Turbulent mixing of passive scalars: evolution of discontinuity fronts and material lines

Jaan Kalda kalda@ioc.ee
CENS, Tallinn University of Technology

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Turbulent mixing — is there any relationship to **waves** and **complexity**?

Passive scalar structures (blobs, high dissipation areas etc.) propagate (according to the underlying velocity field) and evolve (e.g. can develop into dissipative fronts — "shock waves"). This can be considered as a propagation of passive waves.

Process is characterized by self-organization, high intermittency etc — so, there is a lot of complexity.

The problem of passive scalar turbulence has a wide range of direct applications, e.g. pollutant, temperature and food (e.g. plankton) transport in water and atmosphere. There are also less direct applications, such as the nucleation of the rain droplets in warm clouds.

Further, the problem of passive scalar turbulence can be considered as a step towards the complete theory of turbulence (which is the problem of an active vector), the next step being the problem of passive vector turbulence (which is directly applicable to the problem of the kinematic magnetic dynamos — phenomenon leading to the creation of cosmic magnetic fields).

Finally, advances in understanding the turbulence (including the passive scalar turbulence) will benefit the studies in many, seemingly very different, research fields. For instance, turbulence has been the origin and a test field for the nowadays very popular multifractal formalism.

Fully developed turbulence is known to be highly intermittent, characterized by non-Gaussian statistics and non-vanishing probabilities of extreme events. This applies both to the turbulent velocity field itself and to the passive tracer fields (a dye density, temperature, etc.). Such a behaviour is classically described by anomalous (nonlinear) structure function scaling exponents. Qualitatively, one can say that the strength of intermittency should be measured as the significance of the deviations from Gaussianity. For multifractal systems, widely adopted such measures are the anomalous scaling exponents, e.g. $\mu_4 = 2\zeta_2 - \zeta_4$, where ζ_p $(p = 2, 4)$ are the p-th order structure function scaling exponents:

$$\left\langle |\theta(\boldsymbol{r}, t) - \theta(\boldsymbol{r} + \boldsymbol{a}, t)|^p \right\rangle \propto |\boldsymbol{a}|^{\zeta_p}.$$

In the case of passive tracers, such nonlinearity is particularly strong, evidenced by very high values of μ_4 (as well as μ_6 etc). So, the feed-back in the case of active vector problem tends to smooth out the strongest singularities. *This makes the passive scalar problem best suited for studying the origins of intermittency.*

The anomalous scaling of the passive scalar structure functions is caused by the presence of discontinuity fronts. This is why we focus here on the formation and evolution of these fronts.

We present a simple model for the evolution of passive tracers in turbulent flows. Based on that model, we derive an expression for the structure function scaling exponent (which is in a good agreement with existing numerical and experimental data), and reveal the origin of the small-scale anisotropy. We compare these results with the complementing approach of studying the evolution of material lines and show that an important role is played by the reconnection of the tracer isodensity lines.

Passive scalar turbulence is described by a simple linear equation,

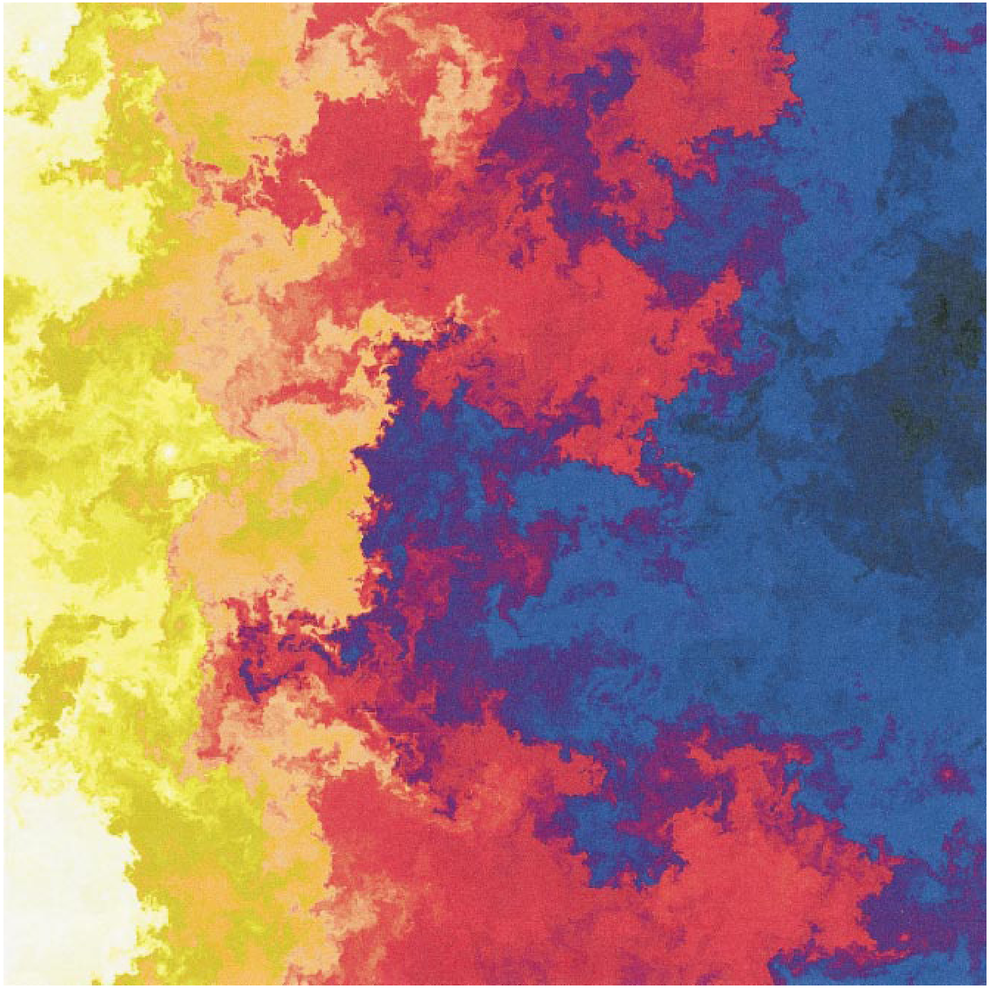
$$\frac{\partial \phi}{\partial t} + \boldsymbol{v} \cdot \nabla \phi = \kappa \nabla^2 \phi + g, \tag{1}$$

where g is a source of the passive scalar ϕ, $\boldsymbol{v}(\boldsymbol{r}, t)$ is a turbulent (possibly incompressible) velocity field, and κ is a seed diffusivity. Alternative description is based on the position $\boldsymbol{r}(t)$ of a passive scalar particle [$f(t)$ is a noise term]:

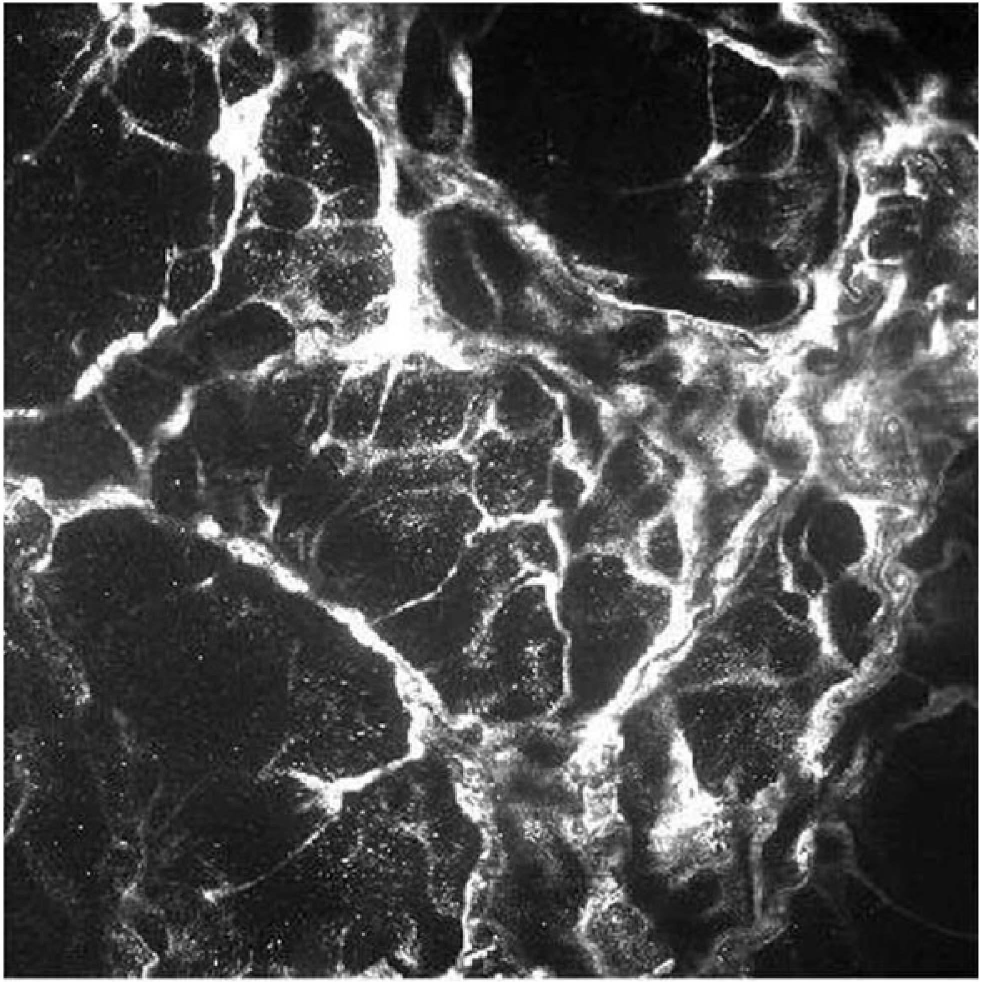
$$\frac{d\boldsymbol{r}}{dt} = \boldsymbol{v}(\boldsymbol{r}, t) + \boldsymbol{f}(t), \tag{2}$$

The problems of passive scalar turbulence are easy to pose, but difficult to solve. This is because (a) the distribution of a passive scalar ϕ tends to be highly intermittent, and (b) the behaviour depends qualitatively on the character of the flow and on the initial conditions. For instance, important cases are:

- rough flows with Kolmogorov spectrum;
- smooth flows (e.g. at the Batchelor scales);
- compressible flows (with the subcases of rough and smooth flows;
- (quasi)stationary flows (e.g. drift wave turbulence);
- stationary mixing;
- decaying tracer;
- tracer from point sources.



[Falkovich, Gawędzki and Vergassola, RMP 2001]



G. Bofetta,
J. Davoudi,
B. Eckhardt,
J.Schumacher,
Phys. Rev. Lett.
93, 134501

Here we focus on the passive scalars evolving in fully developed turbulent flows, when the scalar field becomes everywhere discontinuous: the discontinuity fronts of fractal structure will emerge [A. Celani et al, Phys. Fluids **13**, 1768 (2001)]

These fronts are the very reason for the anomalous scaling of structure functions. However, little is known about the formation and statistics of such fronts.

Mixing effect of turbulent flows is most intuitively characterized by the growth of the distance between two tracer particles r:

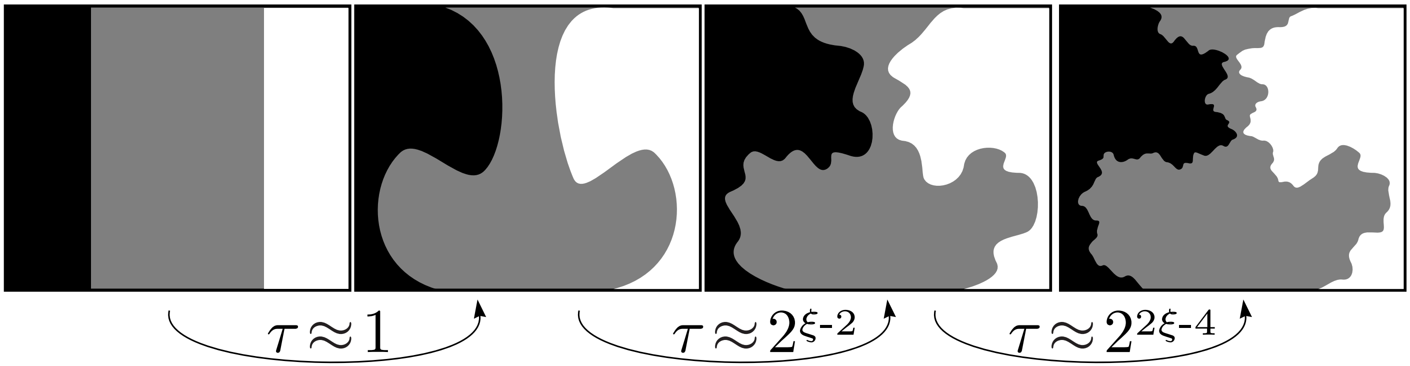
$$\frac{d}{dt}\langle \ln r\rangle \propto \frac{d}{dt}\left\langle [\ln r]^2 \right\rangle \propto r^{\xi-2}.$$

So, with a proper time unit, the distance doubling time is estimated as $\tau \approx r^{2-\xi}$; the Kolmogorov scaling $\tau \approx r^{2/3}$ is matched with $\xi = 4/3$.

Let us decompose the velocity field into components of different characteristic space-scale,

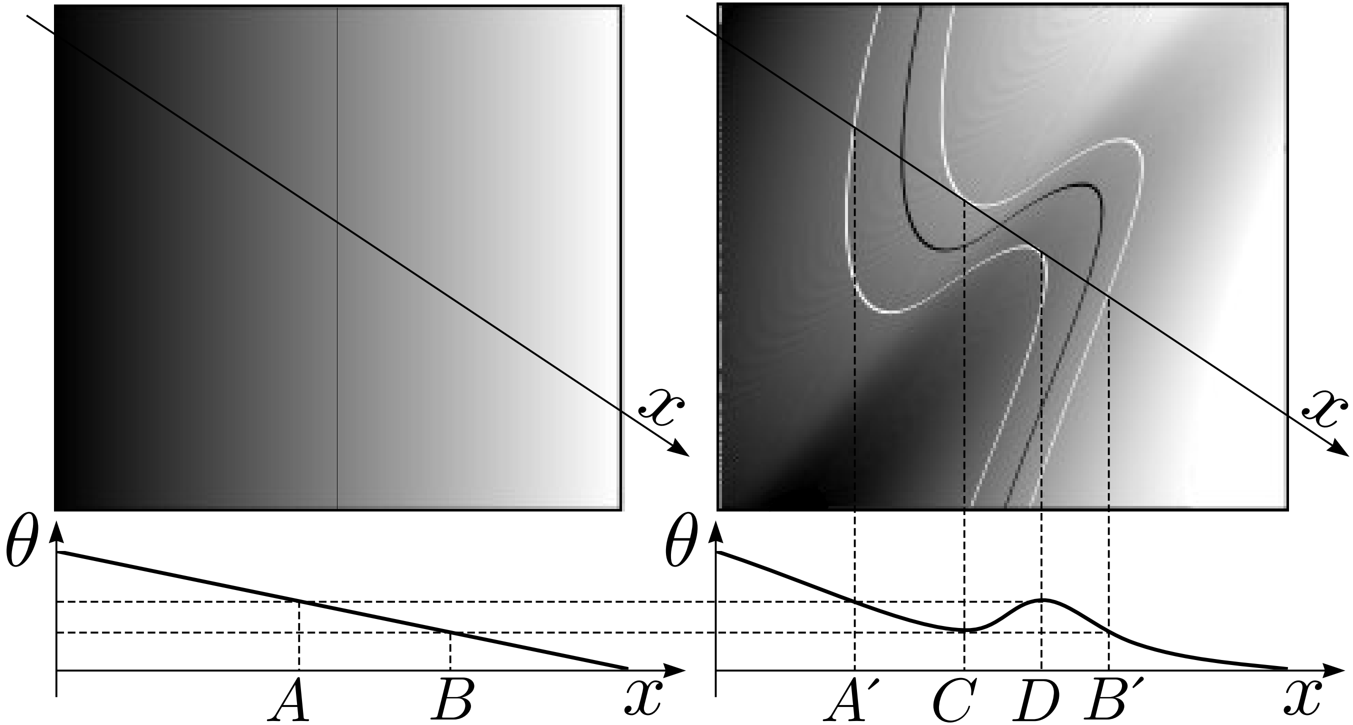
$$\boldsymbol{v}_a(\boldsymbol{r}, t) = \int_{a \le |\boldsymbol{k}| < 2a} \boldsymbol{v}(\boldsymbol{k}, t) e^{i\boldsymbol{kr}} d\boldsymbol{k}.$$

The characteristic time-scale of $\boldsymbol{v}_a$ is estimated as $\tau_a \approx a^{2-\xi}$.

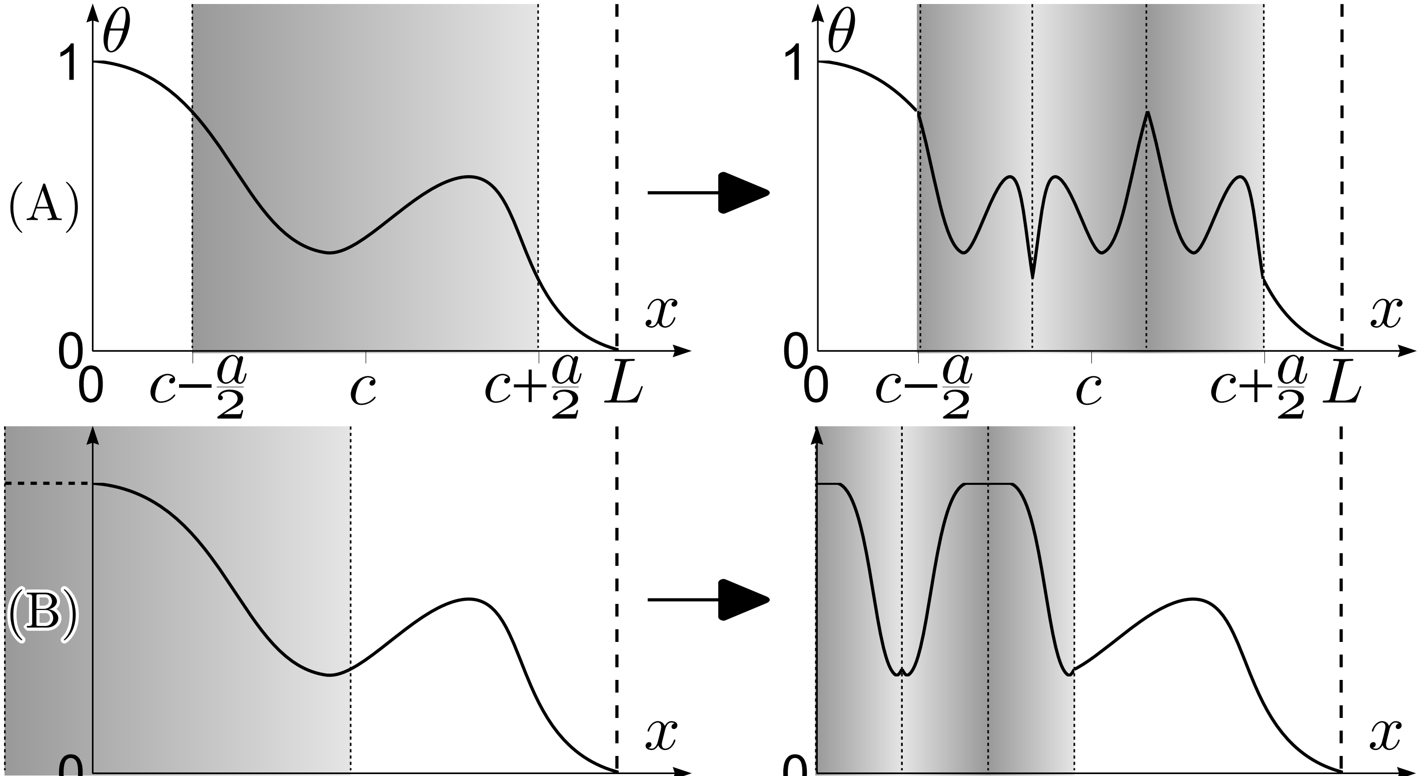


Simplified scheme of the formation of tracer discontinuities. Characteristic time of eddies of size a scales as $\tau \approx a^{2-\xi}$. Due to the combined effect of large and small eddies, low- and high-density regions (black and white, respectively) are brought into contact within a finite time.

We aim to construct a model, which mimics the evolution of the tracer density profile along the x-axis (any 1D cross-section). To begin with, we consider only the effect of an "a-flow" $\boldsymbol{v}_a(\boldsymbol{r}, t)$ (this corresponds to observing the initial tracer field evolution with a spatial resolution a: smaller vortices are not resolved, larger ones are slower and require a longer observation period). In incompressible velocity fields, exponential growth of scalar density gradients is due to exponential stretching of fluid elements, caused by stretching-folding motion of the fluid. Such a stretching-folding motion is provided by a simple shear flow.

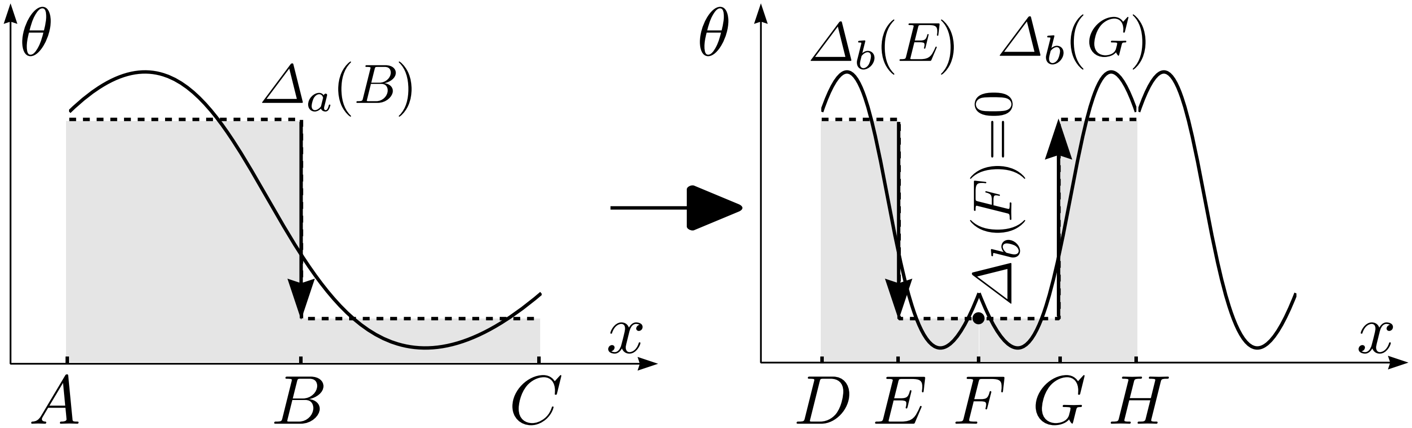


The segment AB of the initial profile $\theta(x)$ evolves into a "kink" of the final profile $\theta'(x)$, consisting of descending, ascending, and descending segments $A'C$, CD, and DB', respectively.

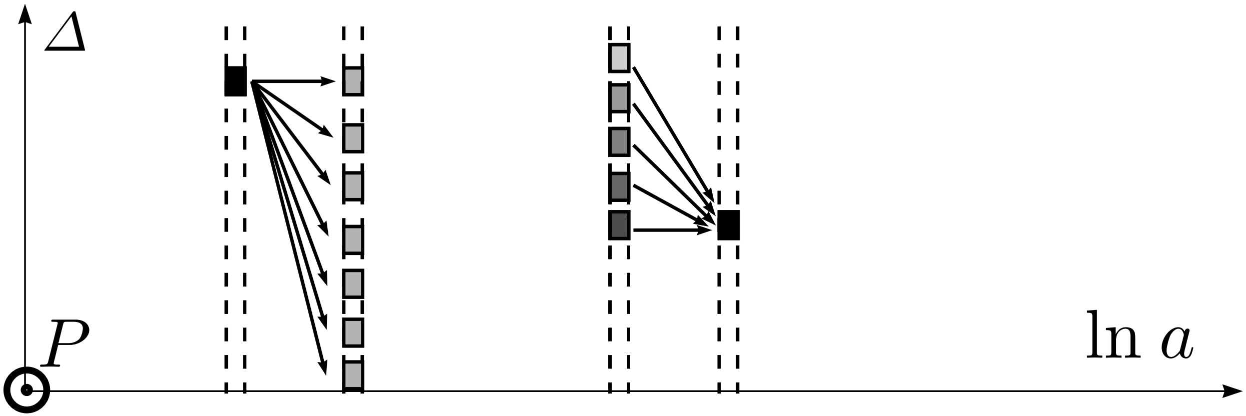


Mapping $\mathcal{M}_{a,c}$ modelling the effect of a single vortex of size a on the tracer profile $\theta(x)$. The vortex may be far from the vessel boundary [Case (A)], or close to it, with $\theta|_{x=0} \equiv 1$ [Case (B)]

For any non-zero diffusivity κ, the diffusion smoothens the tracer density fluctuations at a microscale δ, for which the effective Peclet' number $P_\delta \approx 1$. From the equality of diffusion and mixing times, $\tau_\delta \approx \delta^{2-\xi} \approx \delta^2/\kappa$, we obtain $\delta \approx \kappa^{1/\xi}$. In order to take into account such a smoothing, the mapping $\mathcal{M}_{a_t,c_t}$ is modified so that apart from the effect depicted in previous Figure, it includes also averaging over a sliding window of width δ. For numerical simulations, δ serves as a natural discretisation step.



As a result of the mapping $\mathcal{M}_{2a,B}$, the old value of the mean density difference $\Delta_a(B)$ defines the possible range of new values at trice smaller scale $b = a/3$: the smallest value is $\Delta_b(F) = 0$, and the largest one $\Delta_b(E) = \Delta_b(G) = \Delta_a(B)$



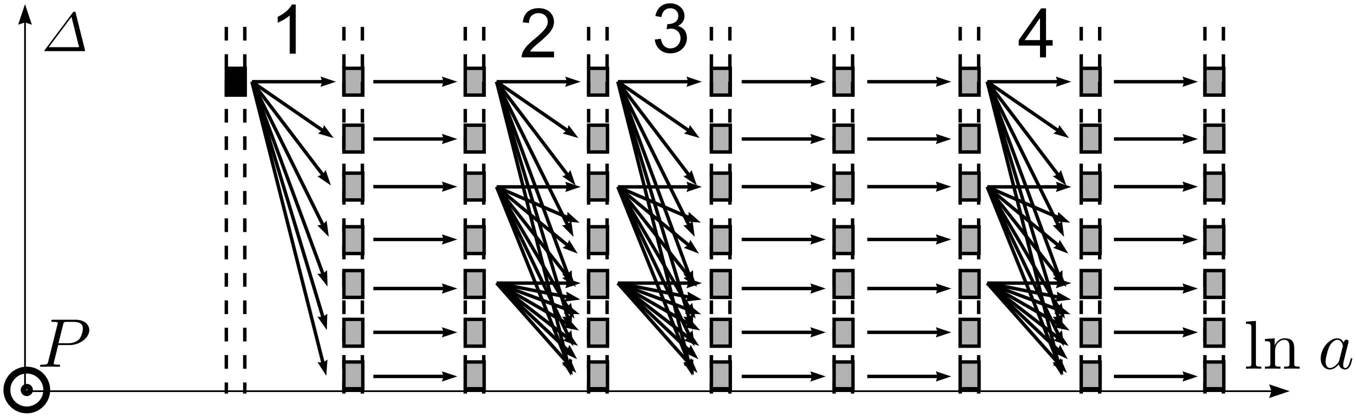
As a result of a mapping, PDF is transferred to smaller scales via a convolution-type re-distribution. Mathematically,

$$f_b(\Delta) = \int_\Delta^1 f_a(\Delta') \frac{d\Delta'}{\Delta'} \qquad (3)$$

(assuming that the maximal value of Δ is 1).

Assumptions: *(i)* We do not consider the effect of those mappings, the size of which is significantly smaller than a and b, because at our scale, they preserve the average density $\bar{\theta}_b$. This is true, if the mapping falls entirely into the segment; if it falls at the edge, $\bar{\theta}_b$ will be changed, but the change remains relatively small. For very small values of $\xi < \xi_0$, when small vortices are much more frequent than the large ones, this assumption will no longer be valid.

(ii) We neglect the effect of larger vortices. This is actually not correct: larger-size mappings compress the profile without reducing the density drop Δ. Such a process corresponds to a direct transfer $f_a(\Delta) \rightarrow f_{a/3}(\Delta)$, without the convolution in Eq (3). So, in average, the profile will be compressed more than trice, before entering the convolution stage. Hence, the effect of larger vortices can be taken into account by using an effective, somewhat increased compression factor $k = a/b > 3$.



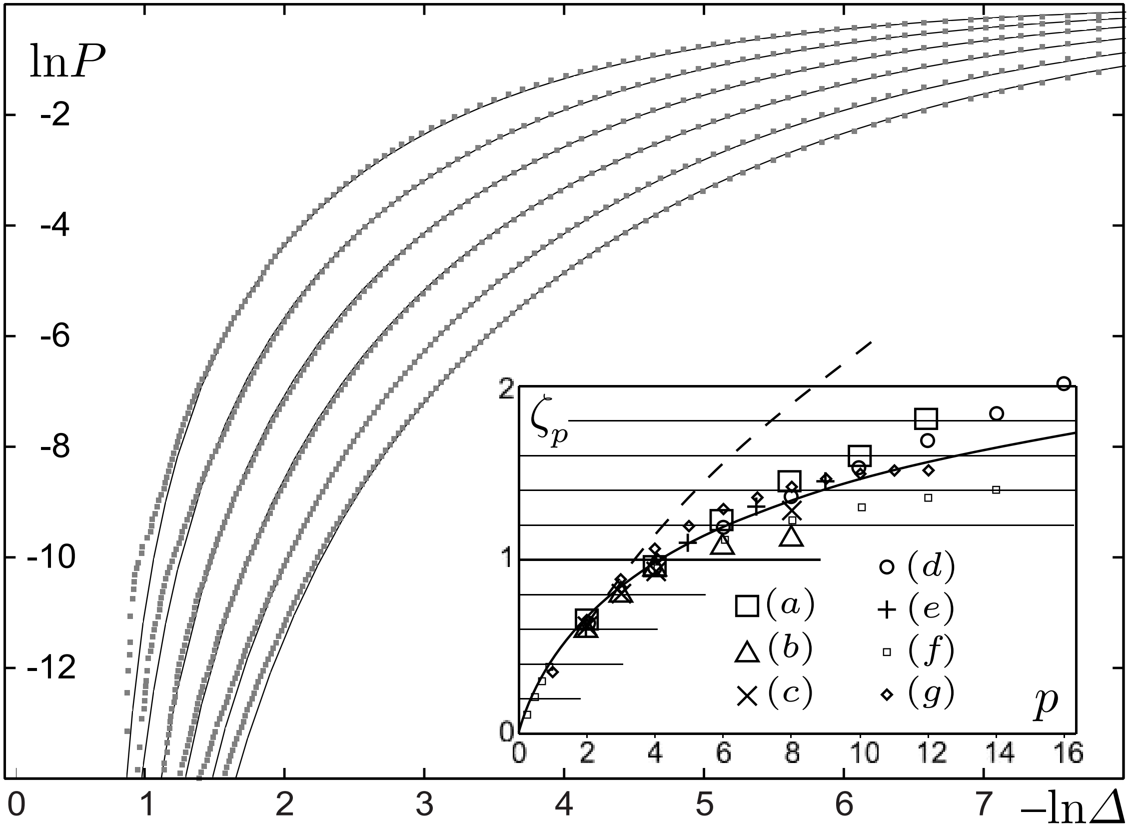
Large mappings lead to a direct transfer of PDF towards small scales, matching-size mappings cause a re-distribution (convolution) of it. For our model, we need to relate the number of convolutions n to the scale reduction factor a_0/a_f. Effective compression factor k is defined via relationship $k^n = a_0/a_f$.

Bearing in mind boundary conditions $\theta(0,t) \equiv 1$ and $\theta(1,t) \equiv 0$, it is reasonable to assume that $f_1(\Delta) \equiv 1$. Then, direct integration results in $f_a(\Delta_a) = |\ln(\Delta_a)|^n/n!$, where $n = -\log_k a$.

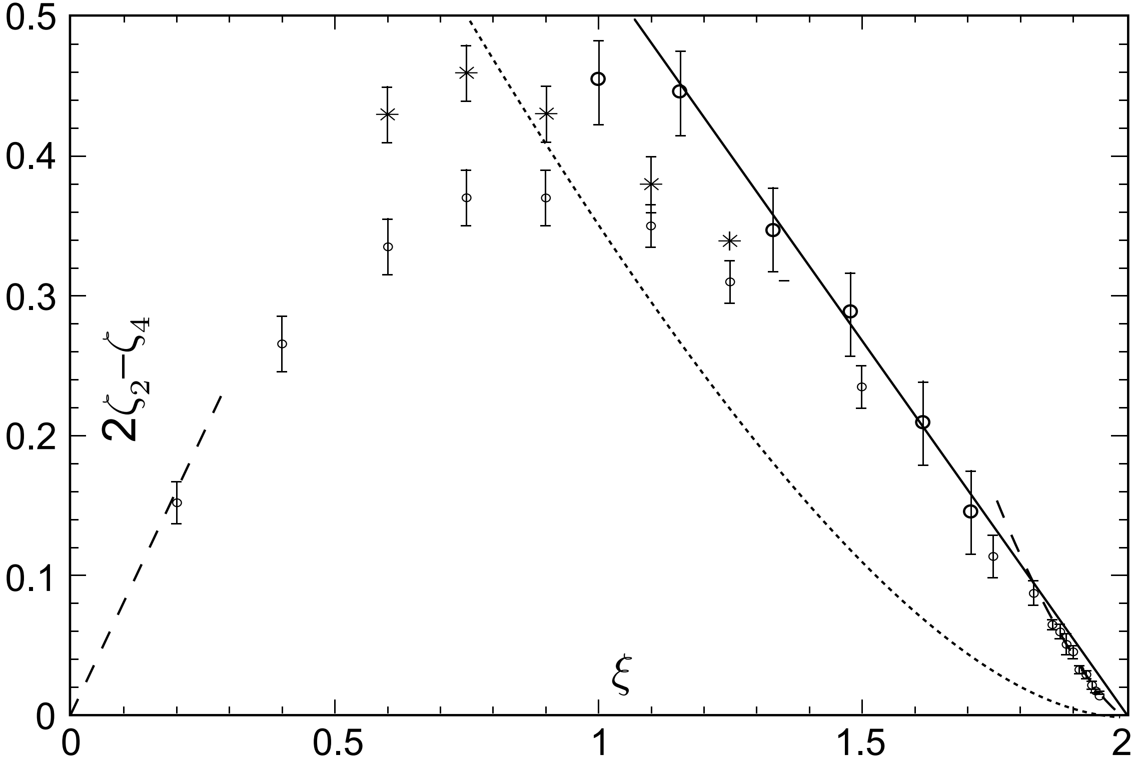
We expect that $\int f_a(\Delta)\Delta^p d\Delta \propto a^{\zeta_p}$; the integral is easily taken, resulting in $\zeta_p = \log_k(p+1)$. Comparing this expression with the classical result $\zeta_2 = 2 - \xi$ (which is valid both for tracer turbulence, and for our 1D model), we obtain $k = 3^{1/(2-\xi)}$. So,

$$\begin{aligned} &f_a(\Delta) = |\ln(\Delta)|^n n!^{-1}, \ n = (2-\xi)|\log_3 a|, \\ &\zeta_p = (2-\xi)\log_3(p+1). \end{aligned} \tag{4}$$

Now, let us recall that we expected $k \geq 3$; this inequality is not satisfied for $\xi < 1$. So, we can conclude that $\xi_0 = 1$, i.e. for $\xi < 1$, the assumption *(i)* is not satisfied. Note that the result $\xi_0 = 1$ is directly applicable only to our 1D model, when all the compression factors are equal to 3. In the case of real 2D or 3D turbulence, the individual (effective) mappings are not obtained by simple 3-fold compression. Hence, the critical value ξ_0 may deviate from 1.

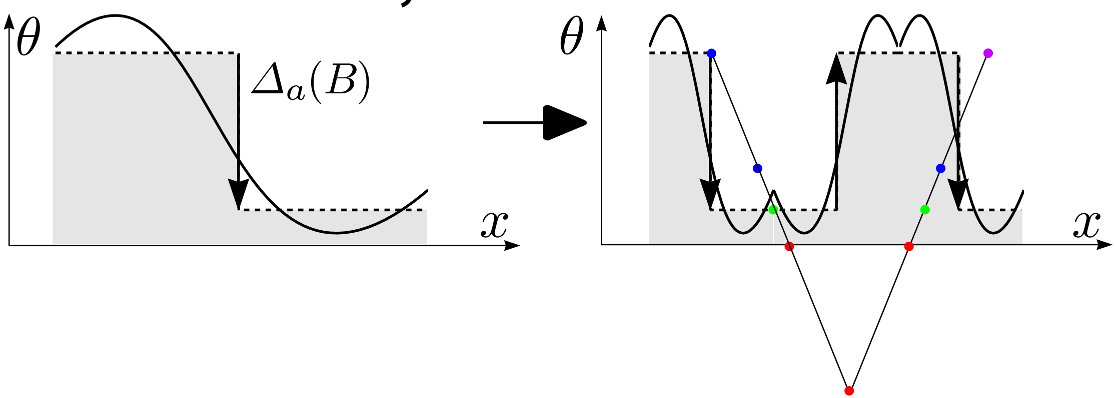


(a) — R.A. Antonia et al, 96; (b) — C. Meneveau et al, 90; (c) — G. Ruiz-Chavarria et al, 96; (d) — L. Mydlarski and Z. Warhaft, 98; (e) — S. Chen and R.H. Kraichnan 98, (f) —A. Celani et al, 01; (g) — F. Moisy et al, 01.



Solid line — theoretical curve $(2-\xi)\log_3 \frac{9}{5}$, dotted line — Kraichnan formula (94), dashed line — perturbation theory [K. Gawędzki and A. Kupiainen, 95], open circles and stars — numerical results of 2D and 3D Lagrangean simulations [U. Frisch et al, 98 & 99], and filled circles — our simulations.

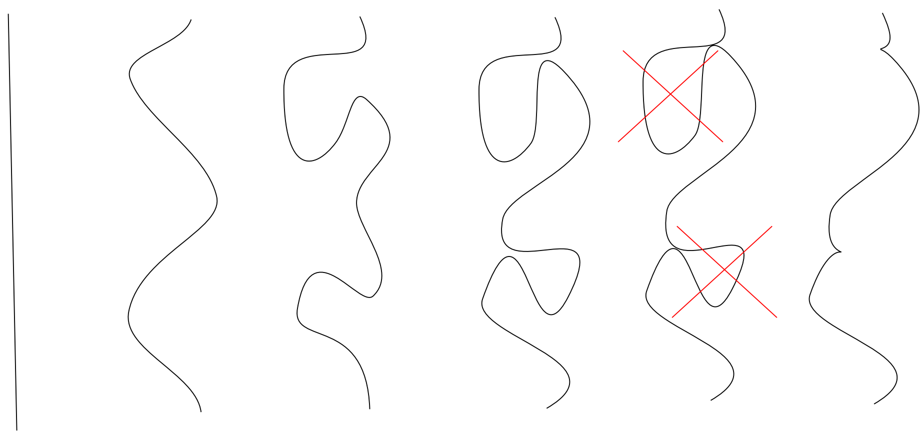
Origin of small-scale anisotropy. Experiments have shown that if one wall is kept warm, and another cold, then the temperature profile obtains ramp-cliff structure. This is reproduced by our 1D model. As a result of the mapping, red and blue parts are symmetric to each other (compensate in S_3); violet part remains uncompensated. Throughout the cascade of mappings, only the largest-Δ-part of the PDF remains non-symmetric.

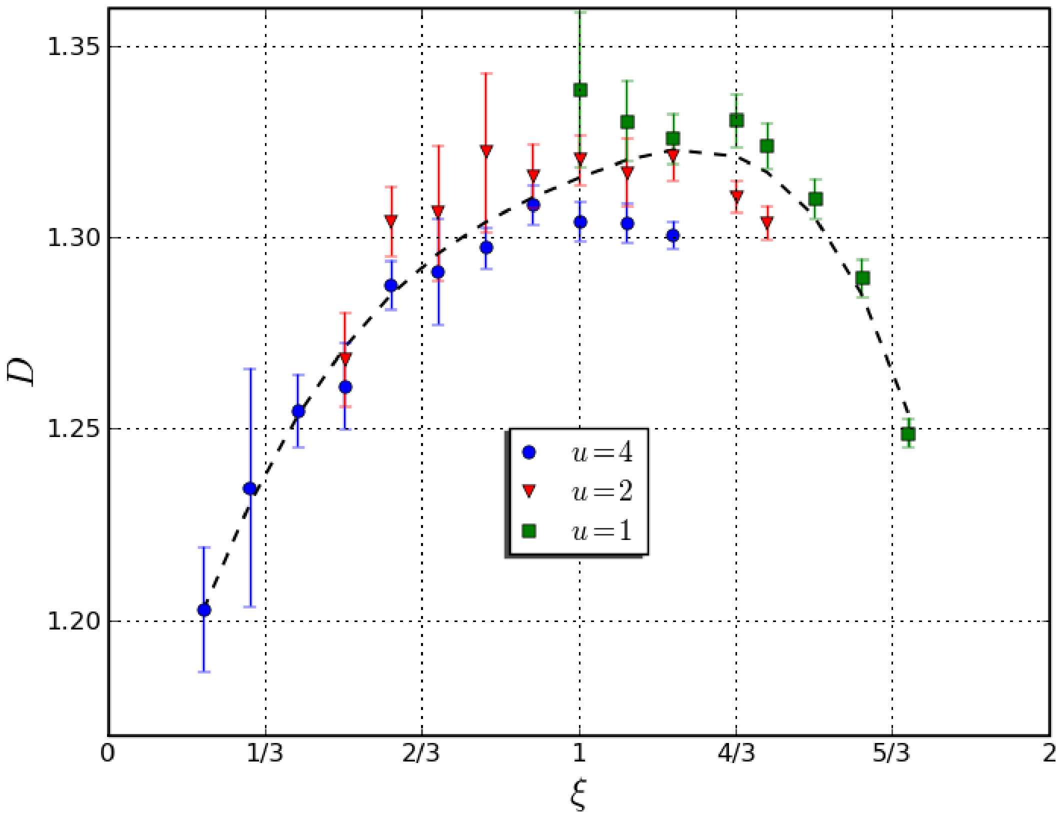


For the signed-Δ-PDF, $\tilde{f}_a(\Delta) - \tilde{f}_a(-\Delta) \neq 0$ only if $|\Delta| > \delta(a)$. The function $\delta(a)$ can be found, assuming that for $|\Delta| > \delta(a)$, $\tilde{f}_a(\Delta) - \tilde{f}_a(-\Delta) \approx \frac{1}{3} f_a(\Delta)$, and bearing in mind that $S_1 = a$. Appropriate calculations result in $\tilde{\zeta}_3 = 1 + 2(2 - \xi)3^{-1/(2-\xi)}/(e \ln 3)$; for the Kolmogorov turbulence ($\xi = \frac{4}{3}$), this yields $\tilde{\zeta}_3 \approx 1.1$ in a reasonable agreement with the experimental data ($\tilde{\zeta}_3 \approx 1 \div 1.1$), and with our simulations with the 1D triplet map model ($\tilde{\zeta}_3 \approx 1.2$).

Another aspect of the tracer field is its statistical topography. For instance, one aspect of it is the fractal dimension of the isodensity lines D. If the tracer field were Gaussian, characterized by the Hurst exponent $H = \zeta_1 = \zeta_2/2$, it would possible to apply the statistical topography of self-affine Gaussian surfaces. In particular, $D \approx 1.5 - H/2$. Since $\zeta_2 = 2 - \xi$, we would obtain $D \approx 1 + \xi/4$. While the scalar field is far from Gaussian, this expression serves as an estimate. According to numerics, $D \approx 1.3$ for $\xi = 4/3$

To clarify the evolution of the isodensity lines, we performed a numerical analysis of the evolution of material lines.





1.35
1.30
1.25
1.20
D
0
1/3
2/3
1
4/3
5/3
2
ξ
u = 4
u = 2
u = 1

Conclusion

While our 1D model and analytics include several simplifications, and hence cannot pretend to provide exact results, they are, with a reasonable accuracy, in agreement with previous experiments and simulations. Therefore, we believe that the model captures the main mechanisms of passive scalar intermittency. It also reveals the origin of the small-scale anisotropy.

The reconnection of isodensity lines plays an important role in the formation of the tracer field. For large values of ξ, the dominating process is the merger of loops, leading to the increase of the fractal dimension D. In the case of smaller values of ξ, the leading mechanism is the smoothing of lines due to the diffusion at the smallest scales, leading to the decrease of D. The Kolmogorov case $\xi = 4/3$ corresponds to the crossover between the two cases.