r_s (r_2 \geq r_s)$.

Proposition 6.6 Assume that the condition (6.21) holds.

(i) There exists a $g_b > 0$ such that when $g = g_b$, system (3.17) has a cusp point at $(r_b, 0)$.

(ii) When $g_b < g < g_*$, system (3.17) has two equilibrium points at $(r_1, 0)$ and $(r_2, 0)$ satisfying $r_1 < r_2 < r_*$; $(r_1, 0)$ is a saddle point and $(r_2, 0)$ is a center point.

(iii) When $g_* \leq g \leq g_m < g_0$, system (3.17) has two equilibrium points at $(r_1, 0)$ and $(r_2, 0)$ with $r_1 < r_* \leq r_2$; $(r_1, 0)$ is a saddle point, $(r_2, 0)$ is a center (saddle) point if $r_2 < r_s (r_2 \geq r_s)$.

(iv) When $g > g_m$, system (3.17) has only one equilibrium point at $(r_2, 0)$ with $r_* < r_2 < \hat{r}$; $(r_2, 0)$ is a center (saddle) point if $r_2 < r_s (r_2 \geq r_s)$.

Proposition 6.7 Suppose that the parameter condition (6.22) holds.

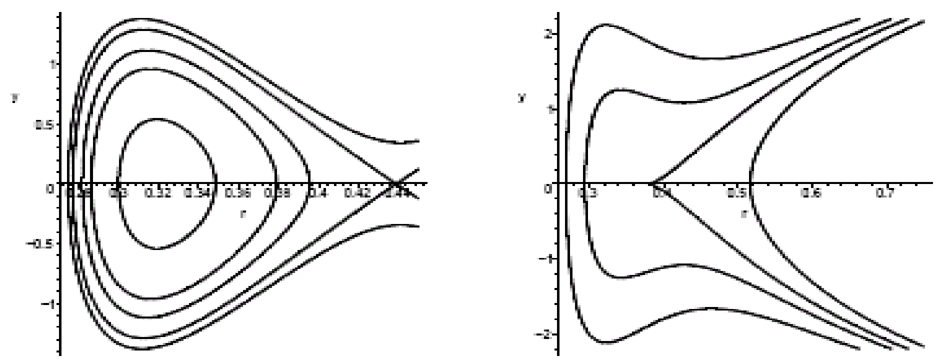
(i) If $g = g_b < g_m < g_0$, then system (3.17) has a cusp point.

(ii) If $g_b < g < g_m < g_0$, then system (3.17) has a saddle point at $(r_1, 0)$, a center point at $(r_2, 0)$, where $r_i = \sqrt{u_i}, i = 1, 2$.

(iii) If $g > g_m$, then system (3.17) has only a center point at $(r_2, 0)$.

Phase Planes

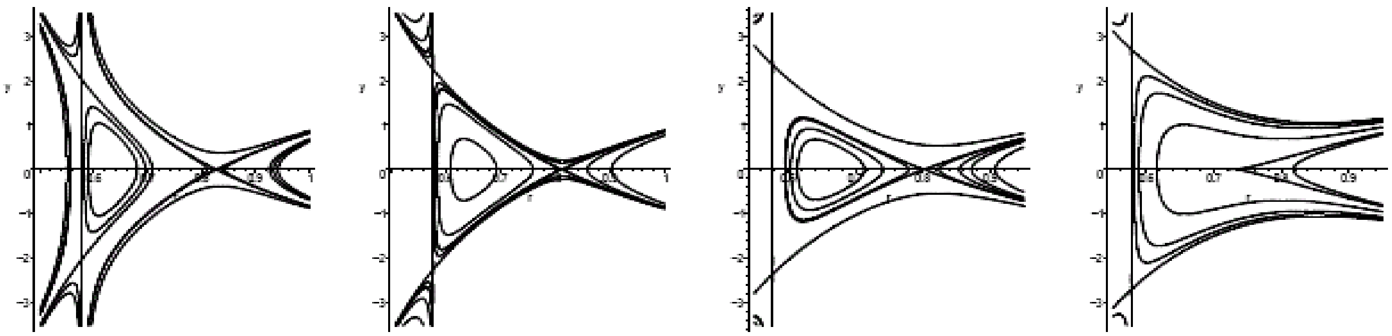
(i) If $a_1 = 0$, then when $g = 0$ there is a saddle point at $(r_2, 0)$; when $0 < g < g_M$ there is a periodic annulus of the center $(r_1, 0)$ which is enclosed by the homoclinic orbit to the saddle point $(r_2, 0)$; when $g = g_M$ there is a cusp point at $(r_{12}, 0)$ (see Fig. 7.1 (1)-(2)).



(1) $g = 0.215 < g_M$. (2) $g = g_M = 0.219358$.

Fig.7.1 The phase portraits of (3.22), when $a_1 = 0$.
(parameters: $k = 4$, $a_1 = 0$, $\beta = 0.25$)

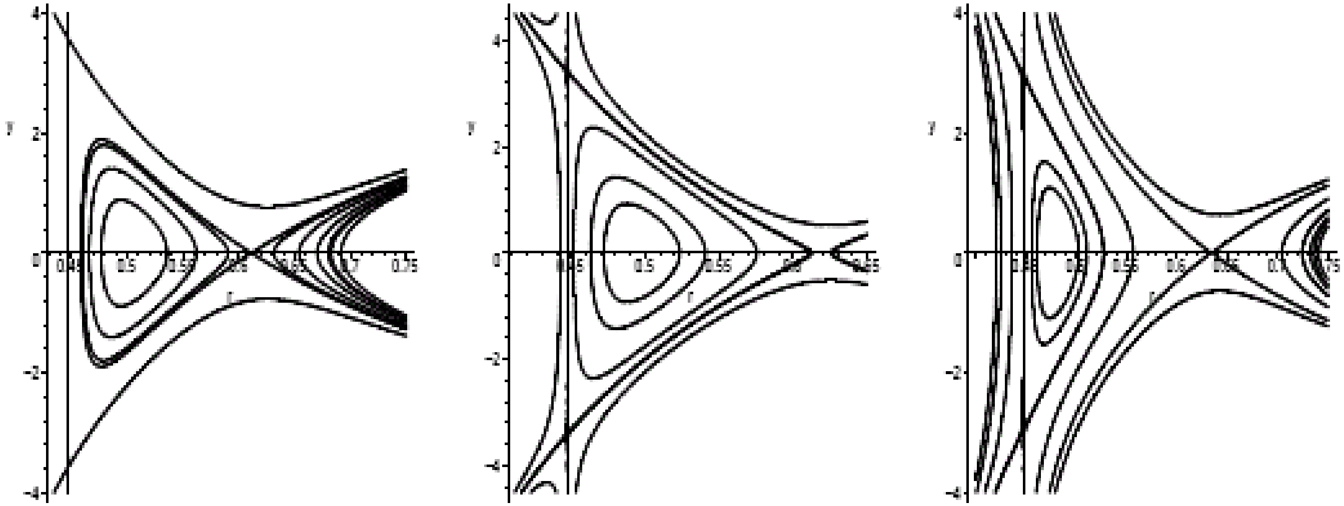
(a) If for $g = 0$ there is a periodic annulus of the center $(r_1, 0)$ which is enclosed by the heteroclinic orbits to two saddle points $(r_s, y_s^{\pm})$, then, there exists a $g = g_M$ such that when $0 < g < g_H$ the same periodic annulus keeps up; when $g = g_H$ the periodic annulus is enclosed by three heteroclinic orbits to three saddle points $(r_2, 0)$ and $(r_s, y_s^{\pm})$; when $g_H < g < g_M$ there is a periodic annulus enclosed by a homoclinic orbit to the saddle point $(r_1, 0)$; when $g = g_M$ there is a cusp point (see Fig. 7.2 (1)-(4)).



(1) $0 \leq g < g_H$ (2) $g = g_H$. (3) $g_H < g < g_M$. (4) $g = g_M$.

Fig.7.2 The phase portraits of (3.22), when $a_1 > 0, r_* > 0$.
(parameters: $k = 19,\ a_1 = 0.25,\ \beta = -0.25, g_H = 0.04969, g_M = 0.09634$.)

(iii) Suppose that $a_1 > 0, \psi_0(\bar{u}) < 0$ for $\bar{u} < \hat{u}$ and the zeros of the function $\psi(r)$ (defined by (6.1)) satisfy $r_s < r_1 < r_2$ for $g = 0$. If for $g = 0$ there is a periodic annulus of the center $(r_1, 0)$ which is enclosed by the homoclinic orbit to the saddle point $(r_2, 0)$, then there exists a $g = g_H$ such that when $0 < g < g_H$, the same annulus keep up; when $g = g_H$, the periodic annulus is enclosed by three heteroclinic orbits connecting to three equilibrium points $(r_2, 0)$ and $(r_s, y_s^{\pm})$; when $g_H < g < g_s$, the periodic annulus is enclosed by two heteroclinic orbits connecting to two saddle points $(r_s, y_s^{\pm})$, where at $g = g_s, r_1 = r_s$; when $g \geq g_s$ there is no center point (see Fig. 7.3 (1)-(3)).

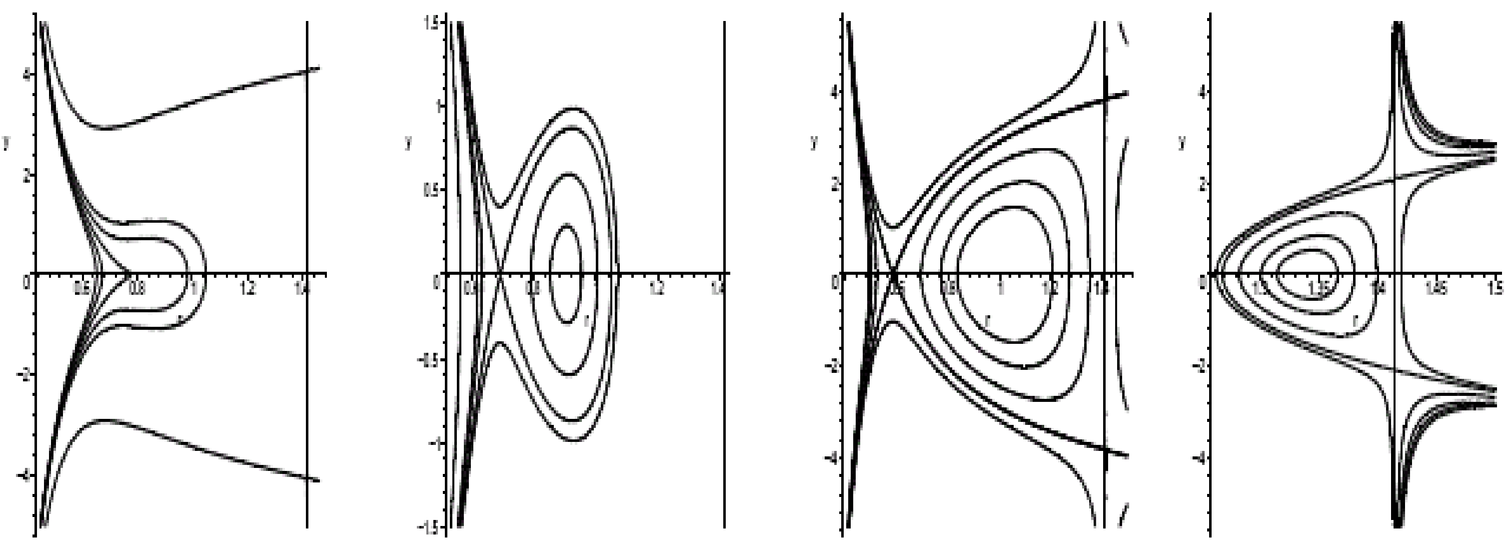


(1) $0 \le g < g_H$ (2) $g = g_H$. (3) $g_H < g < g_s$.

Fig.7.3 The phase portraits of (3.22) in Proposition 7.1 (iii) .

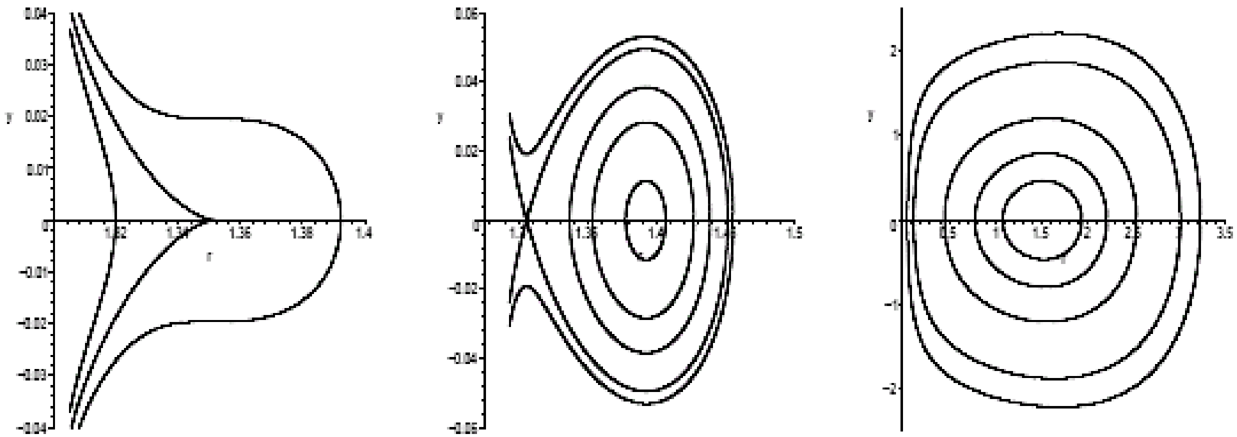
(parameters: $k = 19, \beta = 0.25, a_1 = 0.05, g_H = 0.002937$.)

(iv) Suppose that $a_1 > 0$, $\psi_0(\bar{u}) \geq 0$ and the conditions (6.14) and (6.19) hold. If for $g > g_b$ two zeros of the function $\psi(r)$ defined by (6.1) satisfy $r_1 < r_2 < r_s$, then there exists a $g = g_H$ such that when $g_b < g < g_H < g_0$ there is a periodic annulus of the center $(r_2, 0)$ which is enclosed by a homoclinic orbit to the saddle point $(r_1, 0)$; when $g = g_H$ the periodic annulus keeps up which is enclosed by three heteroclinic orbits connecting three saddle points $(r_1, 0)$ and $(r_s, y_s^{\pm})$; when $g_H < g < g_s$ the periodic annulus keeps up which is enclosed by two heteroclinic orbits connecting two saddle points $(r_s, y_s^{\pm})$; when $g \geq g_s$ the periodic annulus disappears, where at $g = g_s, r_2 = r_s$ (see Fig. 7.4 (1)-(4)).



(1) $g = g_b$ (2) $g_b < g = 1 < g_H$. (3) $g = g_H$. (4) $g_H < g < g_s$.

(v) Suppose that $a_1 > 0$ and the condition (6.22) holds. Then, when $g_b < g \leq g_m < g_0$ there exists a periodic annulus of the center $(r_2, 0)$ which is enclosed by a homoclonic orbit to the saddle point $(r_1, 0)$; when $g > g_m$ there exists a periodic annulus of the center $(r_2, 0)$ in which periodic orbits can expand to the straight line $r = 0$ (see Fig. 7.5 (1)-(3)).



(1) $g = g_b$ (2) $g_b < g = 20 \leq g_m$. (3) $g = 27 > g_m$.

Fig.7.5 The phase portraits of (3.22) in Proposition 7.1 (v) .

(parameters: $k = 4, \beta = -0.25, a_1 = 31.11, g_b = 19.74625, g_m = 26.749$.)

Bounded Traveling Waves

To consider these trajectories in contact with the singular line, we first can prove the following result:

Lemma 8.1 The boundary curves of a periodic annulus are the limit curves of closed orbits inside the annulus. If these boundary curves contain a segment of the singular straight line $r = r_s$ of (3.17), then along this segment, the "time" interval is zero (i.e., there is a jump in $y = r_\xi$).

I. Solitary waves of radial expansion

We first consider the homoclinic orbits as shown in Figs. 7.1 (1), 7.2 (3) and 7.3 (1). It is easy to see that these orbits have the left abscissa at $(r_L, 0)$ satisfying $r_L < r_1 < r_2$ and on these orbits r approaches r_2 as $\xi \to \pm\infty$. Thus, corresponding to these homoclinic orbits, these are solitary wave solutions. We note that r is just the ratio of the radial cylindrical coordinate of a material point in the reference configuration to that in the reference configuration. For these solutions, comparing with the radial stretch at infinity, the radial stretch at a fixed point is smaller, thus we call them to be solitary waves of radial expansion.

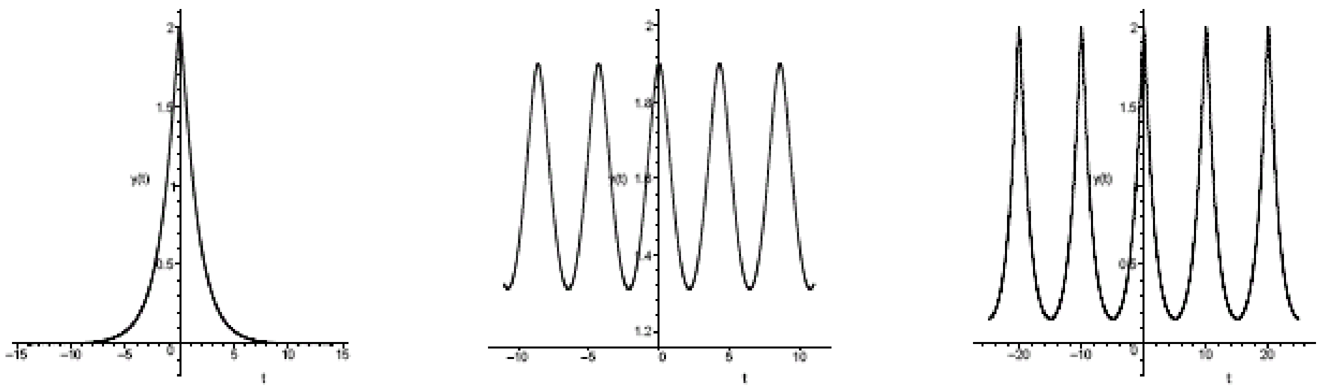
II. Solitary waves of radial contraction

We consider the homoclinic orbits as shown in Fig. 7.4 (2) and Fig. 7.5 (2).

In contrast to the first case, now these orbits have the right abscissa at $(r_R, 0)$ satisfying $r_1 < r_2 < r_R$ and on these orbits r approaches r_1 as $\xi \to \pm\infty$. So that these orbits represent solitary waves of radial contraction.

III. Solitary shock waves of radial expansion

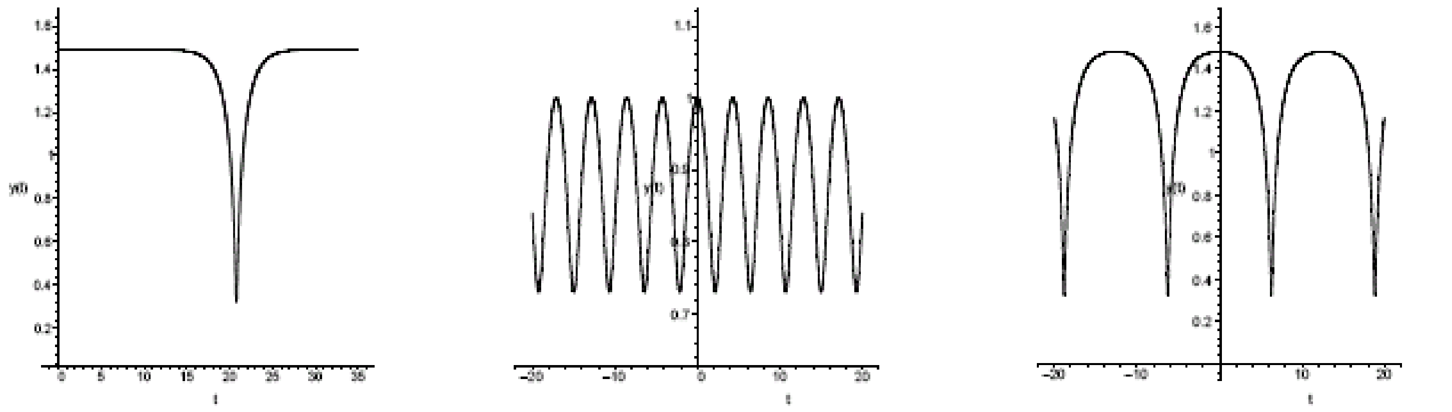
For the periodic annuluses shown as Figures 7.2 (2) and 7.3 (2), their boundary curves contain a segment of the singular straight line $r = r_s$, respectively.



(1) Solitary shock wave (2) Smooth periodic wave (3) Periodic shock wave

Fig.8.1 The profiles of travelling waves of radial-expansion type.

IV. Solitary shock waves of radial contraction



(1) Solitary shock wave (2) Smooth periodic wave (3) Periodic shock wave.

Fig.8.2 The profiles of travelling waves of radial-contraction type.

V. Periodic Waves

VI. Two Types of Periodic shock waves

Conclusions

- By using the techniques of dynamical systems, we show that

1. the traveling waves in a compressible Mooney-Rivlin rod have a lower bound;
2. There seven types of bounded traveling waves, including solitary shock waves and periodic shock waves;
3. The parameters domain for these waves are also established;
4. The solution profiles are plotted from the phase plane trajectories.

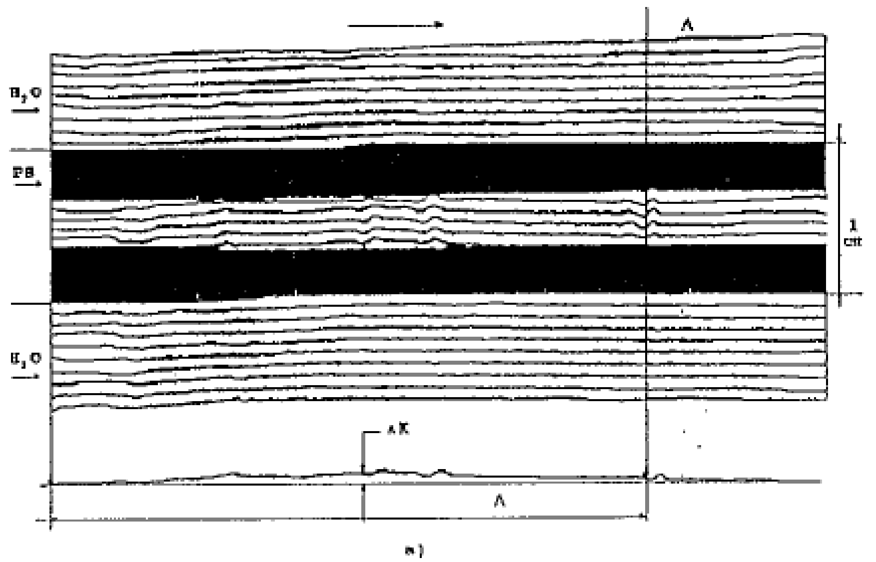


Figure 4.8: Strain soliton in the interval of PS rod of 40-90 mm from the input tip.