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Complexity of Nonlinear Waves

WAVE ENGINEERING

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Traveling waves in strongly inhomogeneous media

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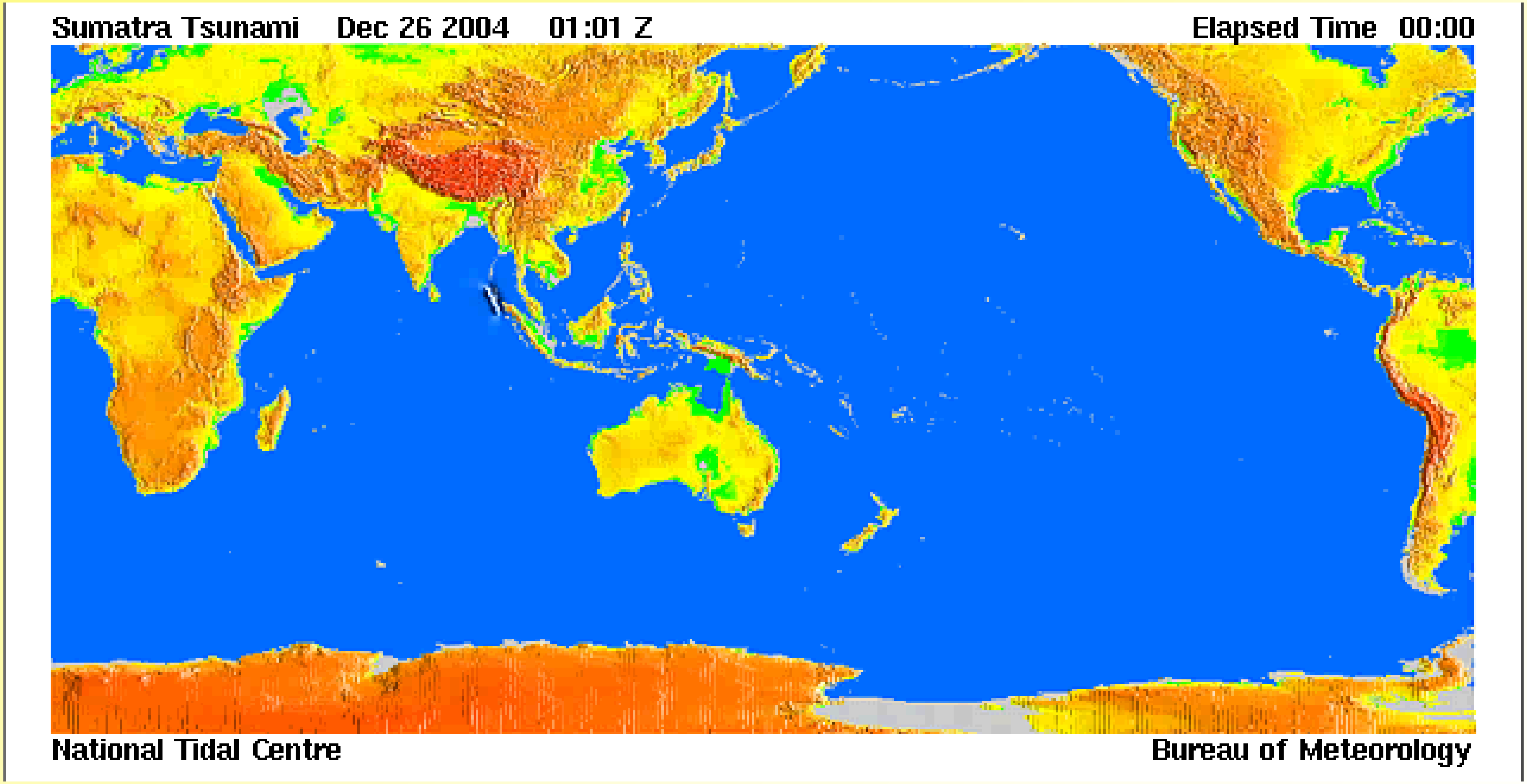


$$\frac{\partial^2 \eta}{\partial t^2} - g\,div\left[h(x,y)\nabla\eta\right] = 0$$

h – water depth

Outline

- Traveling waves in 1D linear case
- Traveling waves in narrow bays and channels
- Nonlinear traveling waves in channels

Simplified 1D linear theory of shallow water waves

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[c^2(x) \frac{\partial \eta}{\partial x} \right] = 0$$

$$c(x) = \sqrt{gh(x)}$$ - wave speed

$\eta(x,t)$ – water displacement

$h(x)$ – water depth

“Non-reflecting” beach with large amplification

Solution of wave equation in the form of a traveling wave

$$\eta(x,t) = A(x)\exp\left[i\left\{\omega t - \Psi(x)\right\}\right]$$

Two unknown functions: A and Ψ

Equations for real and imaginary parts

$$2k\frac{dA}{dx}+A\frac{dk}{dx}+\frac{1}{h}\frac{dh}{dx}kA=0$$

$$\left[\frac{\omega^2}{gh(x)}-k^2(x)\right]A+\left[\frac{d^2A}{dx^2}+\frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right]=0$$

where $k(x)=\frac{d\Psi}{dx}$ ***- wavenumber***

Integration of the first Equation gives us

$$2k\frac{dA}{dx}+A\frac{dk}{dx}+\frac{1}{h}\frac{dh}{dx}kA=0$$

$$A^2(x)k(x)h(x)=const$$

Energy flux conservation

We do not know general analytical solution of the second equation, performed in known functions

$$\left[\frac{\omega^2}{gh(x)} - k^2(x)\right]A + \left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right] = 0$$

It is a variable-coefficient 2d order equation

Not simpler, than initial wave equation

If the depth varies smoothly – WKB Approach

$$\left[\frac{\omega^2}{gh(x)}-k^2(x)\right]A+\left[\frac{d^2A}{dx^2}+\frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right]=0$$

$$k(x)=\frac{\omega}{\sqrt{gh(x)}}$$

eikonal

together

$$A^2(x)k(x)h(x)=\text{const}$$

Try to keep features of the pure propagating wave

$$\left[\frac{\omega^2}{gh(x)} - k^2(x)\right]A + \left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right] = 0$$

$$\left[\frac{\omega^2}{gh(x)} - k^2(x)\right] = 0$$

Overdetermined system

$$\left[\frac{d^2A}{dx^2} + \frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right] = 0$$

$k = k(\omega)$

“Non-reflecting” beach

$$\left[\frac{d^2A}{dx^2}+\frac{1}{h}\frac{dh}{dx}\frac{dA}{dx}\right]=0$$

$$h(x)\frac{dA}{dx}=\text{const}$$

together with

$$A^2(x)k(x)h(x)=\text{const}$$

$$k(x)=\frac{\omega}{\sqrt{gh(x)}}$$

gives

$h(x) \sim x^{4/3}$

“Non-reflecting” beach

$$\eta(x,t) = A\left[\frac{h_0}{h(x)}\right]^{1/4} f\left[t - \tau(x)\right]$$

$$\tau(x) = \int_{-\infty}^{x} \frac{dx'}{\sqrt{gh(x')}}$$

Propagating wave

The shape of the pulse stays the same

Singularity at x = 0 (h = 0)

Velocity field

$$u(x,t) = -g \int_{-\infty}^{t} \frac{\partial \eta}{\partial x} dt' = -g \frac{\partial}{\partial x} \int_{-\infty}^{t} \eta(x,t') dt'$$

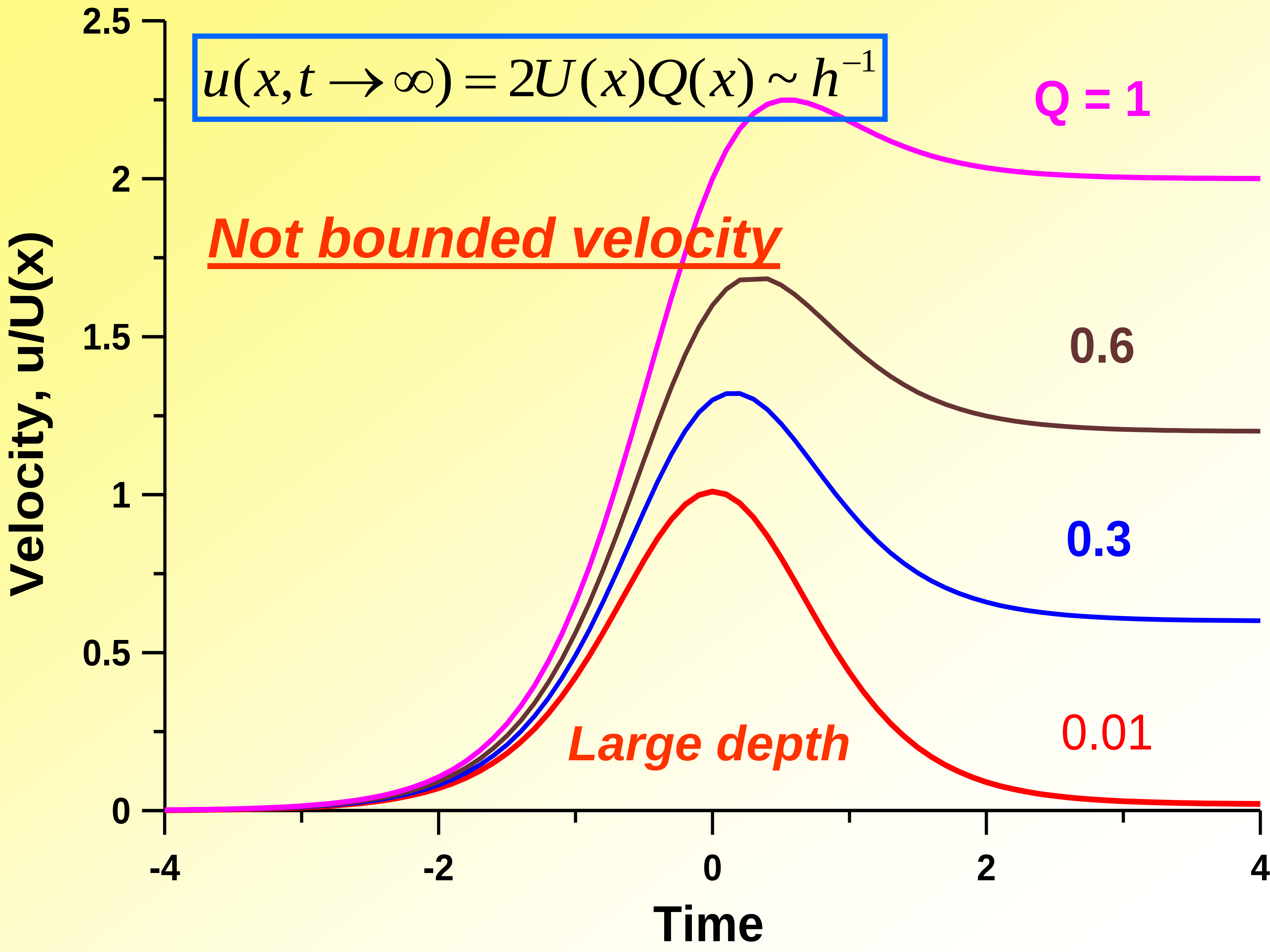
$$\eta(x,t) = A \left[\frac{h_0}{h(x)} \right]^{1/4} \operatorname{sech}^2 \{\Omega[t - \tau(x)]\}$$

$$u(x,t) = U(x) \{\operatorname{sech}^2(T) + Q(x)[\tanh(T) + 1]\}$$

$$U(x) = A \sqrt{\frac{g}{h(x)}} \left[\frac{h_0}{h(x)} \right]^{1/4} \sim h^{-3/4}$$

$$Q(x) = \frac{\sqrt{gh(x)}}{3L\Omega} \left[\frac{h_0}{h(x)} \right]^{3/4} \sim h^{-1/4}$$

WKB amplitude



$u(x,t \to \infty) = 2U(x)Q(x) \sim h^{-1}$
Q = 1
Not bounded velocity
0.6
0.3
0.01
Large depth
Velocity, u/U(x)
Time
2.5
2
1.5
1
0.5
0
-4
-2
0
2
4

Physical solution for sign-variable waves

$$\eta(x,t) = A\left[\frac{h_0}{h(x)}\right]^{1/4} f\left[t - \tau(x)\right]$$

$$u(x) = A\sqrt{\frac{g}{h(x)}}\left[\frac{h_0}{h(x)}\right]^{1/4} f\left[t - \tau(x)\right] + Q(x)\int f(\xi)d\xi$$

Sign-variable pulse $\int_{-\infty}^{+\infty} f(t)dt = 0$

The wave with the largest amplification

Reduction to constant-coefficient wave equation

The solution

$$\eta(x,t) = A(x)H[t,\tau(x)]$$

reduces

$$\frac{\partial^2 H}{\partial t^2} - \frac{\partial^2 H}{\partial \tau^2} = 0$$

If

$$h(x) = x^{4/3}$$

$$A(x) = x^{-1/3} \qquad \tau(x) = 3x^{1/3}$$

It proves uniqueness of the exact travelling wave solutions in inhomogeneous media

Natural condition on the shoreline –

boundedness of water displacement

$$H(\tau = 0, t) = 0$$

As a result, the general solution (Cauchy problem) can be solved

$$\eta(x,t) = \frac{1}{x^{1/3}}\left\{f_+[\tau(x)-t] + f_-[\tau(x)+t] - f_-[-\tau(x)+t]\right\}$$

$$u(x,t) = \sqrt{\frac{g}{p}}\frac{1}{x}\left[f_+(\tau-t) - f\_(\tau+t) - f\_(-\tau+t)\right] -$$

$$-\frac{g}{3x^{4/3}}\left[\Phi_+(\tau-t) - \Phi\_(\tau+t) - \Phi\_(-\tau+t)\right]$$

where

$$\Phi(\xi) = \int f(\xi)d\xi$$

Piston model of wave generation

$$\eta(x,0)=\eta_0(x) \qquad u(x,0)=0$$

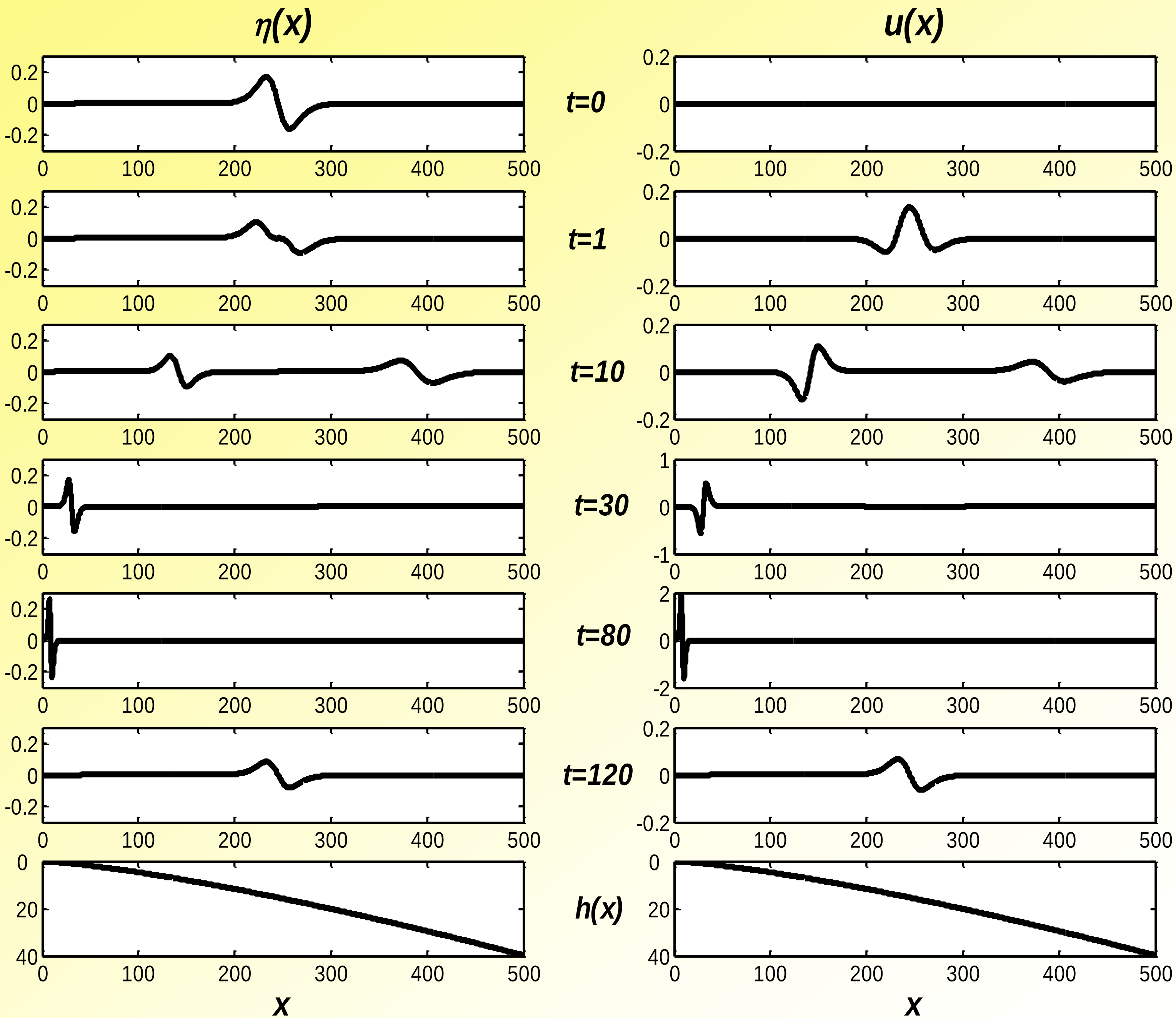
$$\eta(x,t)=\frac{1}{x^{1/3}}\left\{f_0[\tau(x)-t]+f_0[\tau(x)+t]-f_0[-\tau(x)+t]\right\}$$

$$u(x,t)=\sqrt{\frac{g}{p}}\frac{1}{x}\left[f_0(\tau-t)-f_0(\tau+t)-f_0(-\tau+t)\right]-$$

$$-\frac{g}{3x^{4/3}}\left[\Phi_0(\tau-t)-\Phi_0(\tau+t)-\Phi_0(-\tau+t)\right]$$

If the initial disturbance is sign-variable

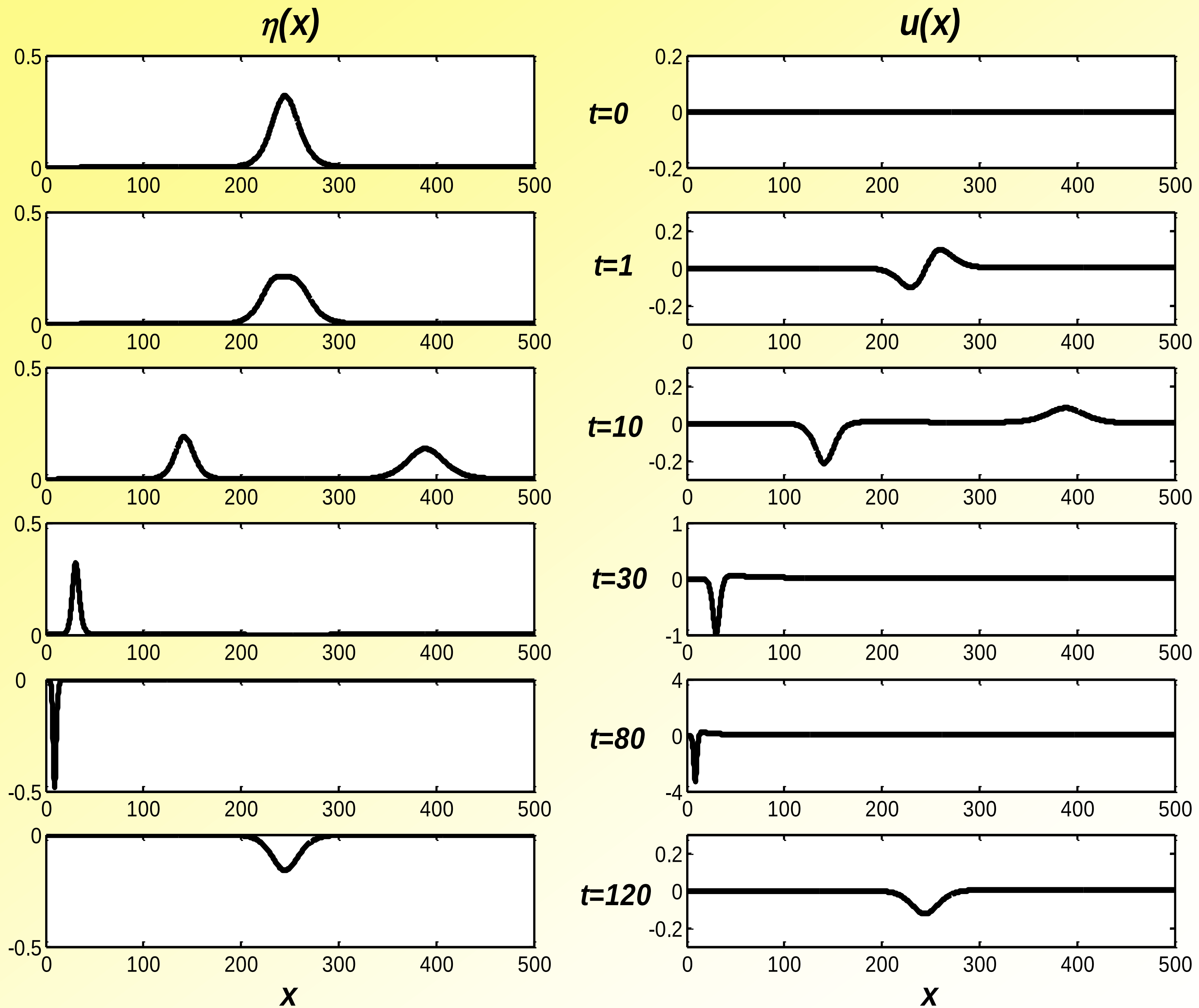
$$f_0(\tau) = -\frac{4}{3}\frac{\tanh[2(\tau-60)/3]}{\cosh^2[2(\tau-60)/3]}$$



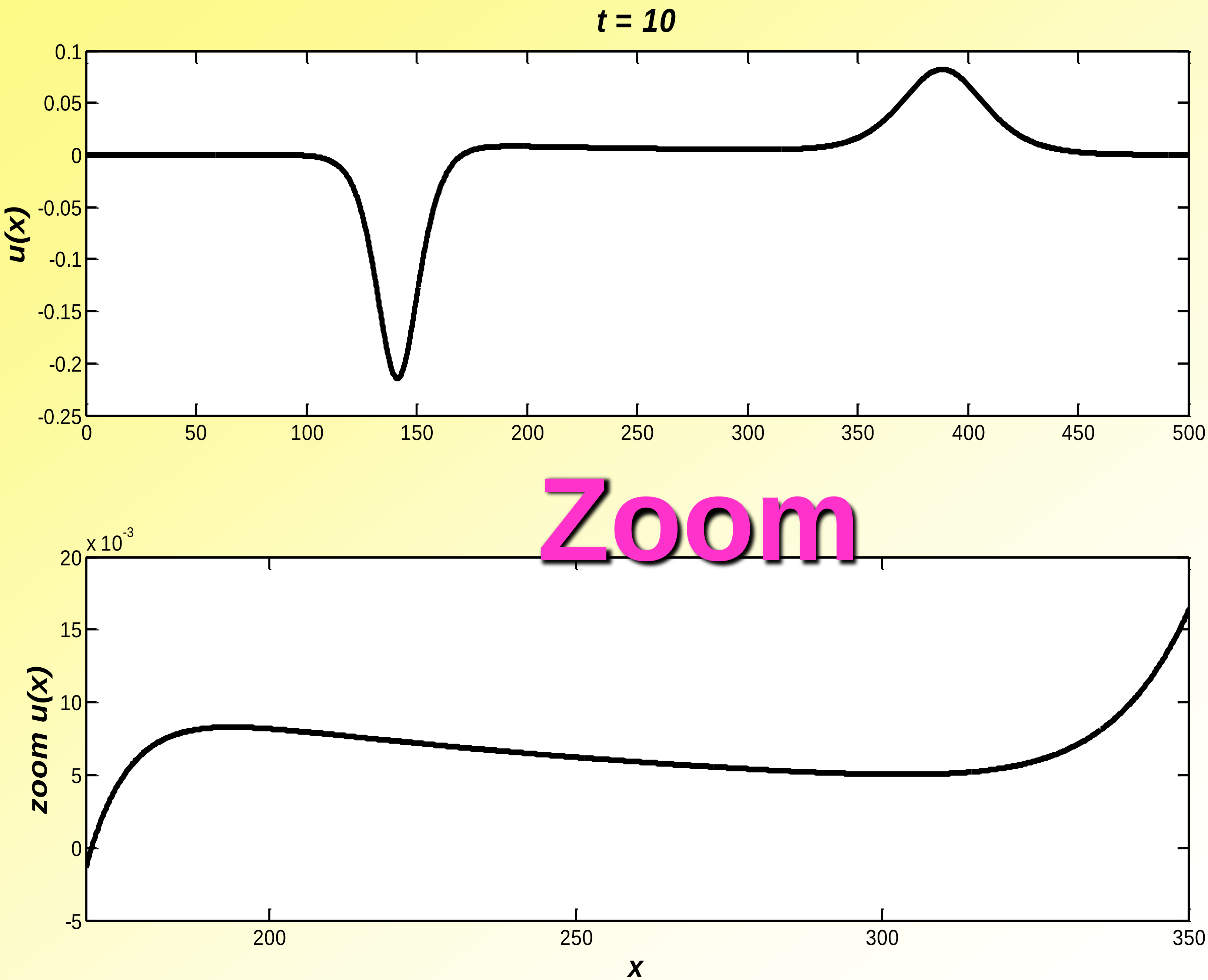
like constant depth

If the initial disturbance is sign-constant

$$f_0(\tau) = \operatorname{sech}^2[2(\tau - 60)/3]$$



Relict current



t = 10
u(x)
0.1
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
0
50
100
150
200
250
300
350
400
450
500
Zoom
x 10^-3
zoom u(x)
20
15
10
5
0
-5
200
250
300
350
x

Runup on a beach $x^{4/3}$

$$\eta(x,t) = A_0 \left[\frac{h_0}{h(x)} \right]^{1/4} \left\{ f[t+\tau(x)] - f[t-\tau(x)] \right\}$$

$$\tau(x) = \int_{-L}^{x} \frac{dy}{\sqrt{gh(y)}} = \frac{3L}{\sqrt{gh_0}} \left[\frac{h(x)}{h_0} \right]^{1/4}$$

Bounded at the shoreline x = 0 (runup)

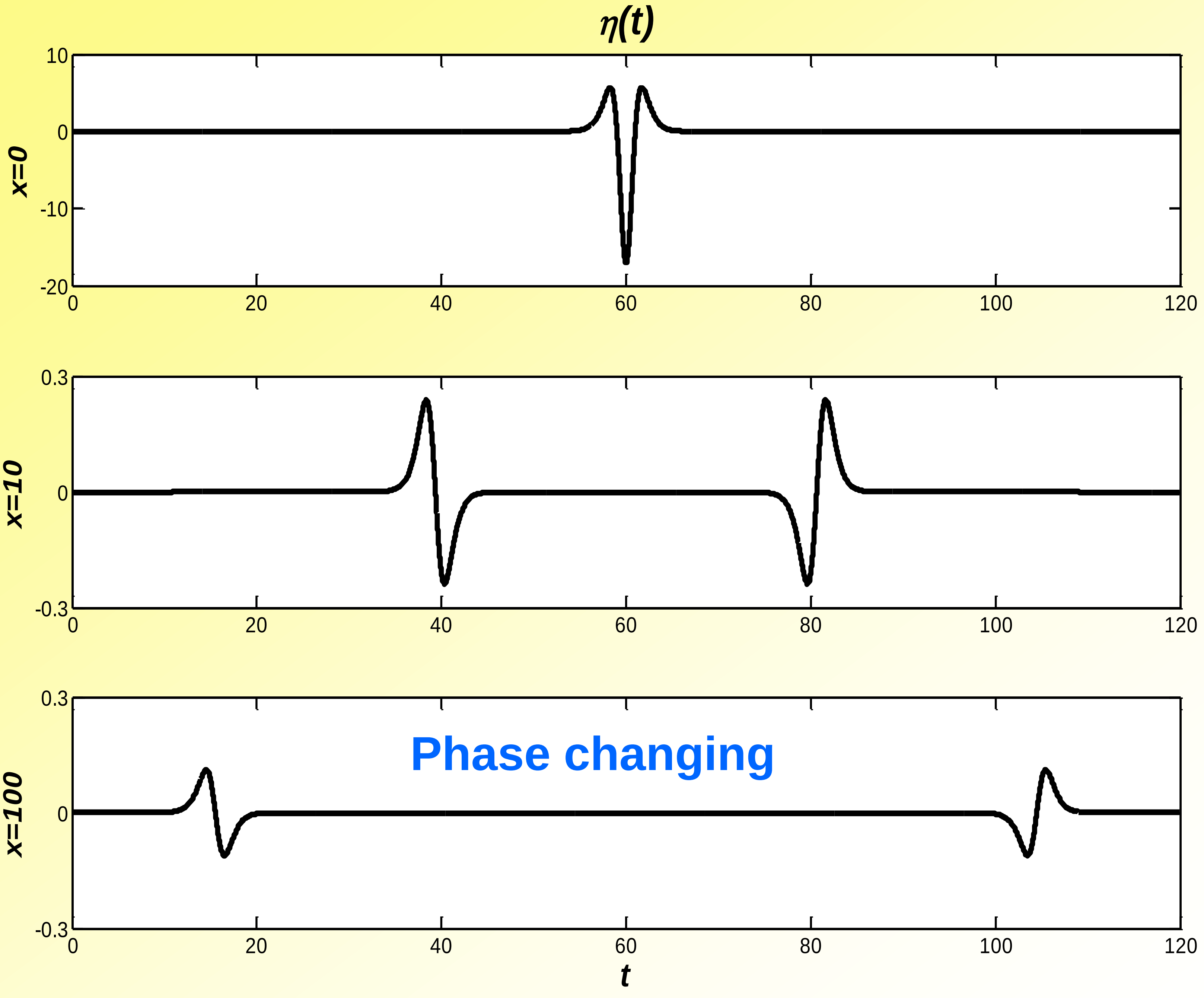
$$R(t) = \eta(x=0,t) = 2\tau_0 \frac{df(t+\tau_0)}{dt}$$

Velocity field at the shoreline

$$u(x \to 0, t) \sim \frac{f(t+\tau_0)}{x}$$

Water discharge

$$h(x)u(x,t) \to x^{1/3} f(t+\tau_0) \to 0$$



$\eta(t)$
x=0
x=10
x=100
Phase changing
t

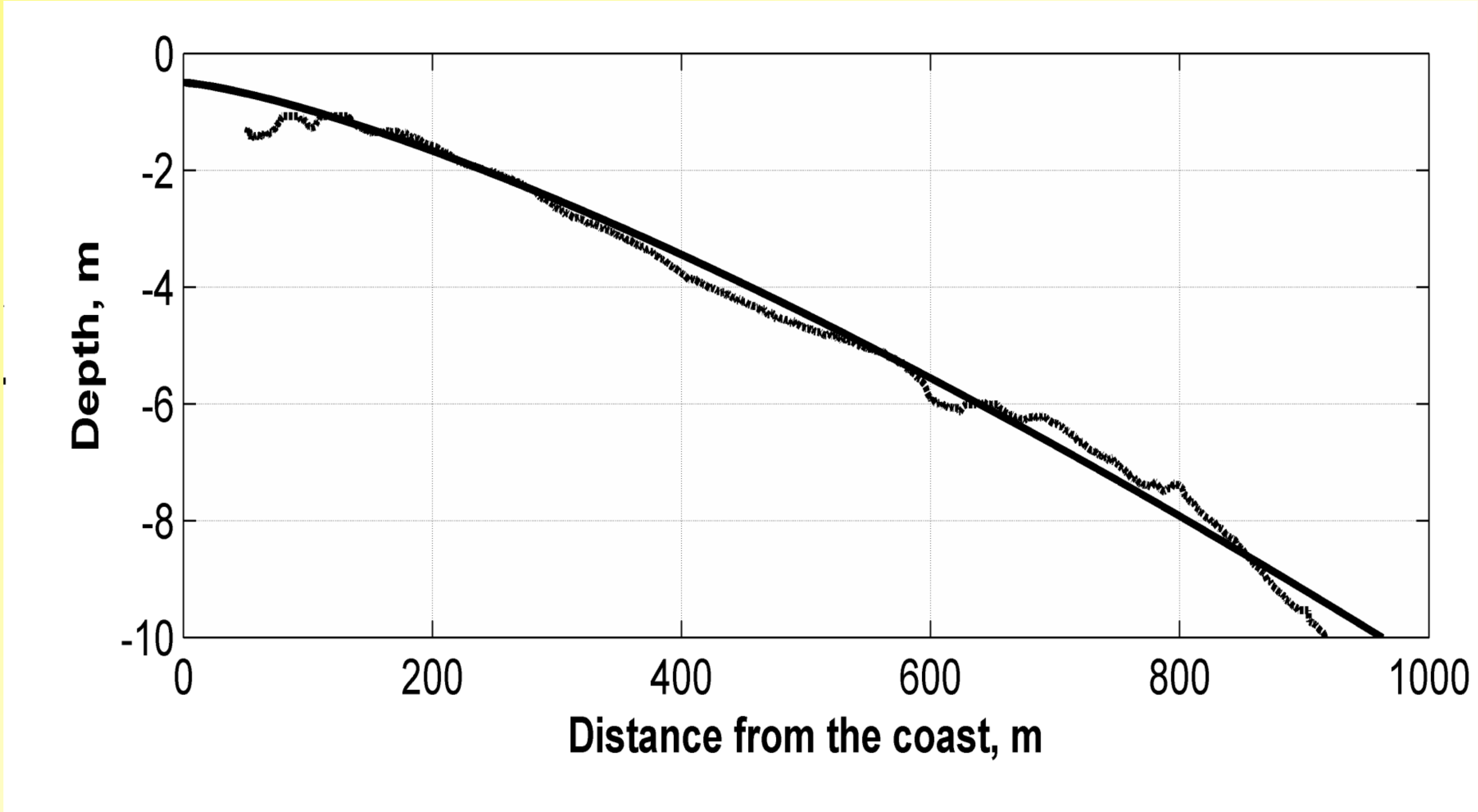
Runup of a soliton

$$R_{max} = 4\frac{A}{\alpha}\sqrt{\frac{A}{h}} \sim A^{3/2}$$

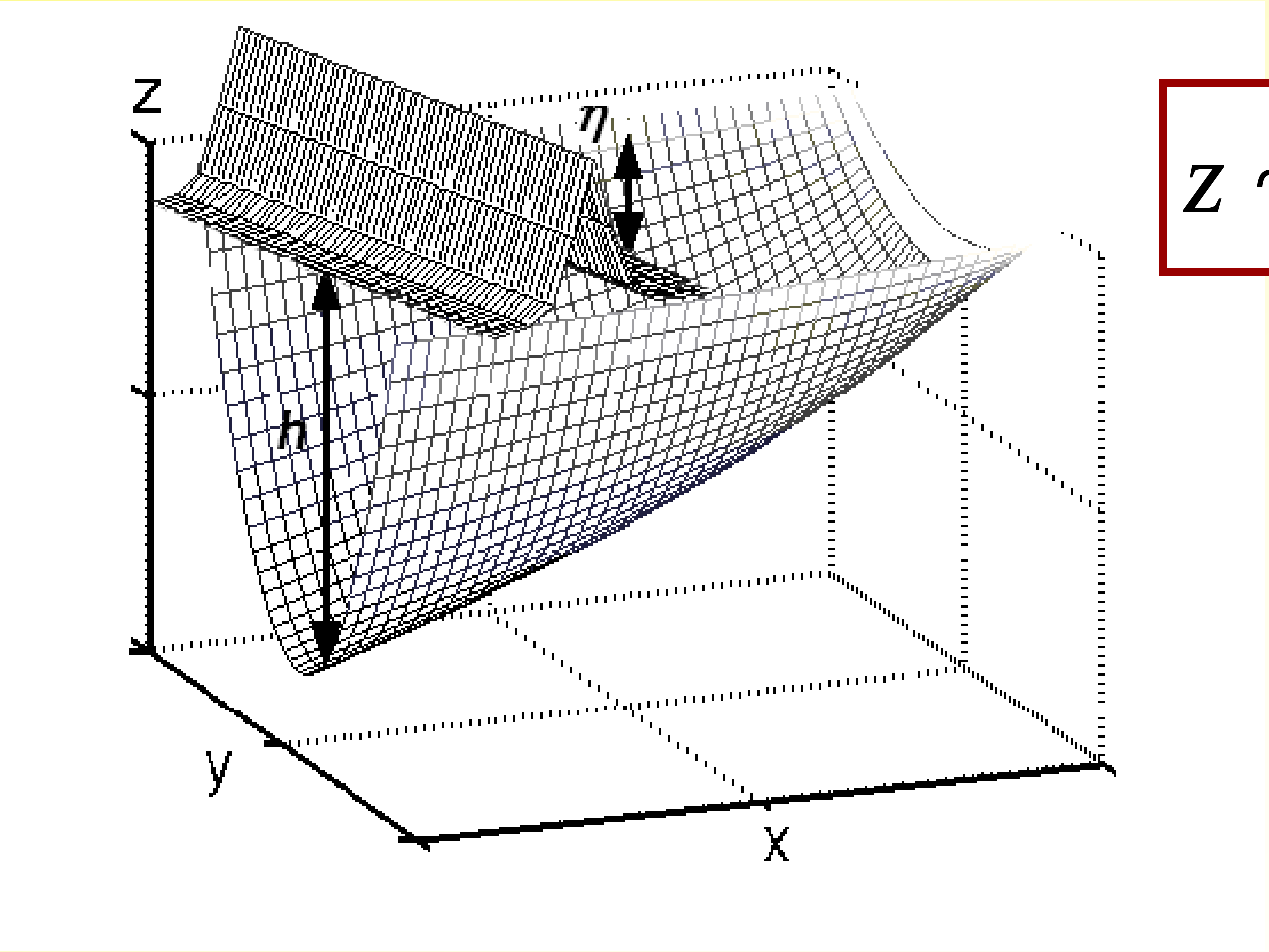
Plane Beach

$$R_{max} = 2.8312\frac{A}{\sqrt{\alpha}}\left(\frac{A}{h}\right)^{1/4} \sim A^{5/4}$$

Pirita beach, Tallinn, Estonia



Traveling waves in narrow bays



$$z \sim |y|^m$$

Basic Equations

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \eta}{\partial x}=0$$

$$\frac{\partial S}{\partial t}+\frac{\partial}{\partial x}\left(Su\right)=0$$

η- water displacement, u - depth-averaged flow,
S - variable water cross-section of the channel

$$S \sim H^q$$

$$q=\frac{m+1}{m}$$

$$\frac{\partial H}{\partial t}+u\frac{\partial H}{\partial x}+\frac{H}{q}\frac{\partial u}{\partial x}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial H}{\partial x}=g\frac{\partial h}{\partial x}$$

Linear problem

$$\frac{\partial^2 \eta}{\partial t^2} - g\frac{dh}{dx}\frac{\partial \eta}{\partial x} - \frac{gh}{q}\frac{\partial^2 \eta}{\partial x^2} = 0$$

Traveling wave solution

$$\eta(x,t) = A(x)\, f\left[t - \tau(x)\right]$$

$$A \sim h^{-\left(\frac{1}{4}+\frac{1}{2m}\right)}$$

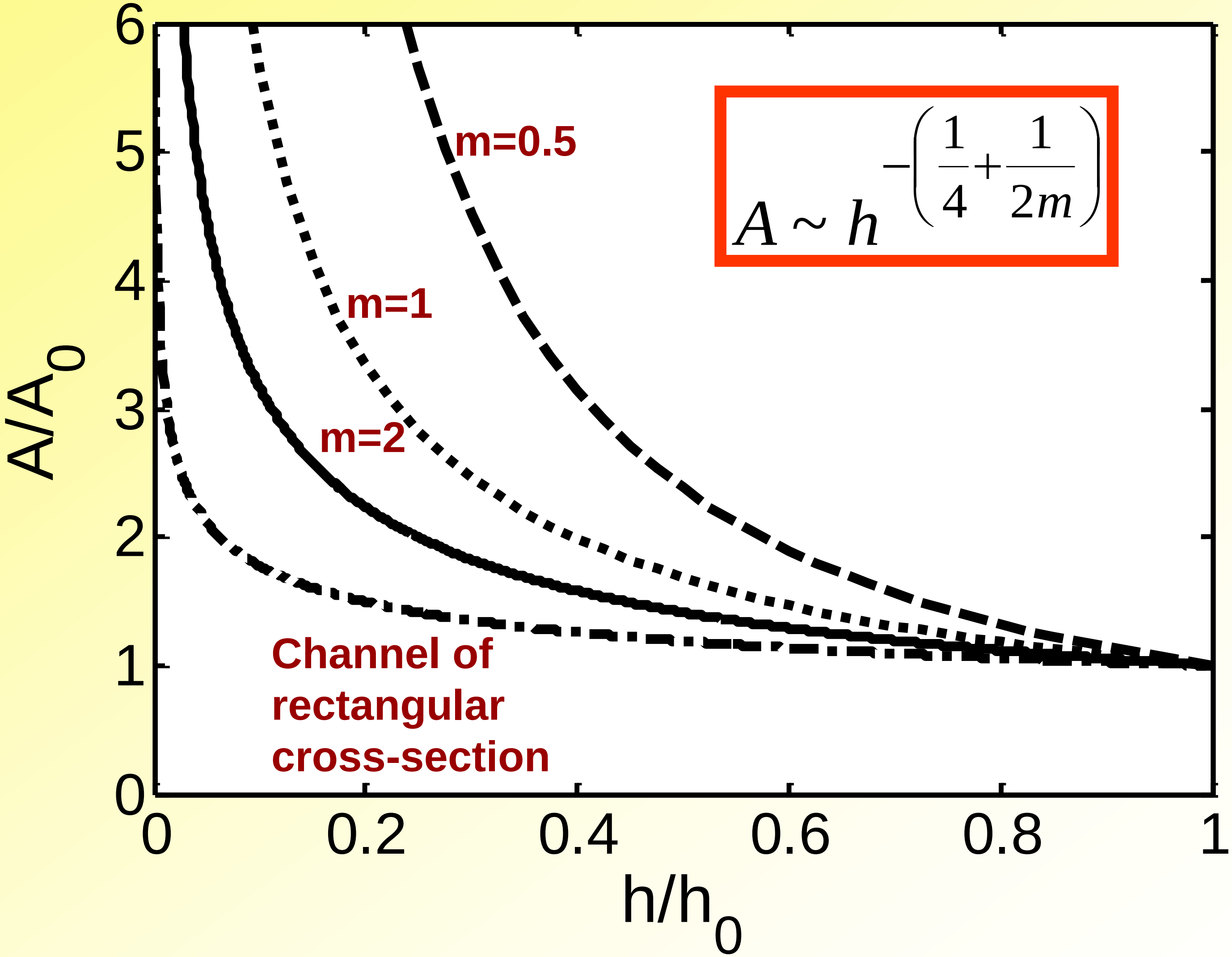
Generalized Green's law

$$\tau(x) = \int \frac{dx}{c(x)}$$

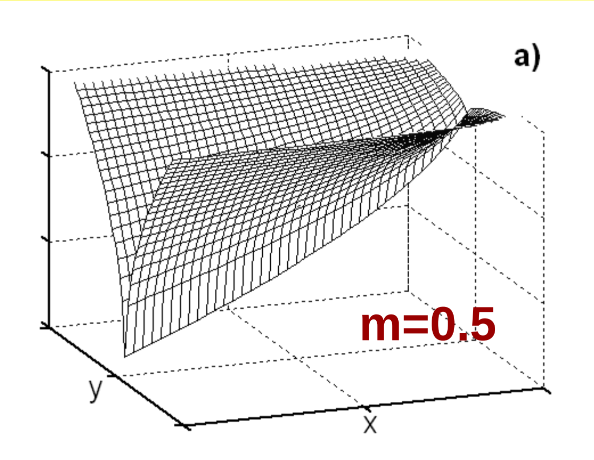
Travel time

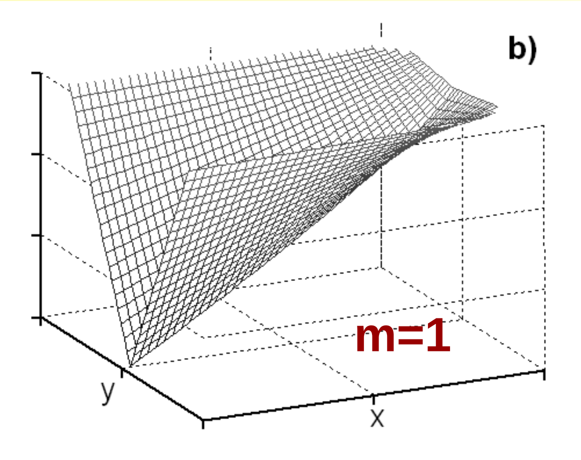
$$c(x) = \sqrt{gh(x)/q}$$

Shoaling effects in U-shaped bays



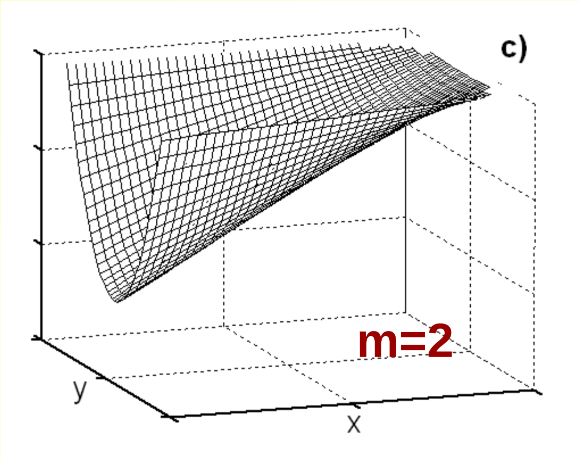
Shapes of “non-reflecting” bays

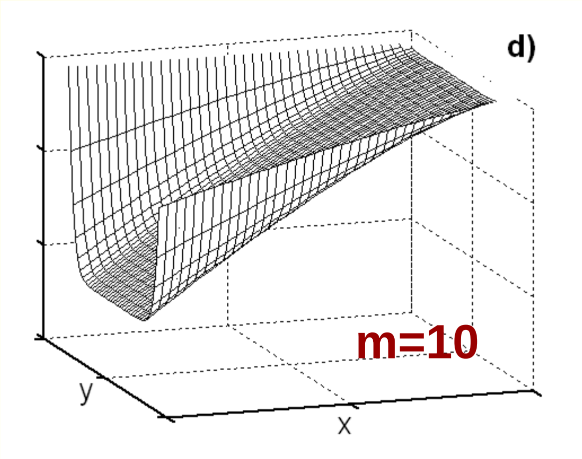




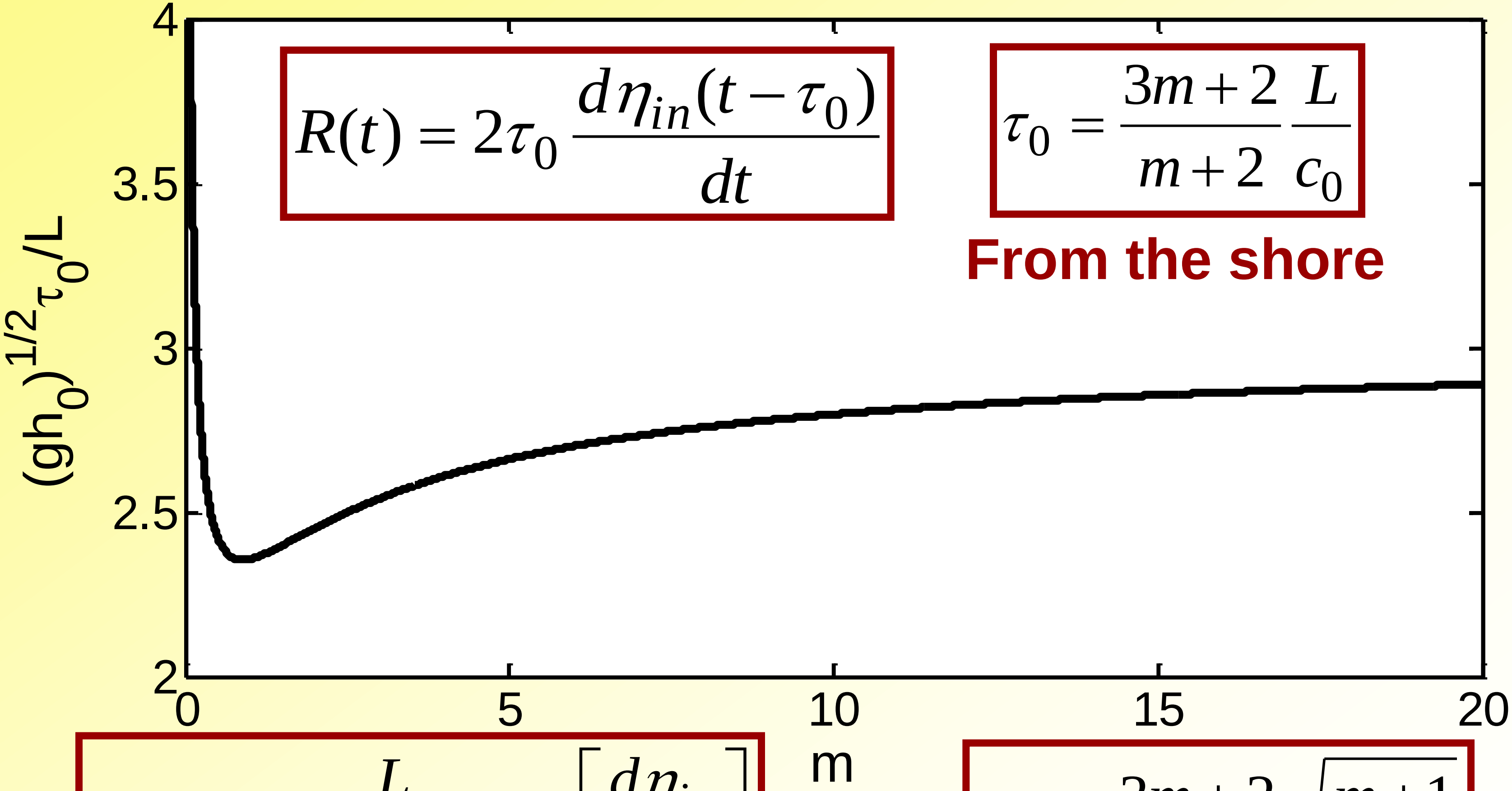
$$h(x) \sim x^{\frac{4m}{3m+2}}$$

$$z \sim |y|^{m}$$





Wave runup in U-shaped bays



$$R(t) = 2\tau_0 \frac{d\eta_{in}(t-\tau_0)}{dt}$$

$$\tau_0 = \frac{3m+2}{m+2}\frac{L}{c_0}$$

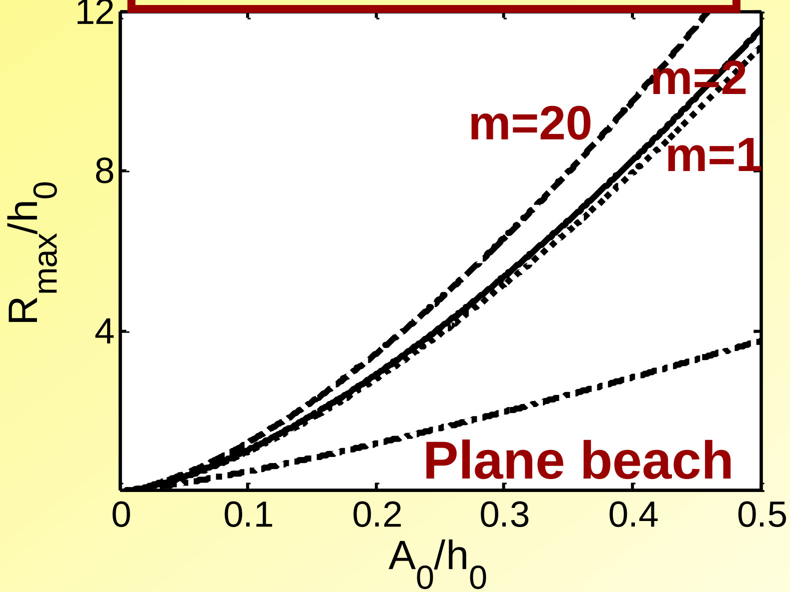
From the shore

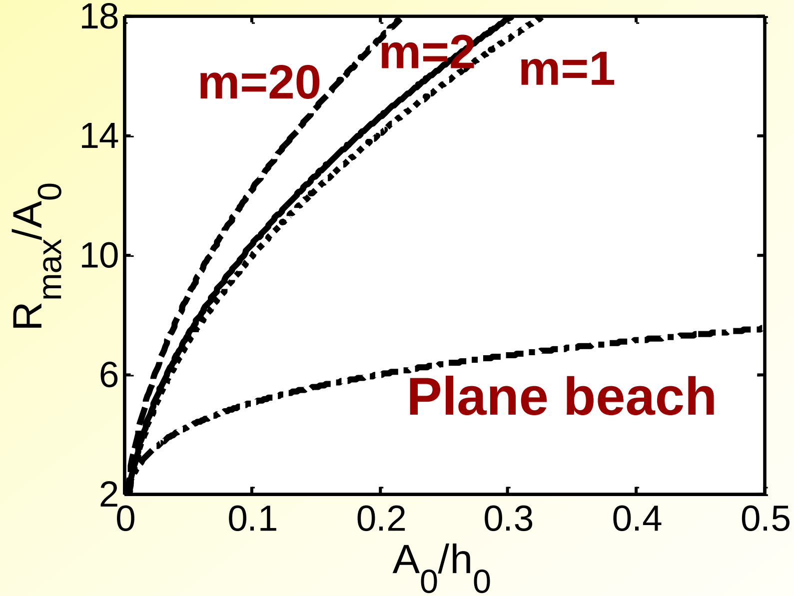
$$R_{\max} = \mu \frac{L}{\sqrt{gh_0}} \max\left[\frac{d\eta_{in}}{dt}\right]$$

$$\mu = 2\frac{3m+2}{m+2}\sqrt{\frac{m+1}{m}}$$

Solitary wave runup

$$\eta(t) = A_0 \operatorname{sech}^2\left[\sqrt{\frac{3A_0 g}{4h_0^2}}\,t\right]$$





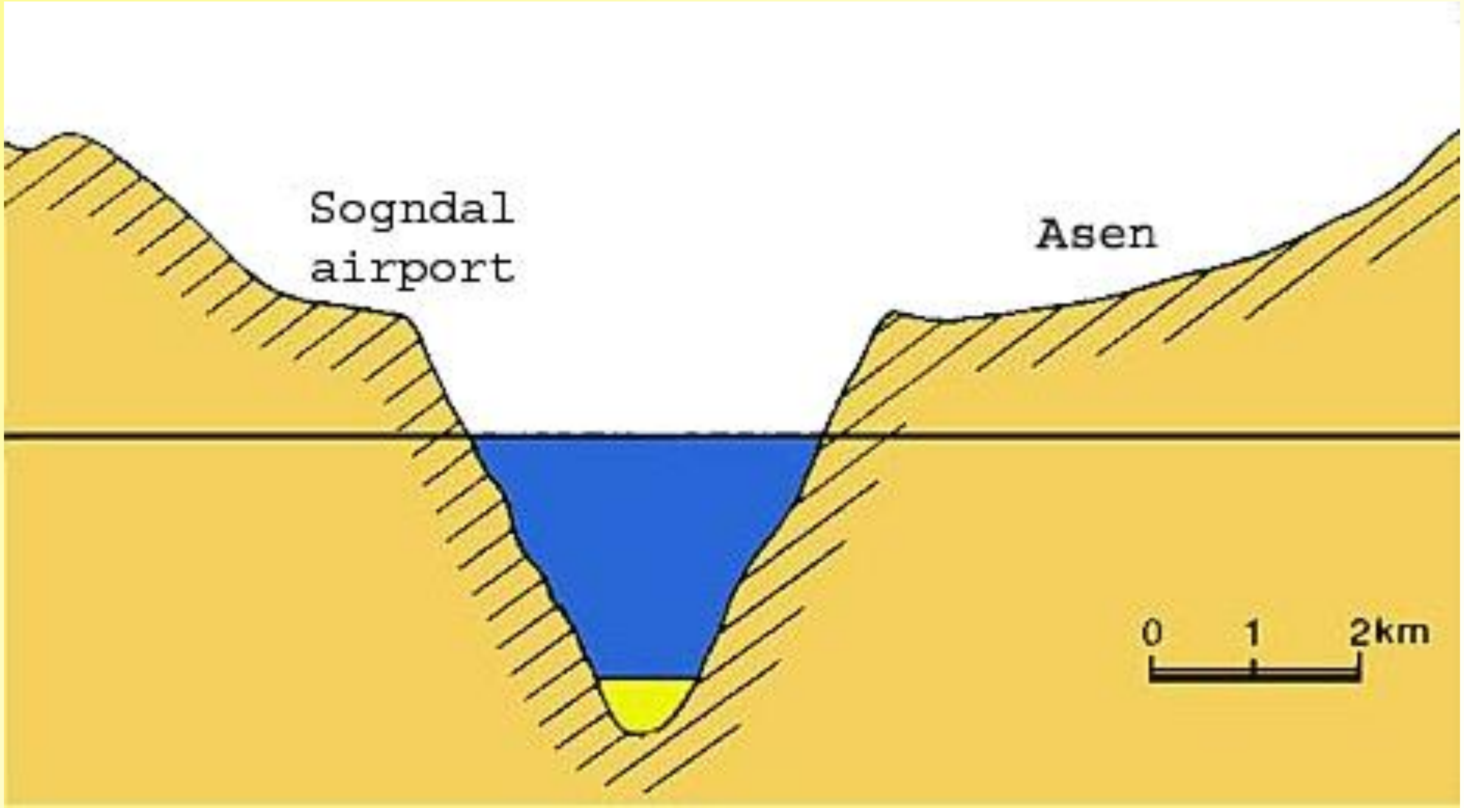
$$R_{max} = \frac{2}{3}\mu L\left(\frac{A_0}{h_0}\right)^{3/2}$$

Narrow bays

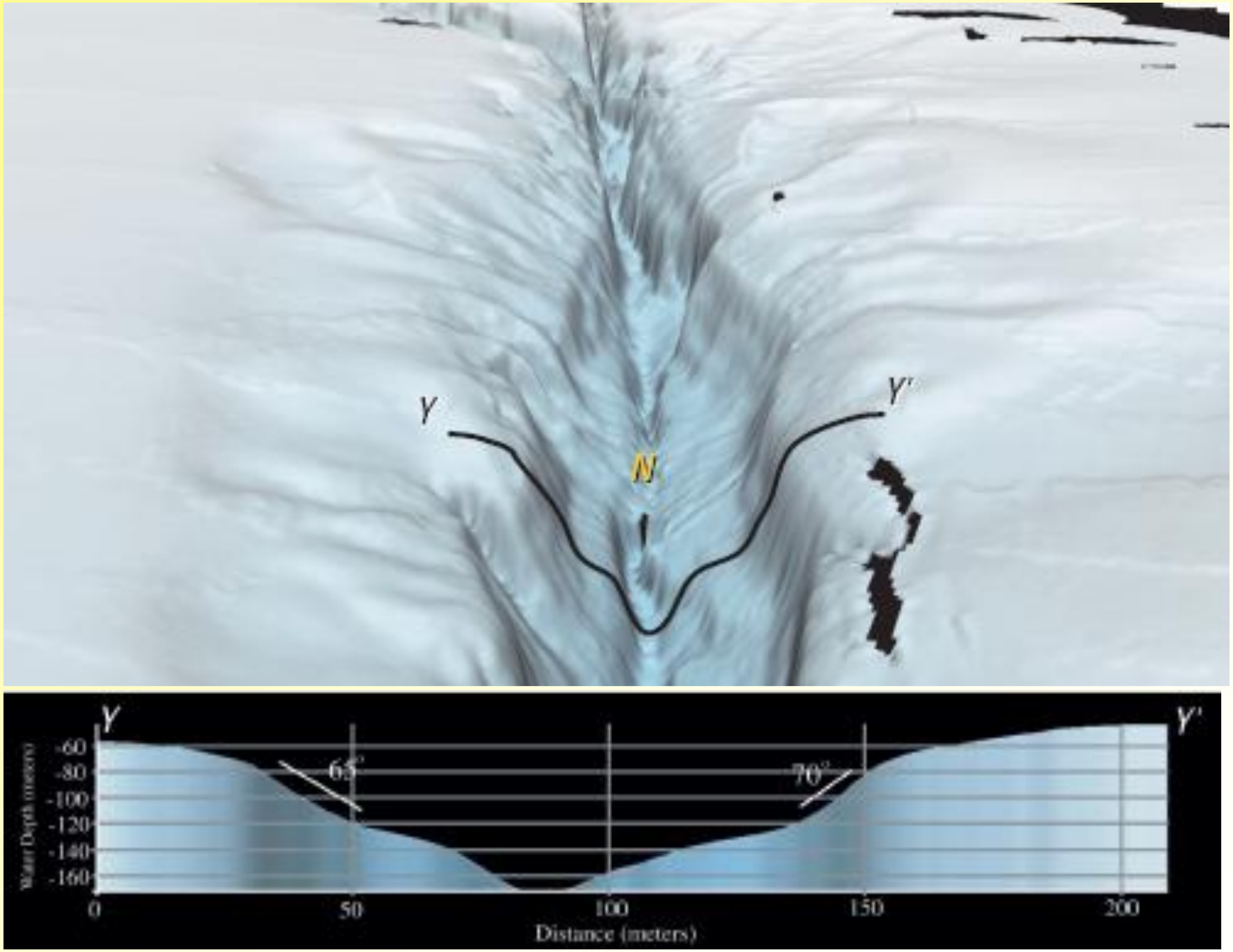
$$R_{max} = 2.8312\sqrt{Lh_0}\left(\frac{A_0}{h_0}\right)^{5/4}$$

Plane beach

Sognefjoren fjord (Norway)

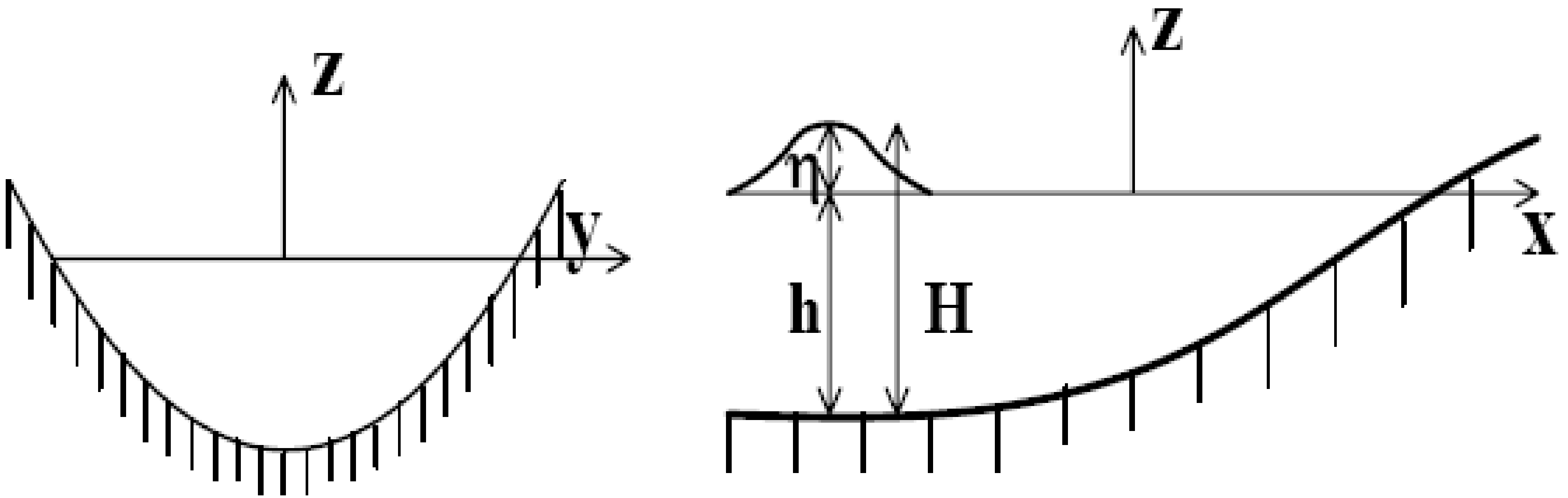


Scripps Canyon (California)



Nonlinear Traveling Waves

Example: inclined channel of parabolic cross-section



Basic Equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial \eta}{\partial x} = 0$$

$$\frac{\partial S}{\partial t} + \frac{\partial}{\partial x}(Su) = 0$$

η- water displacement, u - depth-averaged flow,
S - variable water cross-section of the channel

For parabolic channel $S \sim H^{3/2}$

$$\frac{\partial H}{\partial t} + u\frac{\partial H}{\partial x} + \frac{2H}{3}\frac{\partial u}{\partial x} = 0$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial H}{\partial x} = g\alpha$$

Nonlinear Problem

$$I_{\pm} = u \pm 2\sqrt{\frac{3}{2}gH} + \alpha g t$$

$$\frac{\partial I_{\pm}}{\partial t} + c_{\pm}\frac{\partial I_{\pm}}{\partial x} = 0$$

Riemann invariants

$$c_{\pm} = \frac{2}{3}I_{\pm} + \frac{1}{3}I_{\mp} - \alpha g t$$

Legendre (hodograph) transformation

$$\frac{\partial^2 t}{\partial I_+ \partial I_-} + \frac{2}{I_+ - I_-}\left(\frac{\partial t}{\partial I_-} - \frac{\partial t}{\partial I_+}\right) = 0$$

New variables

$$\sigma = \frac{I_+ - I_-}{2} = \sqrt{6gH}$$

$$\lambda = \frac{I_+ + I_-}{2} = u + \alpha g t$$

And the final linear system

$$\frac{\partial^2\Phi}{\partial\lambda^2}-\frac{\partial^2\Phi}{\partial\sigma^2}-\frac{2}{\sigma}\frac{\partial\Phi}{\partial\sigma}=0$$

$$\sigma\geq 0$$

$$u=\frac{1}{\sigma}\frac{\partial\Phi}{\partial\sigma}$$

$$\eta=\frac{1}{2g}\left[\frac{2}{3}\frac{\partial\Phi}{\partial\lambda}-u^2\right]$$

$$x=\frac{\eta}{\alpha}-\frac{\sigma^2}{6g\alpha}$$

$$t=\frac{\lambda-u}{g\alpha}$$

$$\frac{\partial^2 \Phi}{\partial \lambda^2} - \frac{\partial^2 \Phi}{\partial \sigma^2} - \frac{2}{\sigma}\frac{\partial \Phi}{\partial \sigma} = 0$$

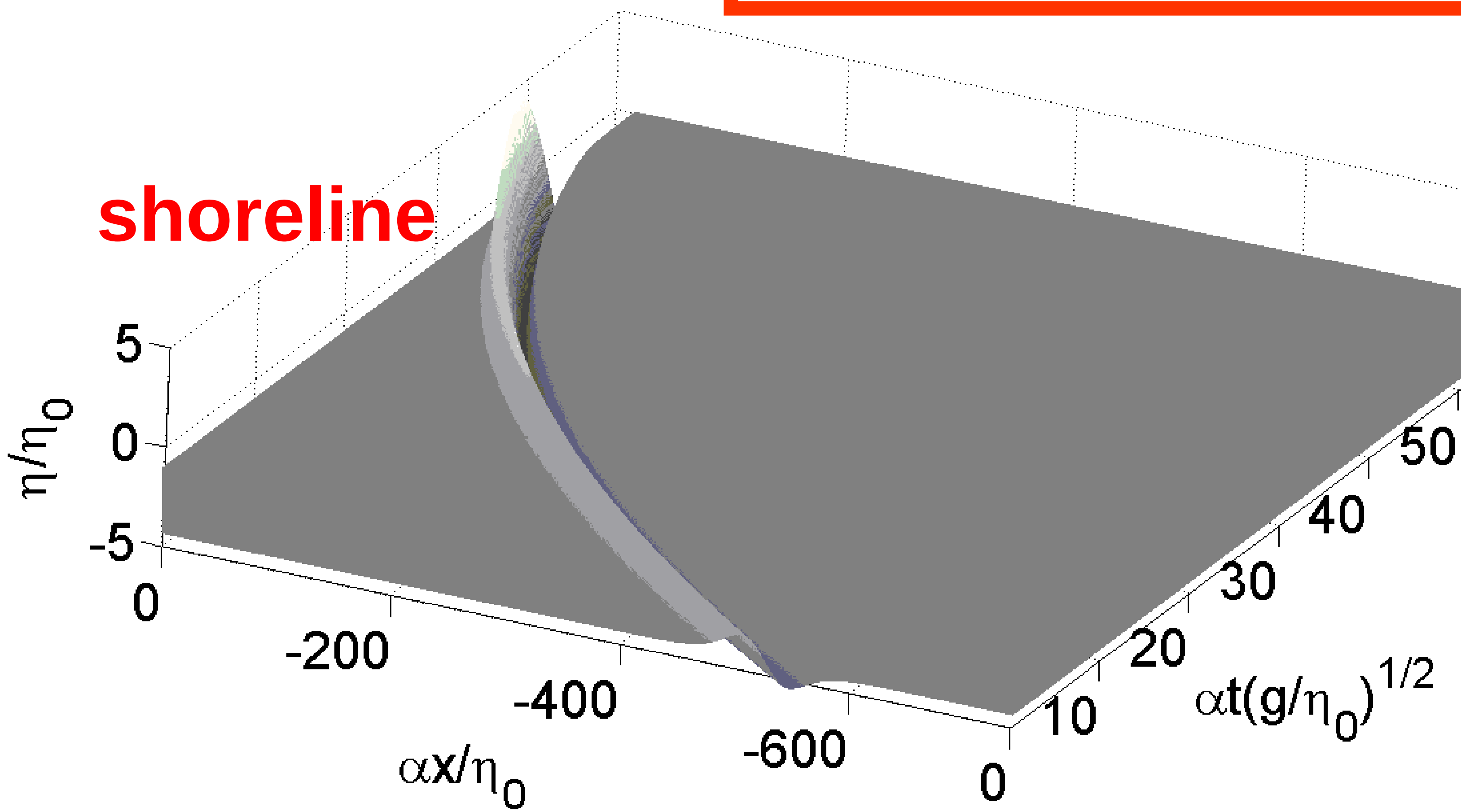
$$\sigma \geq 0$$

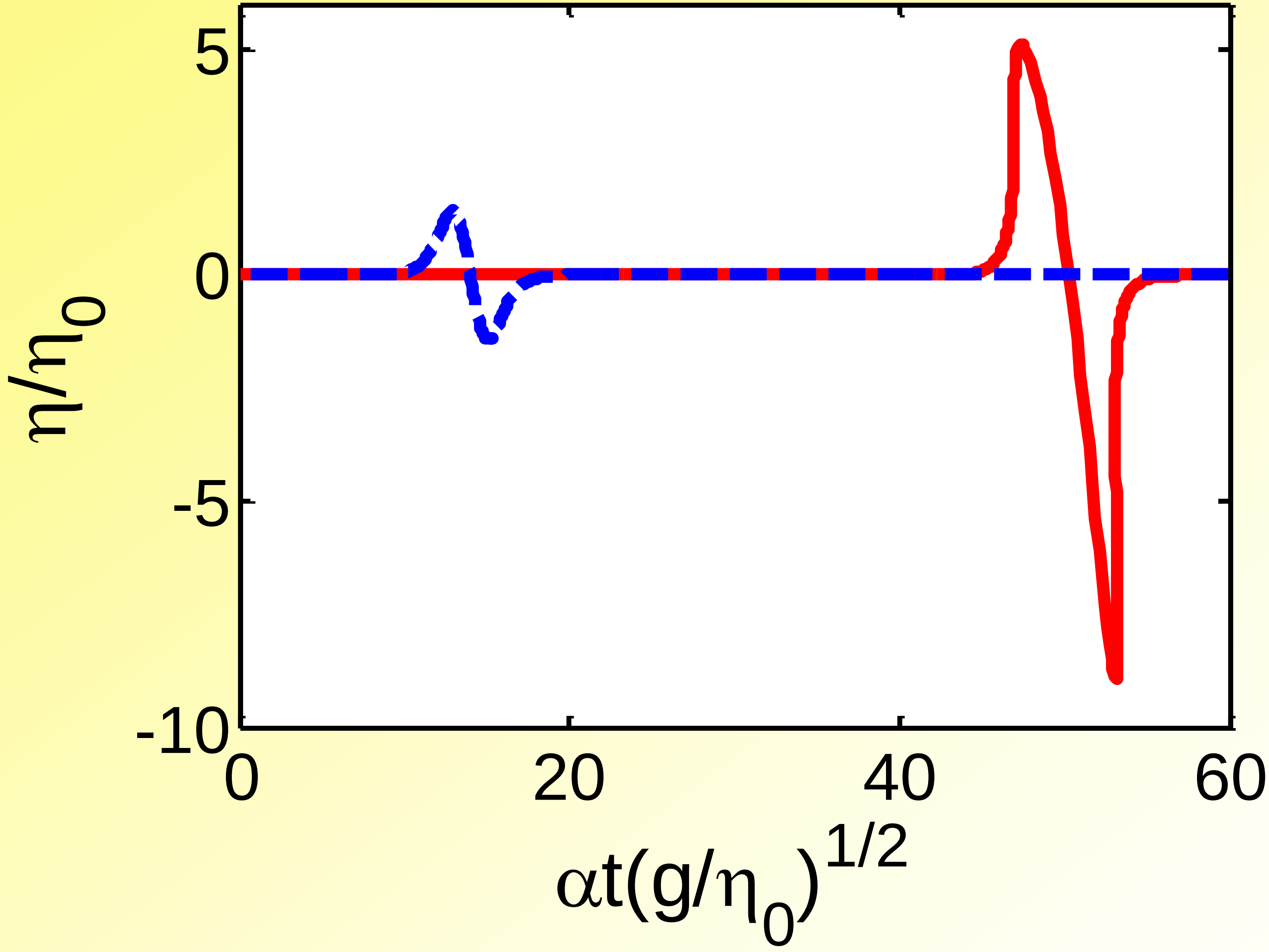
General solution for traveling waves

$$\Phi(\sigma, \lambda) = \frac{\Theta_1(\lambda + \sigma) + \Theta_2(\lambda - \sigma)}{\sigma}$$

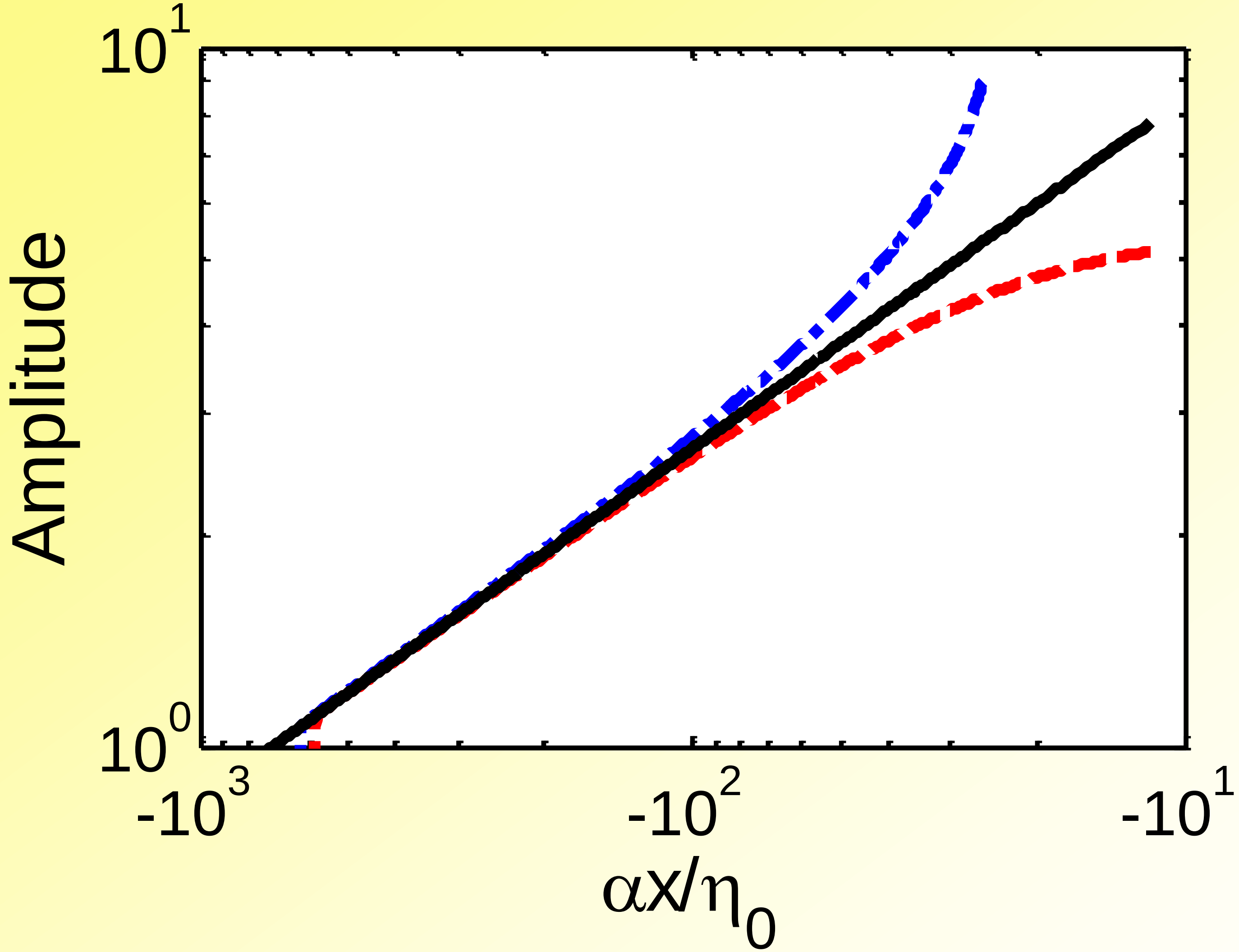
Nonlinear traveling wave

$$\Phi(\sigma,\lambda)=\frac{\Theta(\lambda+\sigma)}{\sigma}$$

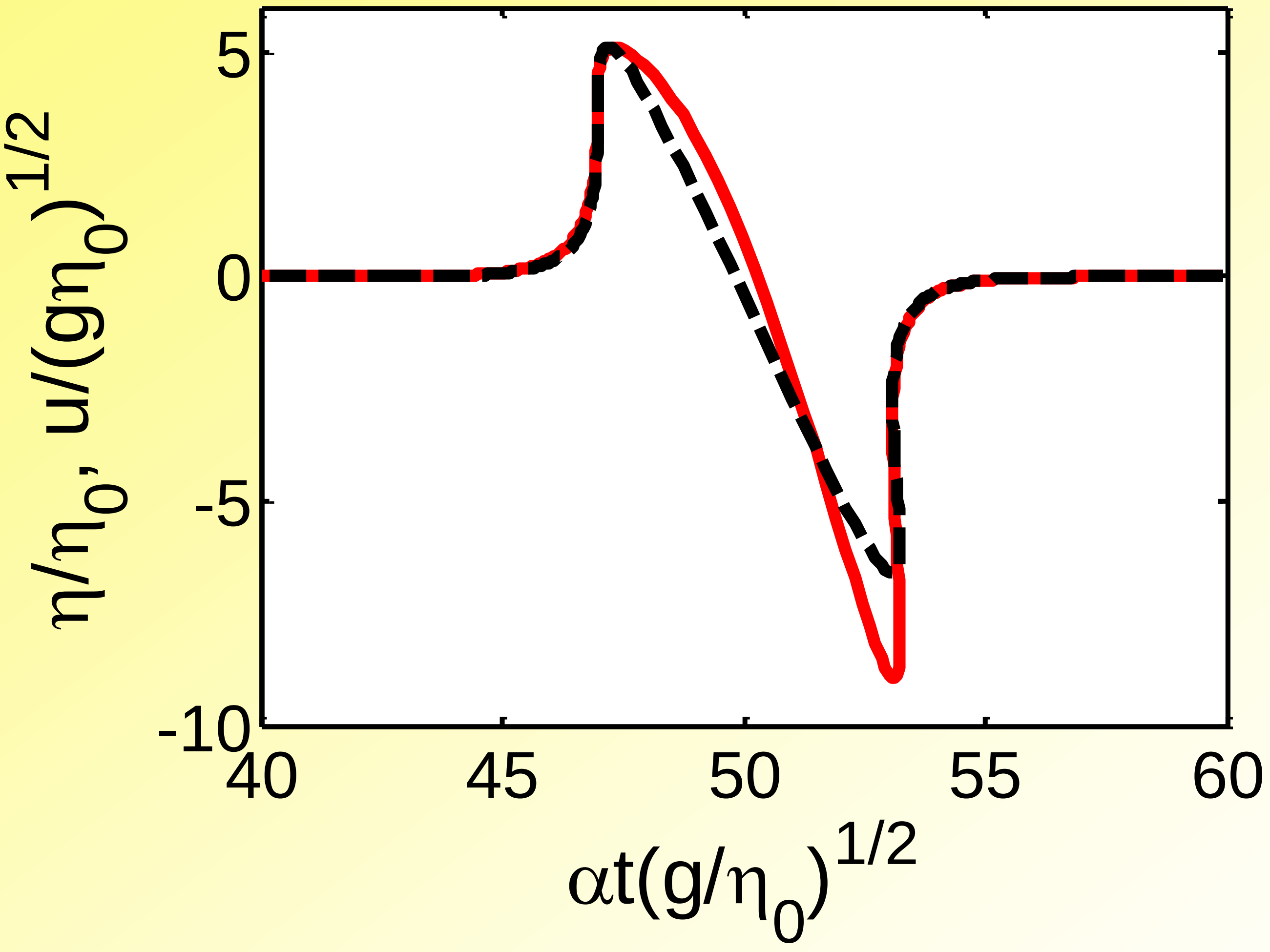




Deformation of the wave shape in approaching wave: blue and red lines correspond to an incident wave and the wave near the shoreline respectively



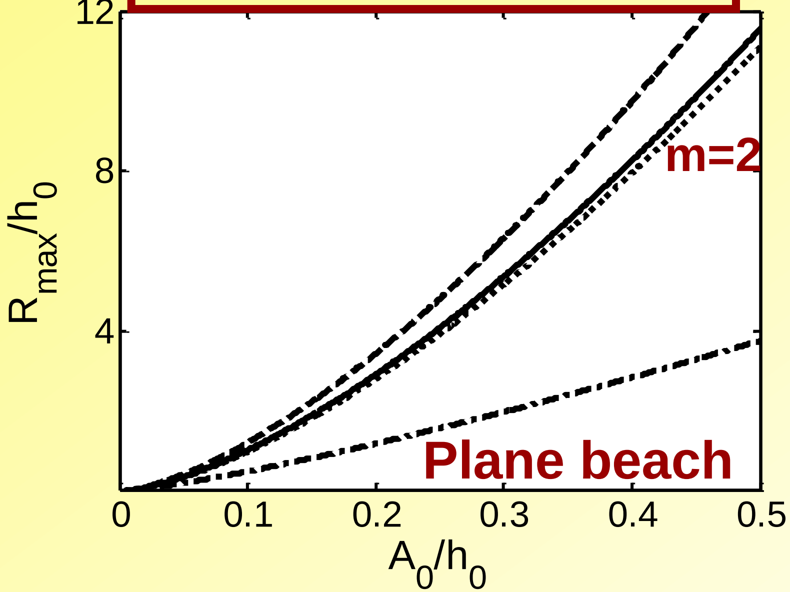
Variation of the positive (red) and negative (blue) amplitudes with distance; black solid line corresponds to the linear Green's law

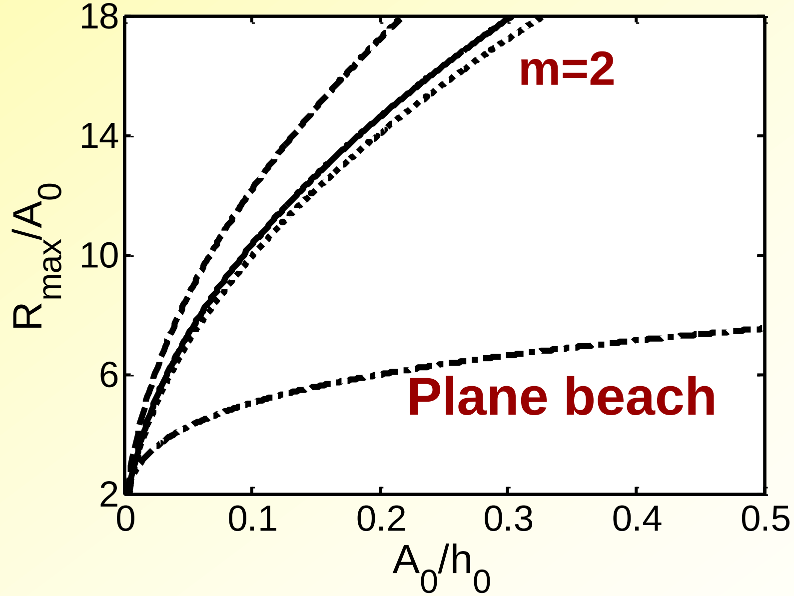


Shapes of water displacement (red) and velocity (black) near the shoreline

Solitary wave runup

$$\eta(t) = A_0 \operatorname{sech}^2\left[\sqrt{\frac{3A_0 g}{4h_0^2}}\, t\right]$$





$$R_{max} = \frac{8}{3}\sqrt{\frac{3}{2}} L \left(\frac{A_0}{h_0}\right)^{3/2}$$

Parabolic bay

$$R_{max} = 2.8312\sqrt{L h_0}\left(\frac{A_0}{h_0}\right)^{5/4}$$

Plane beach

References:

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2. Didenkulova I., Pelinovsky E. and Soomere T. Long wave dynamics along a convex bottom. *J. Geophys. Res.* 2009 114, C07006.
3. Didenkulova I. and Pelinovsky E. Non-dispersive traveling waves in strongly inhomogeneous water channels. *Phys. Lett. A* 2009 373(42), 3883-3887.
4. Didenkulova I., Pelinovsky E. and Soomere T. Exact travelling wave solutions in strongly inhomogeneous media. *Estonian J. Eng.* 2008 14(3), 220-231.
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Conclusions:

“Non-reflecting” configurations:

1. Do EXIST in both linear and nonlinear cases
2. Give LARGE wave amplification and runup
3. Give SIMPLE algorithm for computing wave propagation above complicated bottom relief


Never underestimate the
unpredictability of rough seas!