# Quasilinear Hyperbolic Systems, Nonlinear Superposition and Solitons

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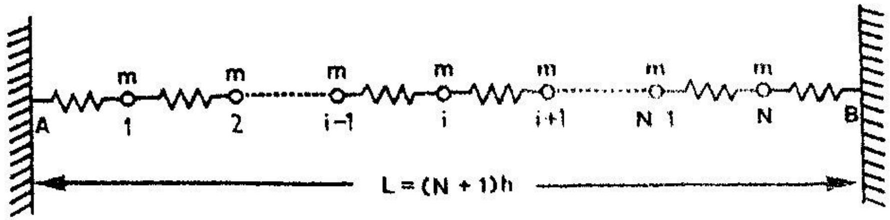


Fig.1. A one-dimensional nonlinear lattice

$$\frac{\partial^2 y}{\partial t^2} = c^2 \left[ 1 + \alpha(1 + p) h^p \left( \partial y / \partial x \right)^p \right] \frac{\partial^2 y}{\partial x^2},$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^3 u}{\partial x^3} = 0,$$

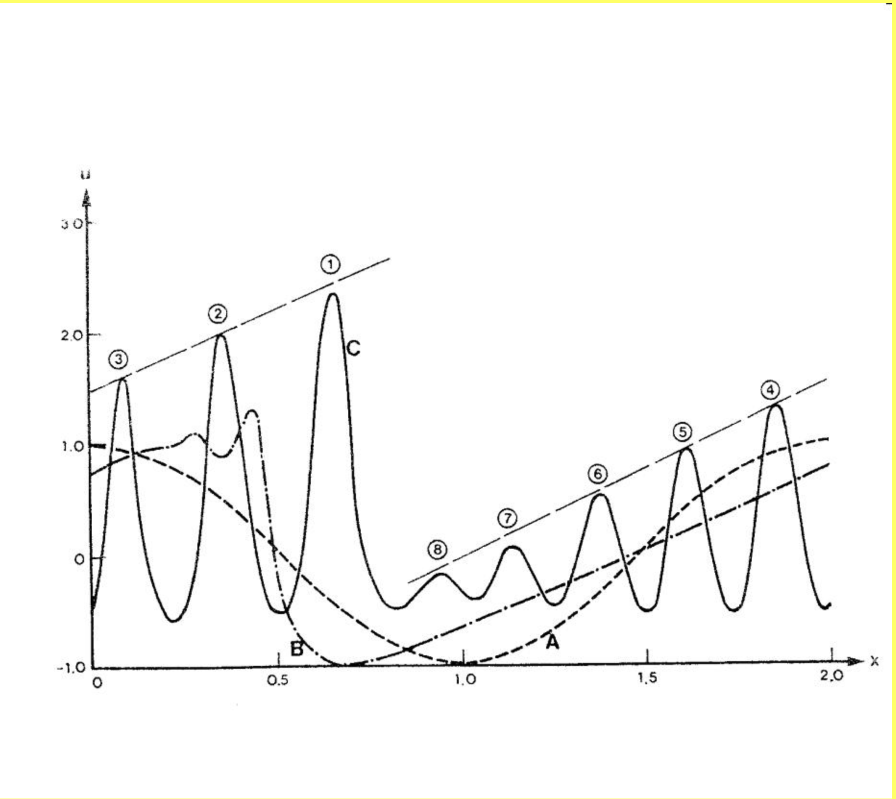


Fig.2 The asymptotic evolution of a train of solitons from an initial sinusoid

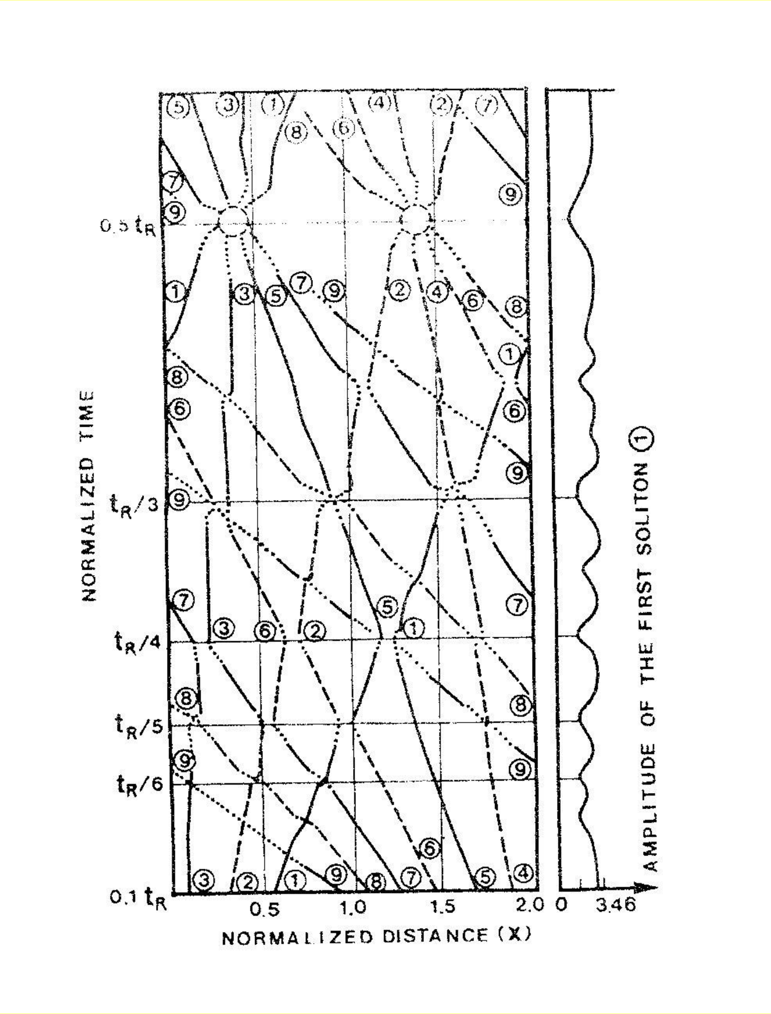


Fig.3. Paths of interacting solitons computed by Zabusky and Kruskal

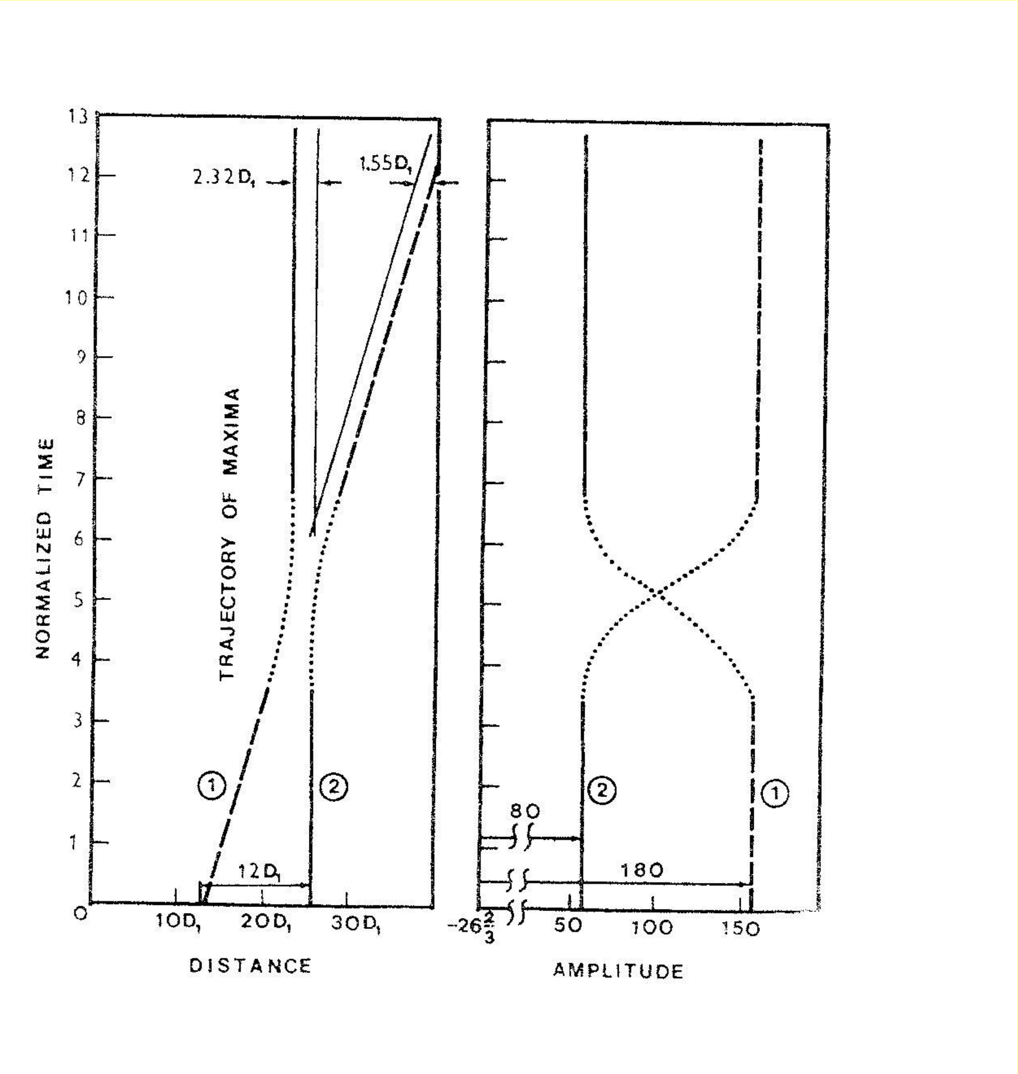


Fig.4 The preservation of soliton shape after soliton interaction

# The Development of Discontinuous Solutions

A natural starting point for this brief review of discontinuous solutions is the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}, \qquad (\mu > 0)$$

$$\frac{\partial u}{\partial t} + \frac{1}{2}\left(u^2\right)_x = 0.$$

## A Weak Solution of Burgers Equation

Let $u_\mu$ be a solution of Burgers equation, then

$$(u_\mu)_t + u_\mu (u_\mu)_x + \mu (u_\mu)_{xx} = 0 \qquad (\mu < 0).$$

With $\varphi$ a $C^1$ test function with compact support, multiply Burgers equation by $\varphi$ and integrate over the half-plane $t > 0$ to switch the derivatives from $u$ to $\varphi$, giving

$$\iint \left( \varphi_t u_\mu + \tfrac{1}{2} \varphi_X u_\mu^2 + \mu \varphi_{xx} u_\mu \right) dx dt = 0.$$

As $\mu \to 0$ this implies that

$$\iint \left( \varphi_t u_\mu + \frac{1}{2} u_\mu^2 \right) dx dt = 0.$$

This is the condition that Burgers equation is satisfied by $u_\mu$ in the weak sense. Classical solutions are a special case of weak solutions, and both satisfy the Rankine-Hugoniot jump condition.

$$u_t + f(u)_x \equiv u_t + f'(u)u_x = 0 \quad \text{(first conservation law)}$$

$$v(u)_t + f'(u)v(u)_x = 0 \quad \text{(second conservation law)}$$

$$v(u)_t + F(u)_x = 0 \quad \text{where} \quad F'(u) = f'(u)v(u)_x$$

Jump condition for first conservation law is

$$s_1 = \left[f(u_+) - f(u_-)\right] / (u_+ - u_-)$$

Jump condition for second conservation law is

$$s_2 = \left[F(u_+) - F(u_-)\right] / [v(u_+) - v(u_-)], \quad \text{so} \quad s_1 \neq s_2.$$

## Layered Solutions

$$u_t + f_x(u) = 0, \qquad u(x,0) = U(x)$$

$$|U| \le U_0, \quad |U'| \le U_1$$

$$f_1 = \max_{|s| \le U_0} |f'(s)|, \quad f_2 = \max_{|s| \le U_0} |f''(s)|, \quad |u| \le U_0$$

## Time interval to be used

$$h = 1/(2 f_2 U_1)$$

i. Introduce a special smoothing operation for data on a line $t =$ constant, denoted by $S\{.\}$.

ii. Using the smoothed initial data $S\{U\} = u_0(x,t)$, find a strict solution $u_1(x,t)$ in the layer $0 \le t \le h.$

iii. In a second layer $h \le t \le 2h$, find a strict solution $u_2(x,t)$ subject to smoothed initial data such $u_2(x,h) = S\{u_1(x,h)\}.$

iv. Continue this layering process to obtain a sectionally continuous smoothed layered approximation for $t > 0$ such that

$$u(x,t) = \begin{cases} u_1(x,t) \ \text{ for } \ 0 \le t < h, \\ u_2(x,t) \text{ for } h \le t < 2h, \\ u_3(x,t) \text{ for } 2h \le t < 3h, \\ \vdots \end{cases}.$$

v. Then for a suitable smoothing operation, it can be proved that this layered approximate solution converges to a **unique weak solution** of (11) subject to the initial condition

$$u(x,0) = U(x).$$

The smoothing process contains two steps. The first involves converting a bounded and continuous solution $v(x)$ into a sequence of step function $R_\varepsilon\{v(x)\}$ that together form a sequence of Riemann problems. This is achieved by defining

$$R_\varepsilon\{v(x)\} = \frac{1}{2\varepsilon}\int_{(i-2)\varepsilon}^{i\varepsilon} v(\xi)\,d\xi \qquad \text{for} \qquad (i-2)\varepsilon \le x < i\varepsilon,$$

$$Av_\varepsilon\left\{R_\varepsilon\{v(x)\}\right\} = \tfrac{1}{2}\int_{-1}^{1} R_\varepsilon\{v(x+\varepsilon\tau)\}\,d\tau.$$

$$\frac{f(u_+) - f(u_-)}{u_+ - u_-} \le \frac{f(u_0) - f(u_-)}{u_0 - u_-},$$

# Nonlinear Superposition and the Riccati Equation.

$$y' + Qy + Ry^2 = P$$

The change of variable $y = u'/Ru$ gives

,

$$Ru'' - (R' - QR)u' - PR^2u = 0.$$

$$C = \frac{(y_1 - y_3)(y_2 - y_4)}{(y_1 - y_4)(y_2 - y_3)}$$

# The Burgers Equation – a Dissipative Equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \mu\frac{\partial^2 u}{\partial x^2}, \qquad (\mu > 0)$$

with $$u(x,0) = U(x)$$

Do solutions $u_\mu(x, t)$ approach a limit as $\mu \to 0$, and if so does the limit satisfy the limiting differential equation, namely,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u_t + \left(\tfrac{1}{2}u^2 - \mu u_x\right)_x = 0,$$

which may be considered to be a compatibility condition for the existence of a function $\psi$ with the properties that

$$u = \psi_x \quad \text{and} \quad \mu u_x - \tfrac{1}{2}u^2 = \psi_t .$$

The substitution for $u$ in the second equation then leads to the result

$$\mu\psi_{xx} - \tfrac{1}{2}\psi_x^2 = \psi_t .$$

Next, the introduction of what is now called the Hopf-Cole transformation $\psi = -2\mu \ln\theta$ shows that

$$u = \psi_x = -2\mu\frac{\theta_x}{\theta},$$

after which the Burger's Equation is transformed into the linear heat equation.

$$\frac{\partial\theta}{\partial t} = \frac{\partial^2\theta}{\partial x^2}.$$

We mention in passing that the Burgers equation describes the steady traveling wave solution called the **Burgers shock wave**

$$u(\zeta) = \tfrac{1}{2}\left(u_\infty^- + u_\infty^+\right) - \tfrac{1}{2}\left(u_\infty^- - u_\infty^+\right)\tanh\left[\left(u_\infty^- - u_\infty^+\right)\zeta/(4\mu)\right],$$

with $\zeta = x - ct$ and $c = \tfrac{1}{2}\left(u_\infty^- + u_\infty^+\right),\ u_\infty^- > u_\infty^+$

# The KdV Equation – A Nonlinear Dispersive Equation.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \mu\frac{\partial^3 u}{\partial x^3} = 0,$$

The IST approach can be represented diagramatically as follows:

$\begin{bmatrix} \text{initial condition} \\ u(x,0) = u_0(x) \end{bmatrix} \rightarrow$ direct Schrödinger scattering $\rightarrow \begin{bmatrix} \text{scattering data} \\ \text{at } t = 0 \end{bmatrix}$

$\downarrow$ $\downarrow$

time evolution in configuration space

time evolution in spectral space

$\downarrow$ $\downarrow$

$\begin{bmatrix} \text{solution } u(x,t) \\ \text{at } t > 0 \end{bmatrix} \leftarrow$ inverse Schrödinger scattering $\leftarrow \begin{bmatrix} \text{scatering data} \\ \text{at } t > 0 \end{bmatrix}$

The soliton property is not confined to the KdV equation and, for example, it applies to the equation

$$y_{xt} + \operatorname{sign}\left(\frac{dx}{ds}\right)\left[\frac{y_{xx}}{\left(1+y_x^2\right)^{3/2}}\right]_{xx} = 0$$

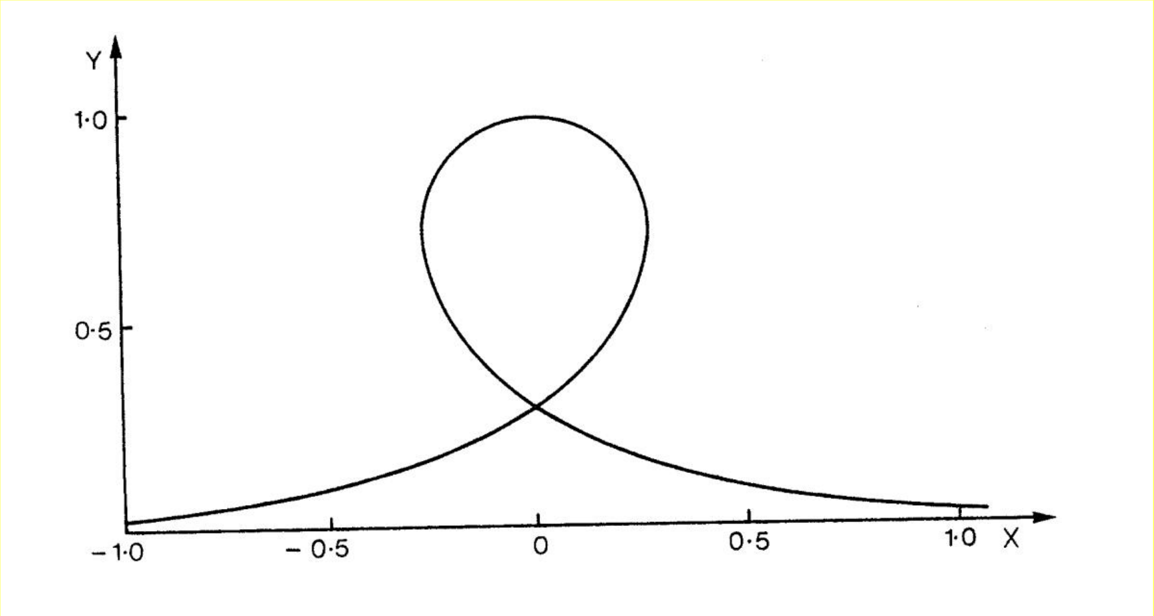


Fig.5 A single loop soliton

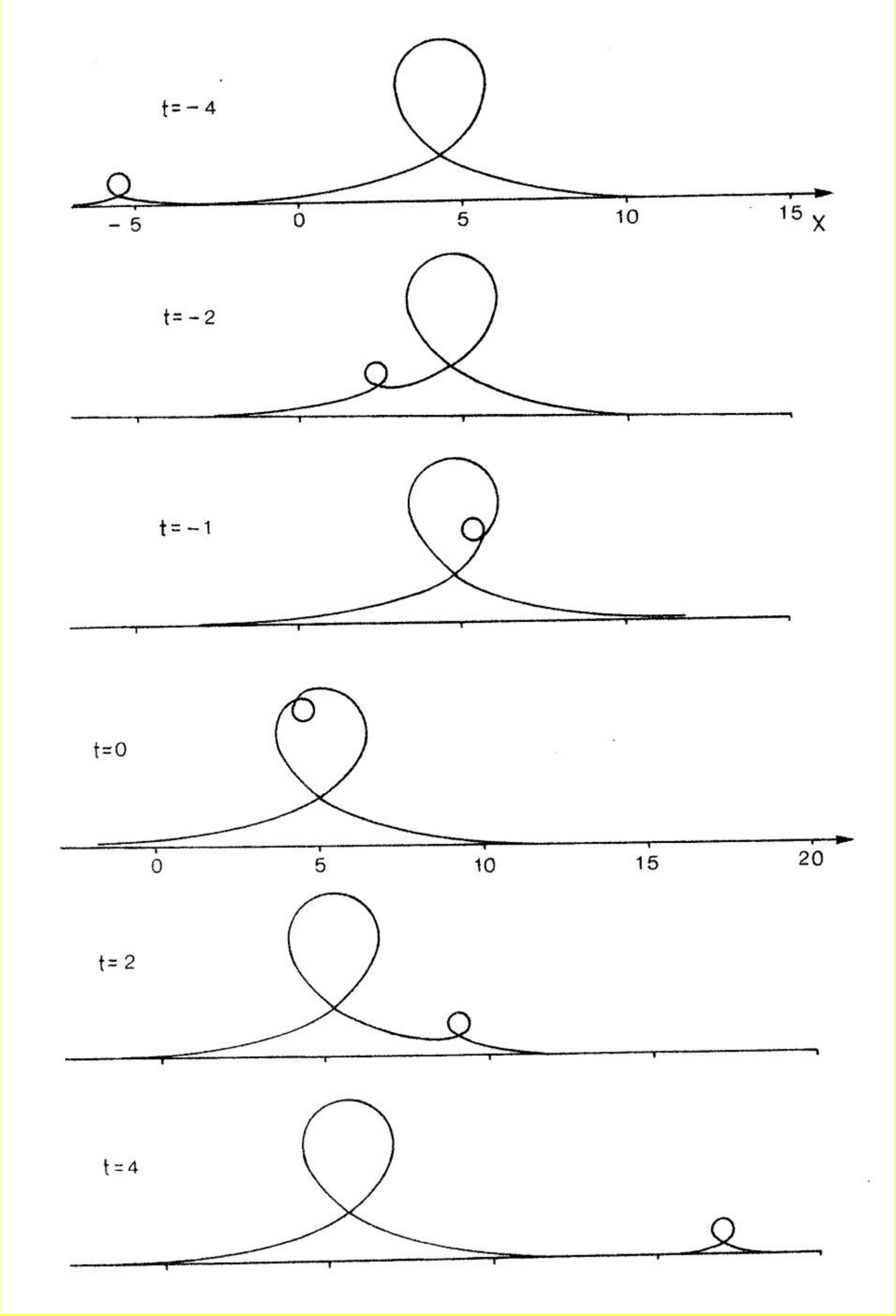


Fig.6. The interaction of two loop solitons

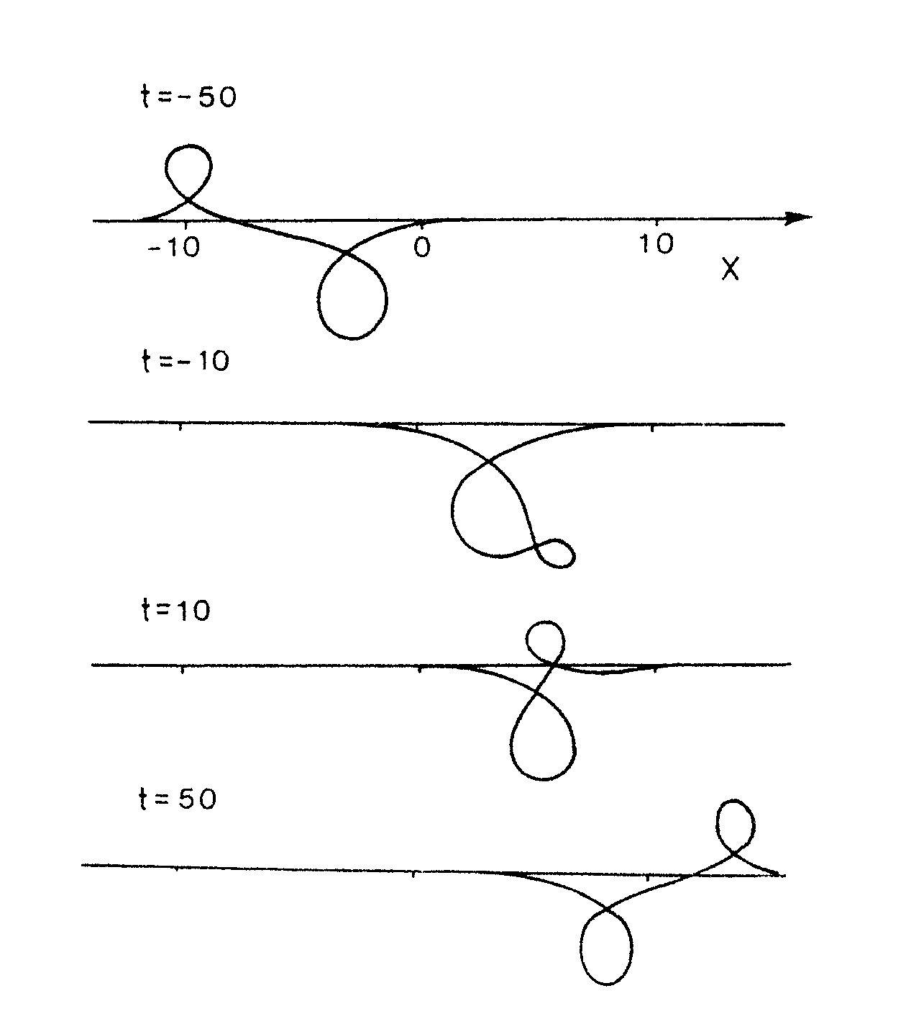


Fig.7. The interaction of a loop soliton and an anti-loop soliton

# Backlund Transformations

In brief, a Bäcklund transformation for a second-order equation for a dependent variable $\varphi(\xi,\eta)$ is best described as the pair of relationships

$$\frac{\partial\varphi'}{\partial\xi} = P\left(\varphi', \varphi, \varphi_\xi, \varphi_\eta, \xi, \eta\right) \quad \text{and} \quad \frac{\partial\varphi'}{\partial\eta} = Q\left(\varphi', \varphi, \varphi_\xi, \varphi_\eta, \xi, \eta\right)$$

where the consistency condition for these two equations provides a new equation for $\varphi'$.

If it possible to find such transformations that map into themselves, then any known solution $\varphi$ may be used to find a new solution $\varphi'$.

One of the simplest examples of a Bäcklund transformation, though not related to wave propagation, is provided by the Cauchy-Riemann equations

$$\left.\begin{array}{c}\text{Laplace equation}\\ u_{xx}+u_{yy}=0\\ \text{Harmonic conjugate}\\ v_{xx}+v_{yy}=0\end{array}\right\}\Leftrightarrow\left\{\begin{array}{c}\text{Bäcklund transformation}\\ \text{Cauchy-Riemann equations}\\ u_x=v_y\\ u_y=-v_x\end{array}\right.$$

Notice that in this rather special case the conjugate PDE for $v$ happens to be the same as the original PDE for $u$, namely the Laplace equation.

# Linearity and Nonlinear Superposition.

$$\frac{\partial u}{\partial x} + a(x,y)\frac{\partial u}{\partial y} + b(x,y)f(u) = 0.$$

Now let us seek a superposition law $g$, such that $u = g(v,w),$

$$\frac{\partial g}{\partial v}\left[\frac{\partial v}{\partial x} + a(x,y)\frac{\partial v}{\partial y} + b(x,y)f(v)\right] + \frac{\partial g}{\partial w}\left[\frac{\partial w}{\partial x} + a(x,y)\frac{\partial w}{\partial y} + b(x,y)f(w)\right]$$

$$+ b(x,y)\left[f(g) - f(v)\frac{\partial g}{\partial v} - \frac{\partial f}{\partial w}\frac{\partial g}{\partial w}\right] = 0.$$

However, as v and w are solutions of the equation, it will be satisfied if

$$f(v)\frac{\partial g}{\partial v} + f(w)\frac{\partial g}{\partial w} = f(g).$$

The general solution of this equation is

$$\int \frac{dg}{f(g)} = \int \frac{dv}{f(v)} + K\left[\int \frac{dv}{f(v)} - \int \frac{dw}{f(w)}\right],$$

Now consider the special case $f(u) = u,$ when

$$g(v,w) = wK\left(v/w\right).$$

The semilinear equation (41) then reduces to a **linear equation**, and the usual linear superposition becomes possible if

$$K\left(v/w\right) = A\left(v/w\right) + B,$$

where A and B are constants

$$u = g(v,w) = Av + Bw.$$

# Asymptotic Methods

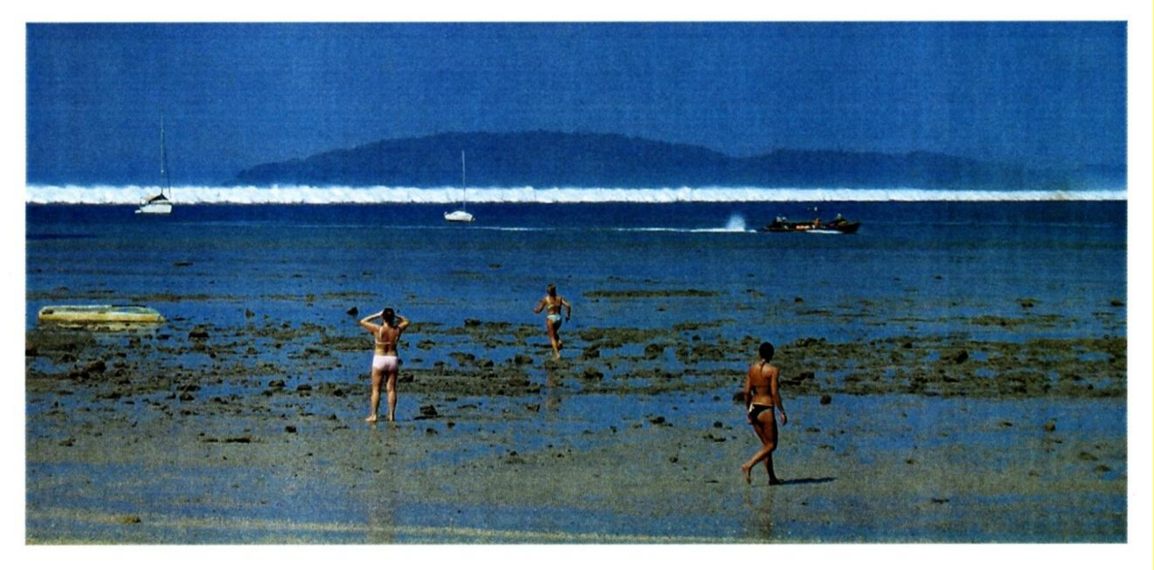

Fig.8 Two distant incoming tsunami

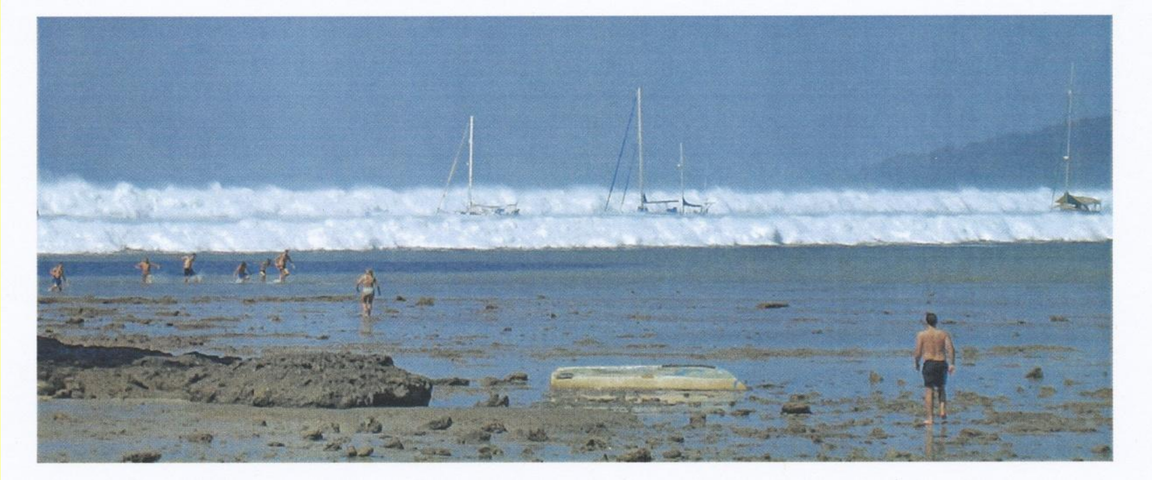

Fig.9 The two tsunami about to coalesce