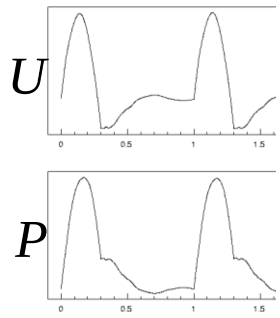
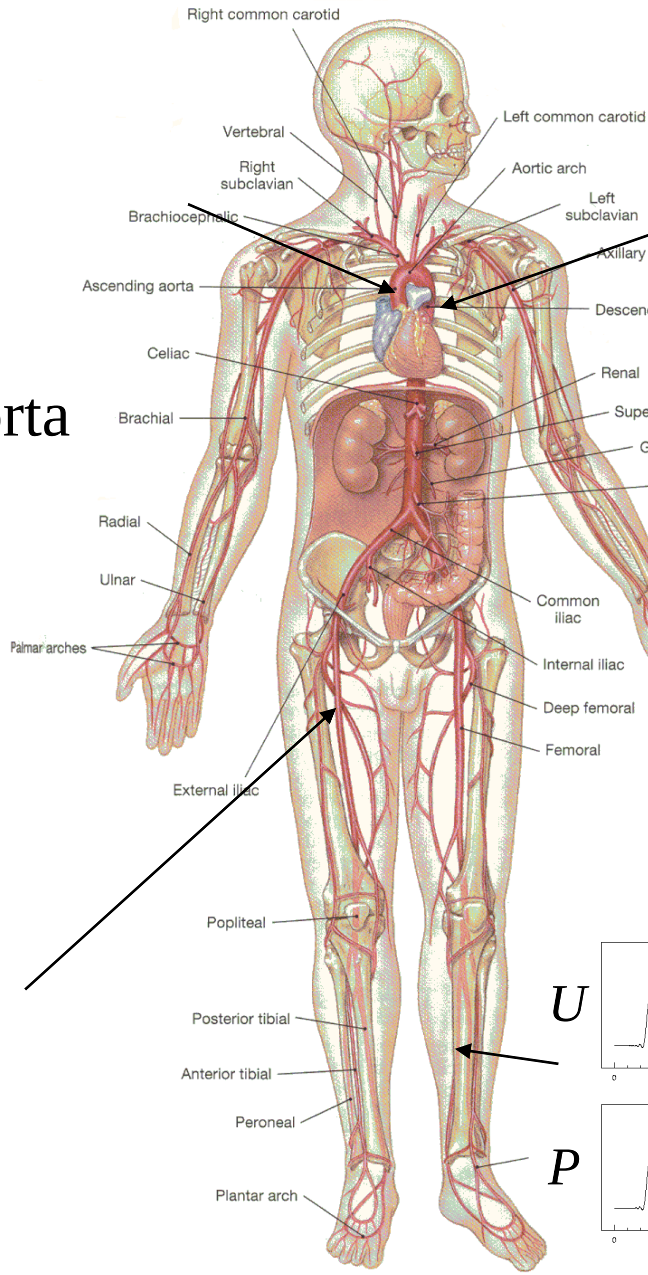
Numerical Simulation of Wave Propagation in Multilayered Viscoelastic Tubes

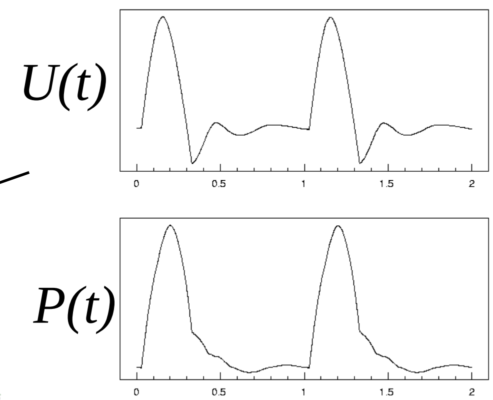
Natalya Kizilova

***Kharkov National University*, Ukraine**



Ascending aorta

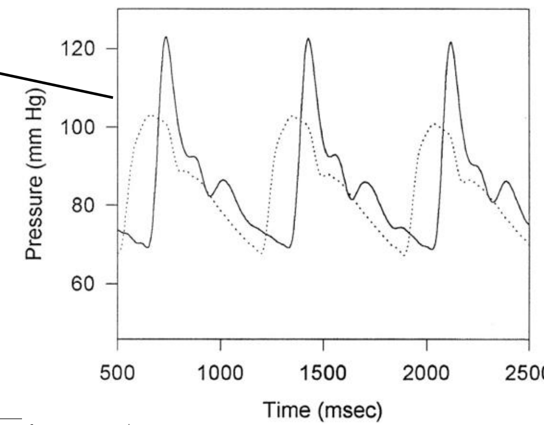




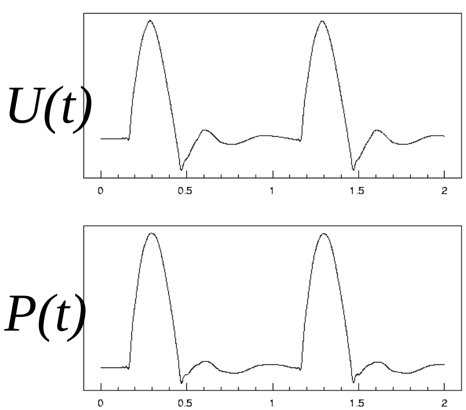
Thoracic aorta

Y=US/P - admittance

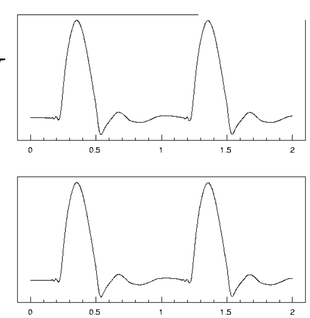
Z=P/US - impedance



Radial

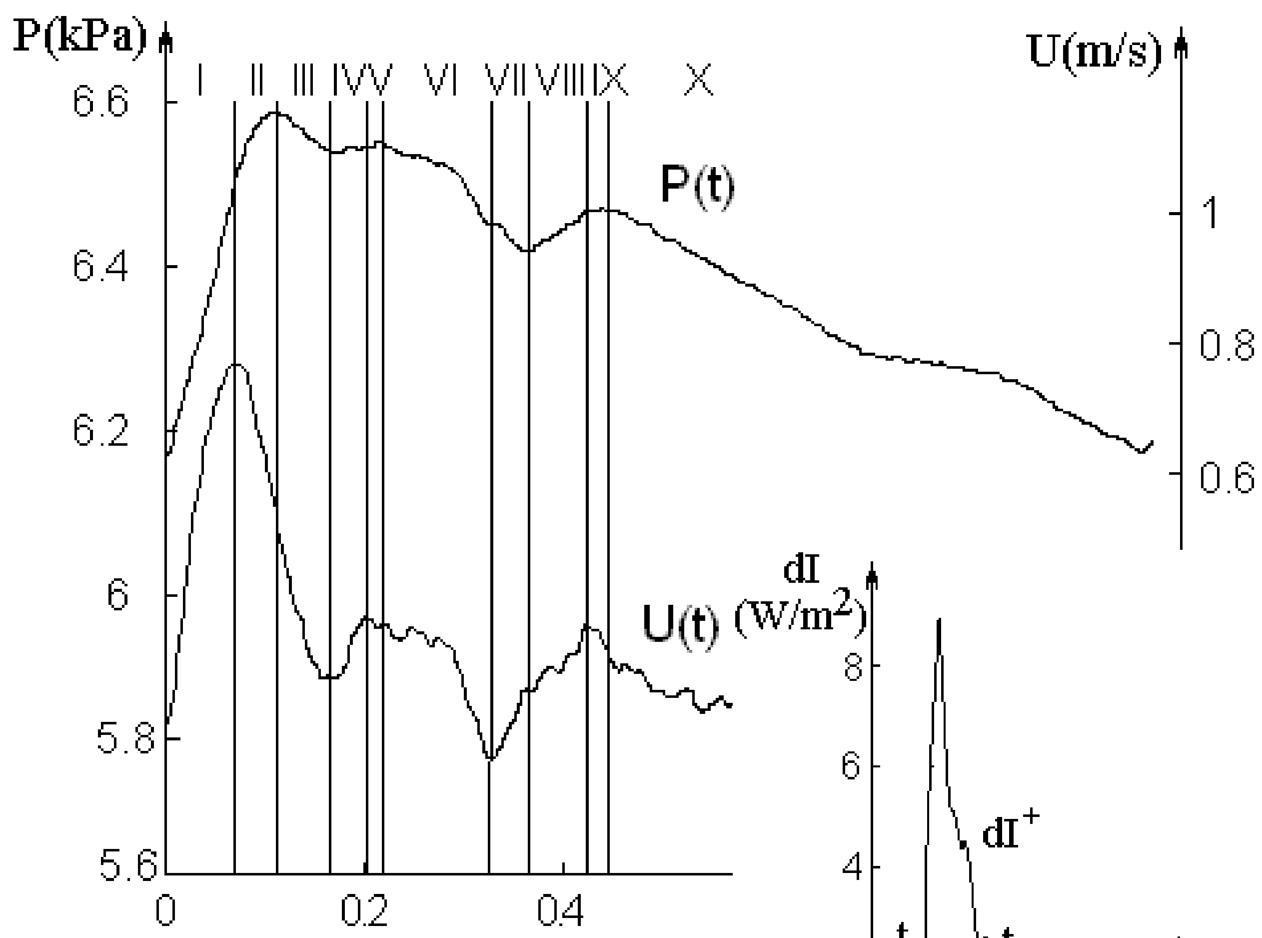


Femoral



Tibial

Pressure-flow curves

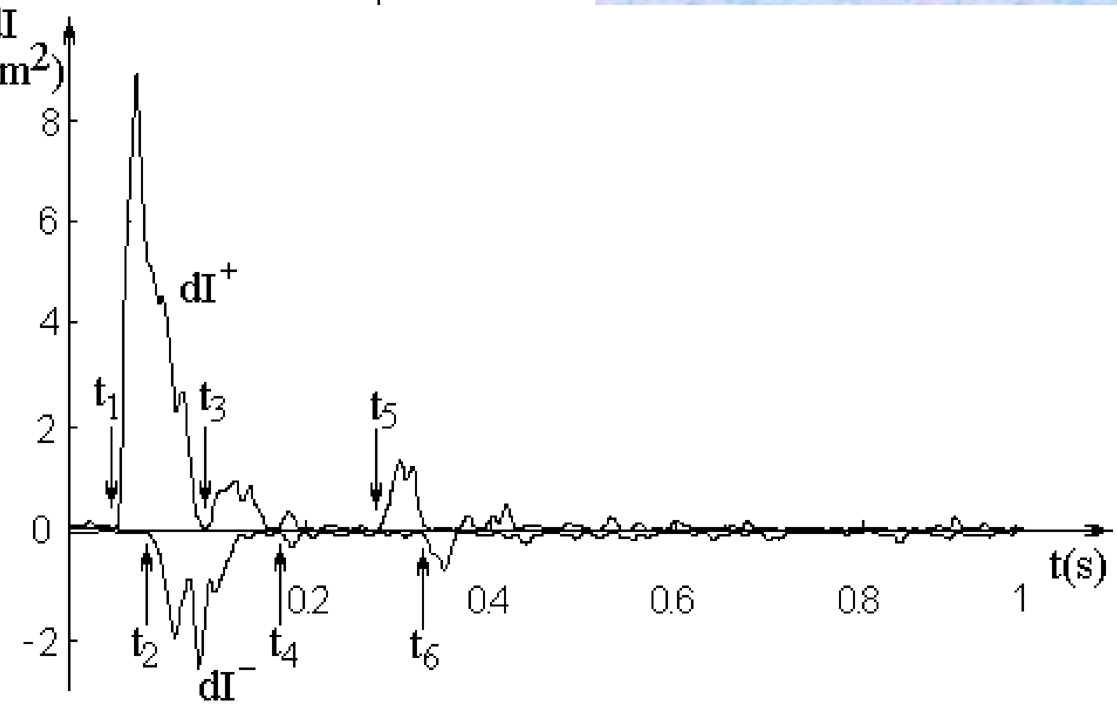


$$P^+ = \frac{P + \rho c U}{2}$$

$$P^- = \frac{P - \rho c U}{2}$$

$$c = \frac{1}{\rho}\frac{dP}{dU}$$

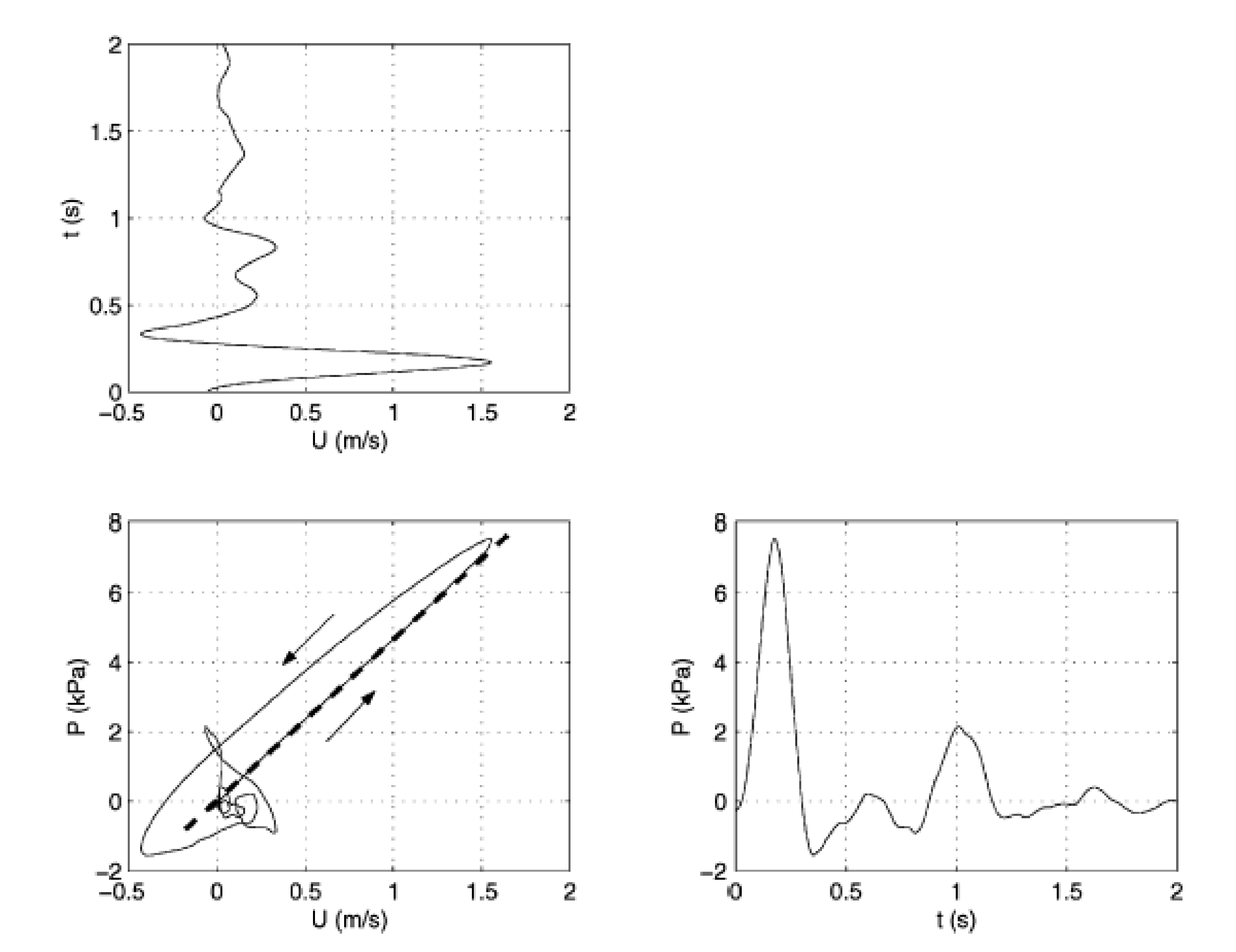
$$Z = \rho c / S$$



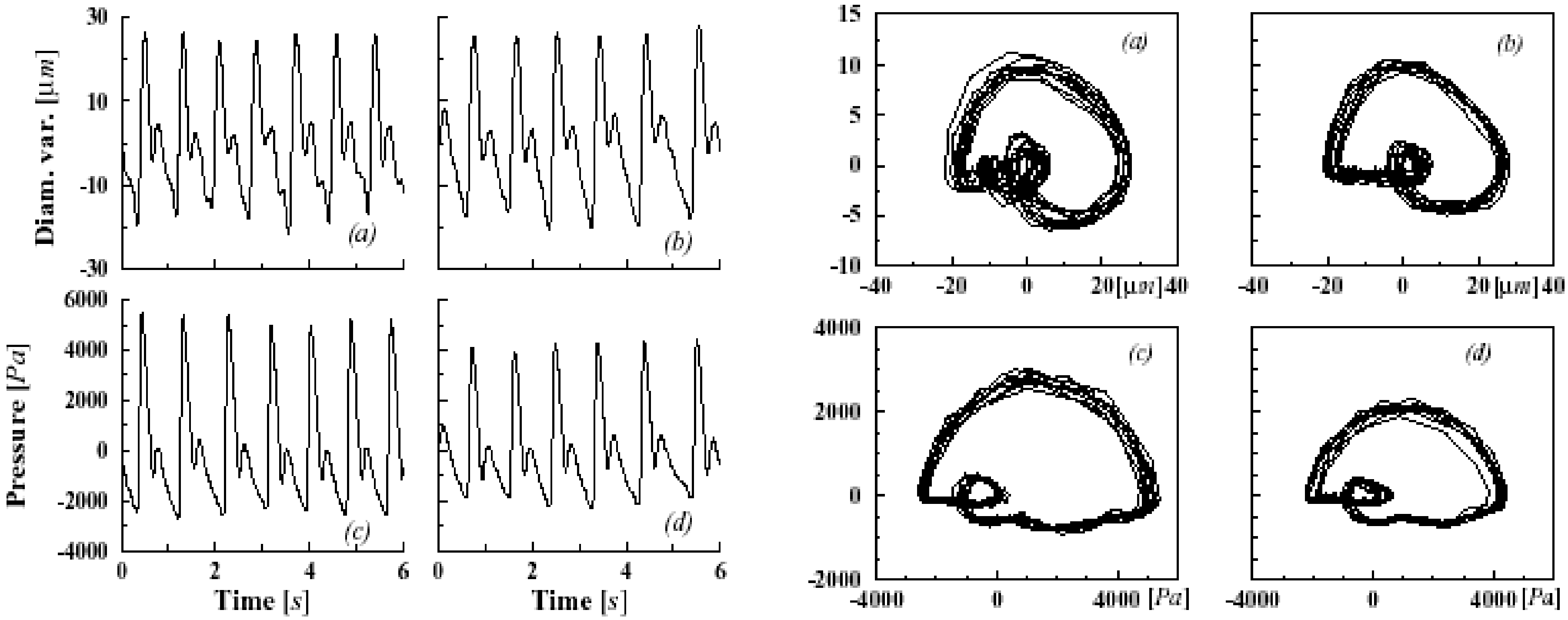
$$I^{\pm} = P^{\pm} U^{\pm}$$

$$\Delta I^{\pm}(t) = \pm \frac{1}{4\rho c}\left(\Delta P(t) \pm \rho c \Delta U(t)\right)^2$$

Analysis of pressure-flow loops

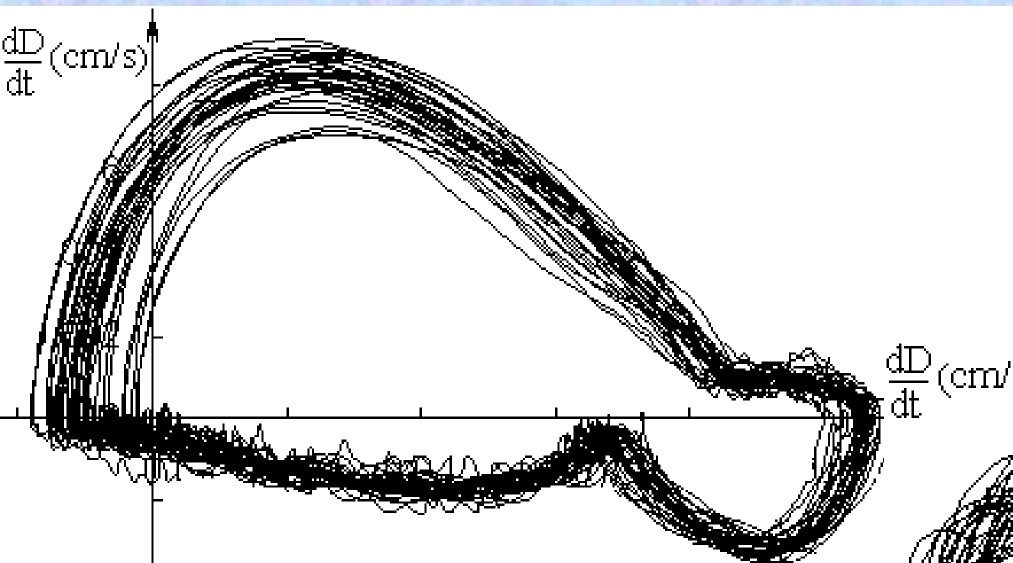


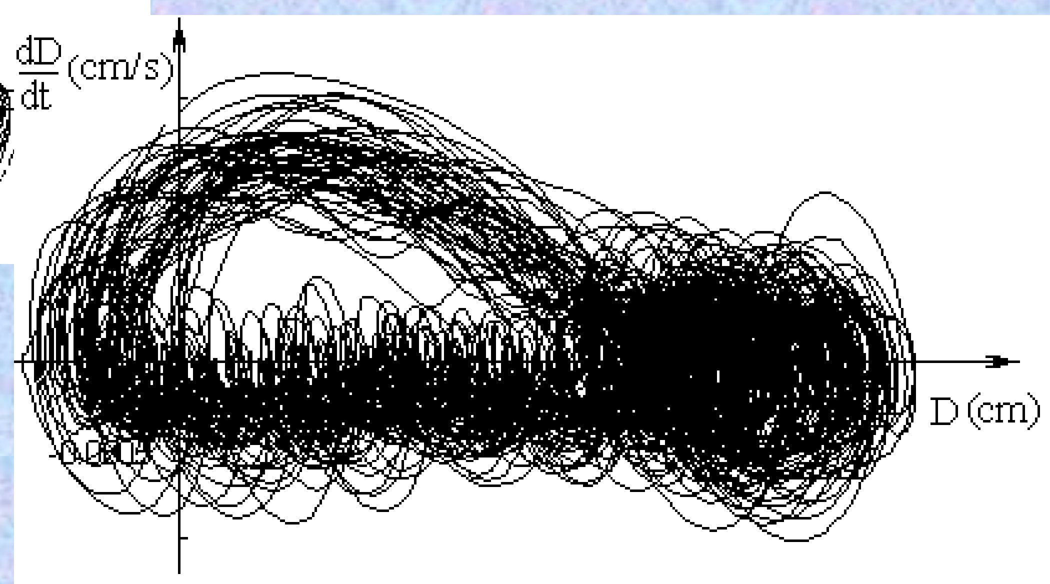
Ultrasound measurements of the vessel wall (pressure) and flow oscillations



N. Michoux, R. Joannid, G. Gouesbet, C. Thuillez, B. Maheu, and L. Le Sceller Physical determinism in human arterial dynamics. *Eur. Phys. J.* Ser.A (1999) 8:265-268.

D/(D) curves





$s = \frac{\Sigma p}{c + p}$

L.Euler (1707-1783)

Principia pro motu sanguinis per arterias determinando (1755)

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(AU) = 0$$

$$A = \alpha(1 - \exp^{-P/c})$$

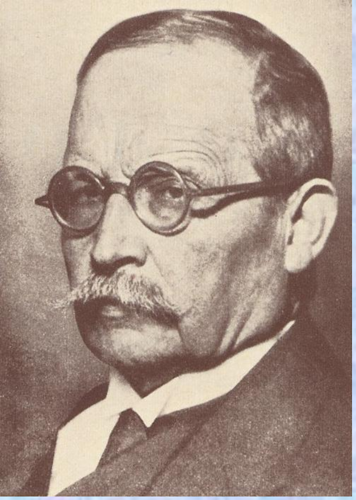
$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$A = \frac{\beta P}{\gamma + P}$$

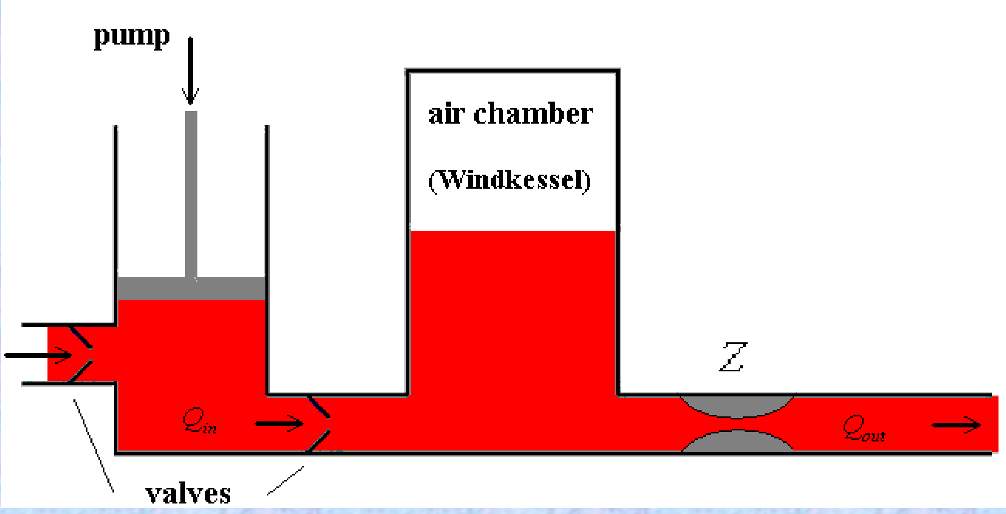
$$A = A(P, x)$$

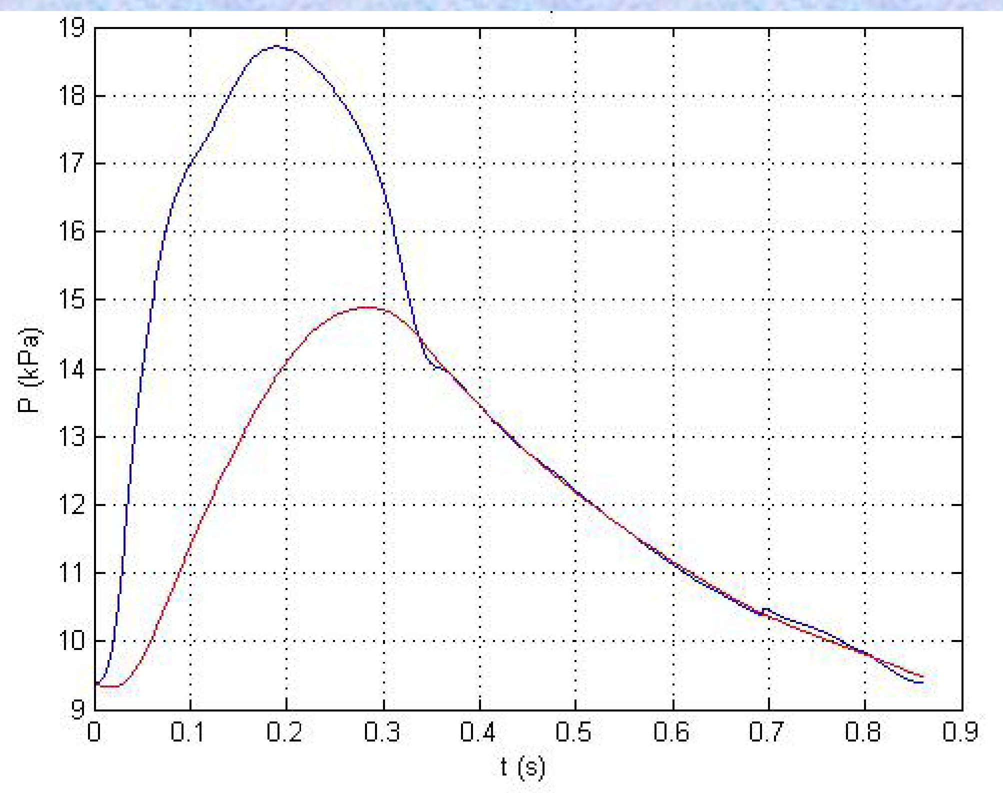
$$P = P(t, x); U = U(t, x)$$

43. In motu igitur sanguinis explicando easdem offendimus insuperabiles difficultates, quae nos impediunt omnia plane opera Creatoris accuratius perscrutari; ubi perpetuo multo magis summam sapientiam cum omnipotentia coniunctam admirari ac venerari debemus, cum ne summum quidem ingenium humanum vel levissimae vibrillae veram structuram percipere atque explicare valeat.



O.Frank
(1815-1944)





$$\frac{dV}{dt} = Q_{in} - Q_{out}$$

$$V(t) = V_0 + kP(t); \quad Q_{out} = \frac{P}{Z}$$

$$k\frac{dP}{dt} + \frac{P}{Z} = Q_{in}(t)$$

$$P(t) = e^{-t/Zk}\left(P_0 + \int_0^t Q_{in}(\tau) e^{\tau/Zk}\, d\tau\right)$$

1d linear theory of the waves in arteries

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(AU) = 0$$

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial P}{\partial x}$$

$$A = A(P, x)$$

$$P = P(t, x); U = U(t, x)$$

$$A = A(P)$$

$$U = U_0 + U', \quad P = P_0 + P'$$

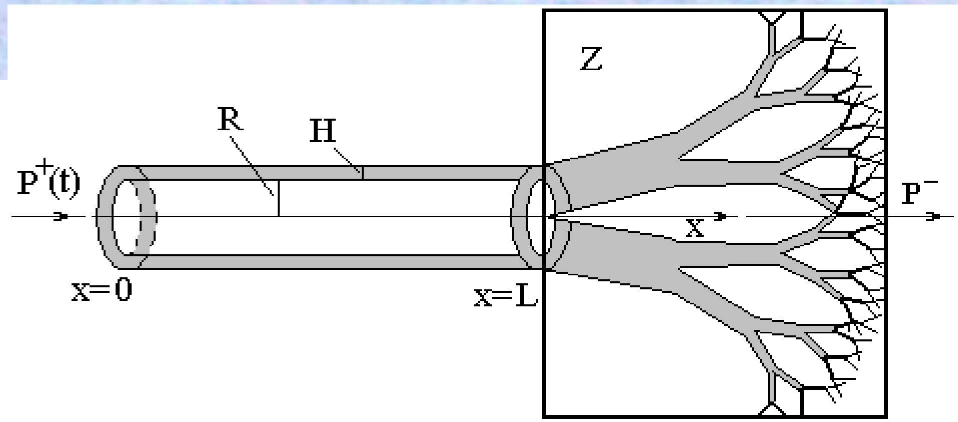
$$D\frac{\partial P'}{\partial t} + \frac{\partial U'}{\partial x} = 0$$

$$\rho\frac{\partial U'}{\partial t} + \frac{\partial P'}{\partial x} = 0$$

$$D = \left(A\frac{dP}{dA}\right)^{-1}$$

Lighthill, M.J. (1978) *Waves in fluids*. Cambridge University Press, Cambridge

2d axially symmetric wave propagating in thick wall viscoelastic tube



$$\mathrm{div}\,\vec{v} = 0, \quad \rho\left(\frac{\partial \vec{v}}{\partial t} + (\vec{v}\nabla)\vec{v}\right) = -\nabla p + \mu\Delta\vec{v}$$

$$\mathrm{div}\,\vec{u} = 0, \quad \rho_s \frac{\partial^2 \vec{u}}{\partial t^2} = -\nabla p_s + \mathrm{div}\,\hat{\sigma}$$

$$r = 0: \quad v_r = 0, \quad |v_x| < \infty$$

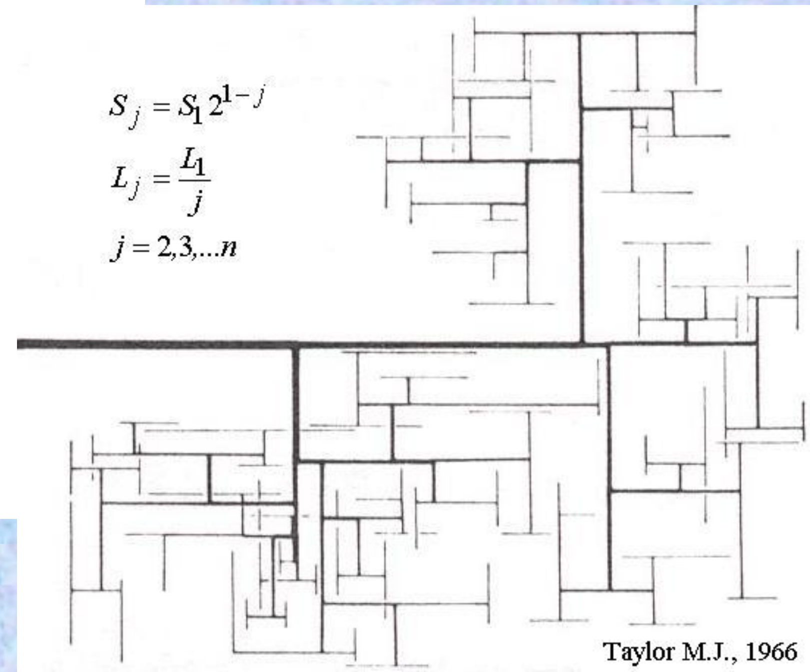
$$r = R': \quad \frac{\partial \vec{u}}{\partial t} = \vec{v}, \quad -p + \mu\frac{\partial v_r}{\partial r} = -p_s + \sigma_{rr}, \quad \mu\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right) = \sigma_{rx}$$

$$r = R_0 + H: \quad \vec{u} = 0$$

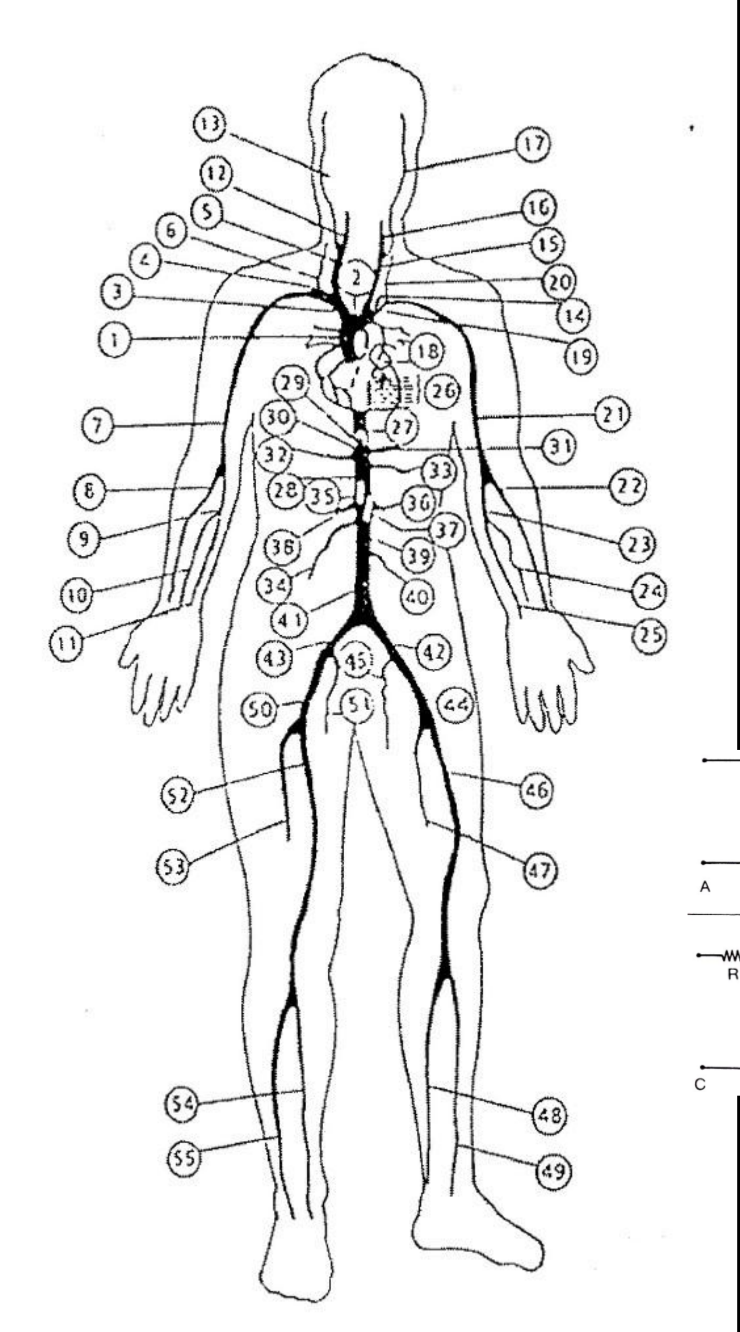
$$\hat{\sigma} = \hat{E}\hat{\varepsilon} + \mu_s \frac{d\hat{\varepsilon}}{dt}$$

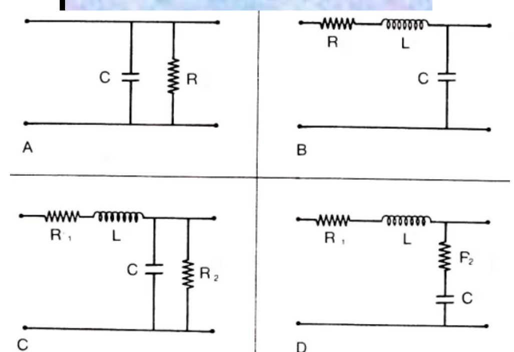
$$x = 0: \quad P = \sum_{k=0}^{\infty} P_k^0(r)\, e^{i\omega_k t}$$

$$x = L: \quad \int_0^{R'} r p\, dr = \pi Z^-(R')^2 \int_0^{R'} r v_x\, dr$$

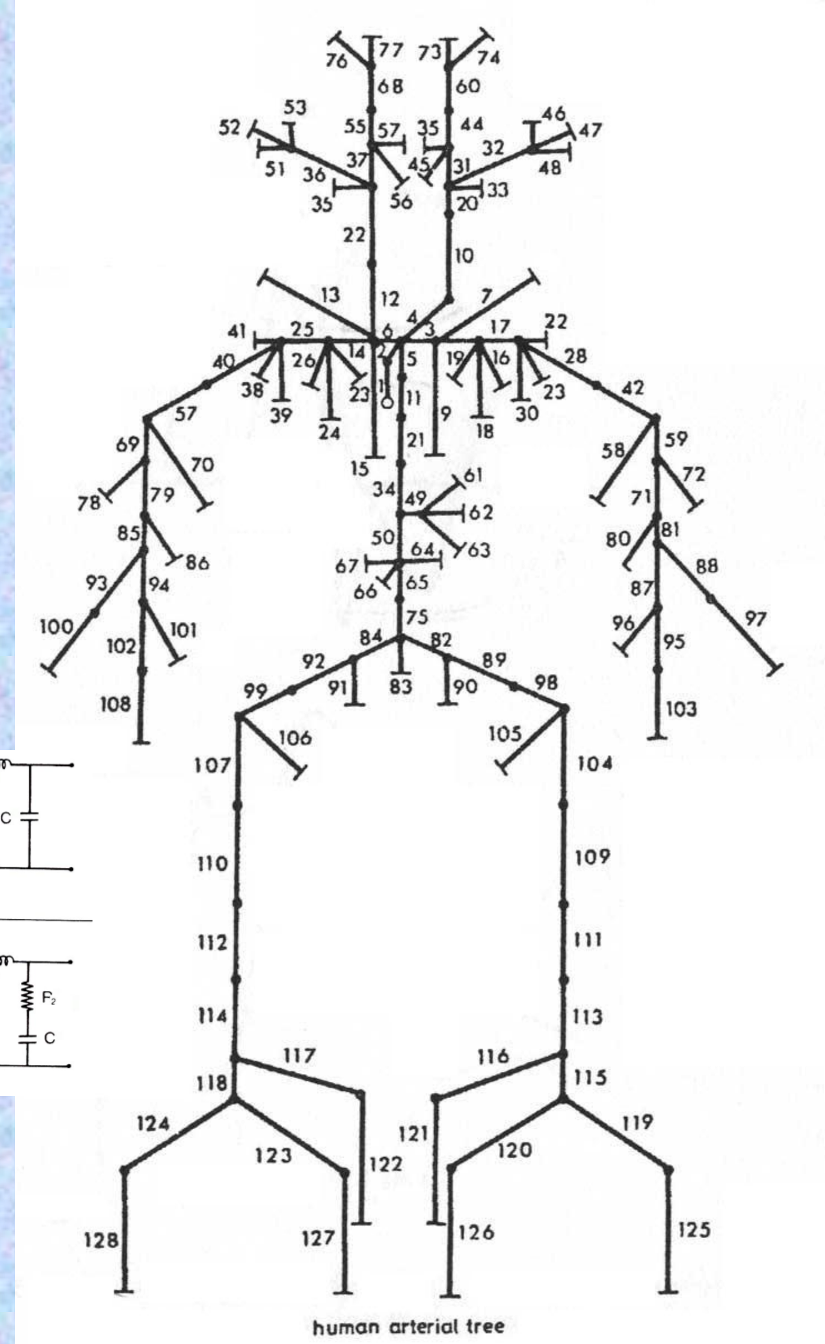


J.R. Womersley (1957)



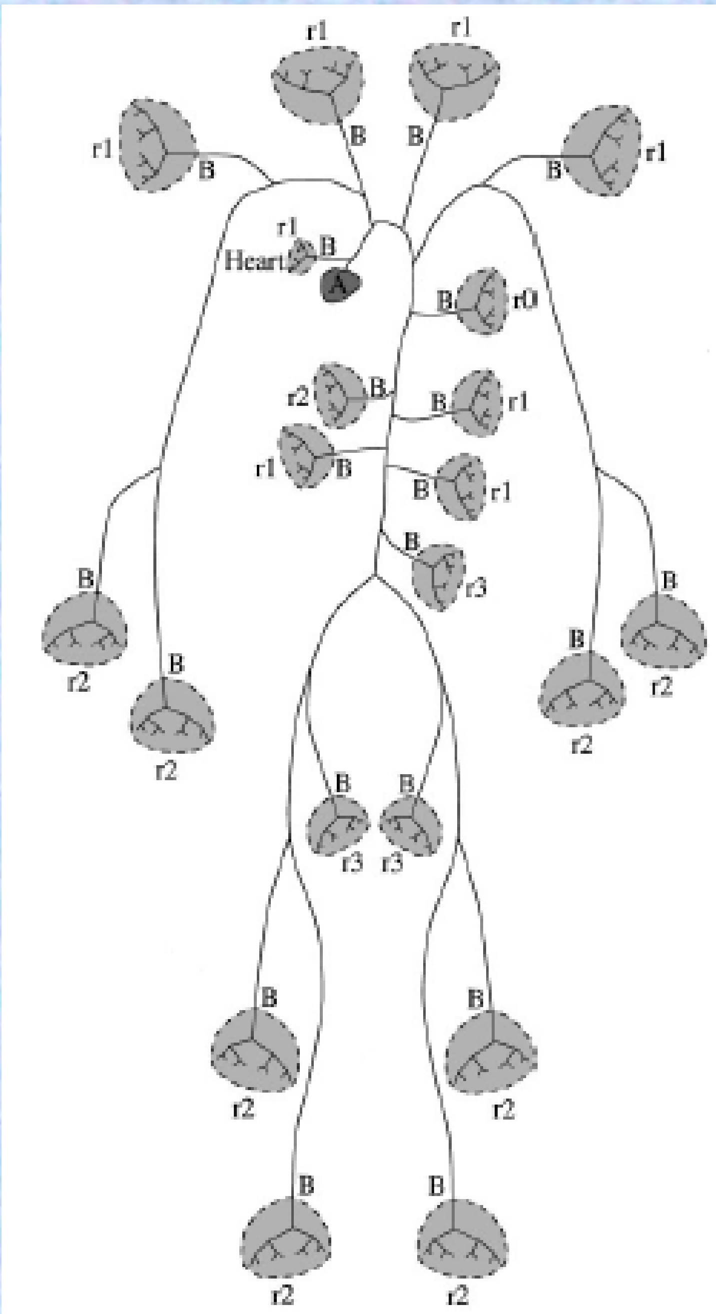


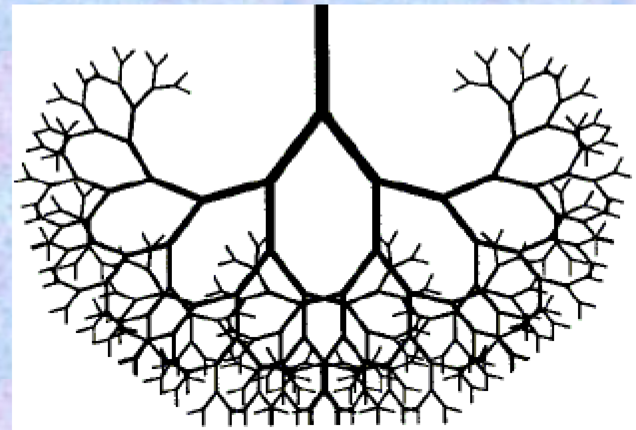
55-tube in vitro model. [Westerhof et al 1968]



76-tube in vitro model. [Avolio et al 1980]

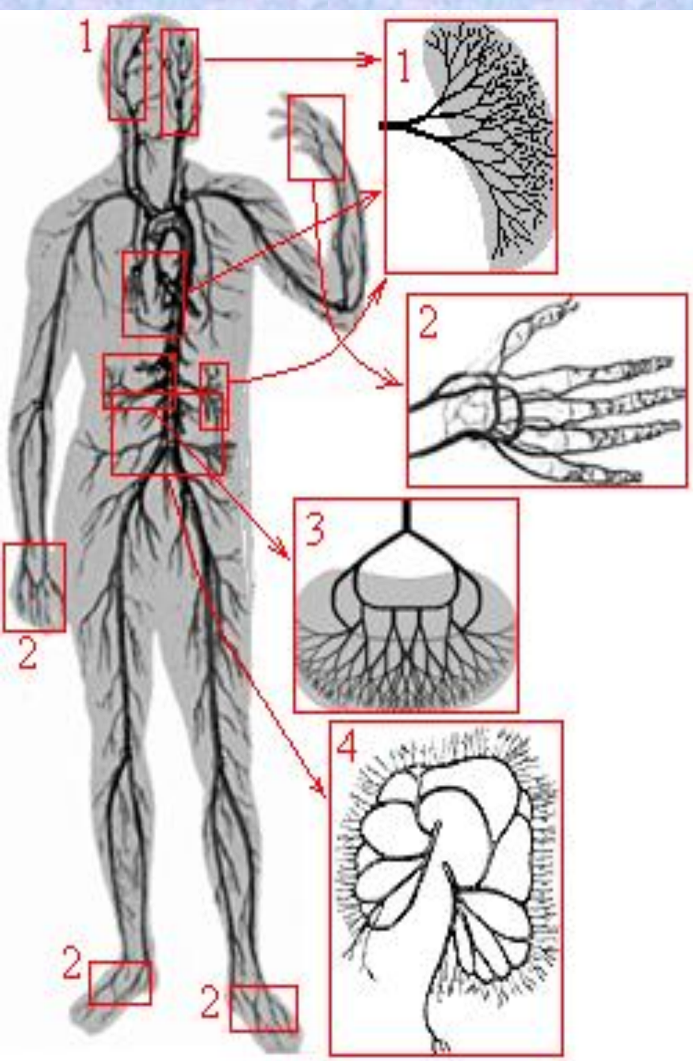
Полная модель сосудистого русла человека

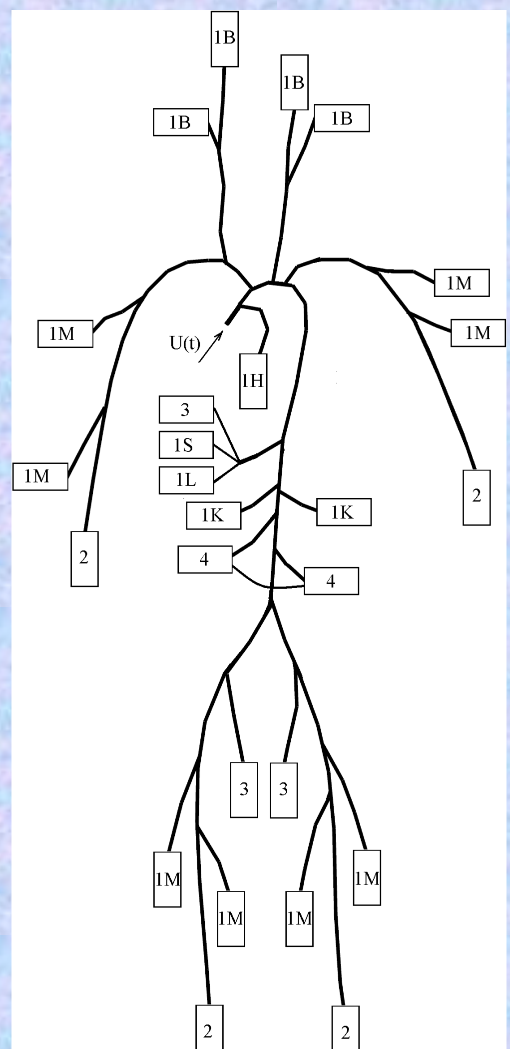




55-tube model with tree-like terminal elements
[Olufsen M.1998]

78-tube in vivo model of the human systemic arterial tree [Kizilova N.N., Zenin O.K. 2005]





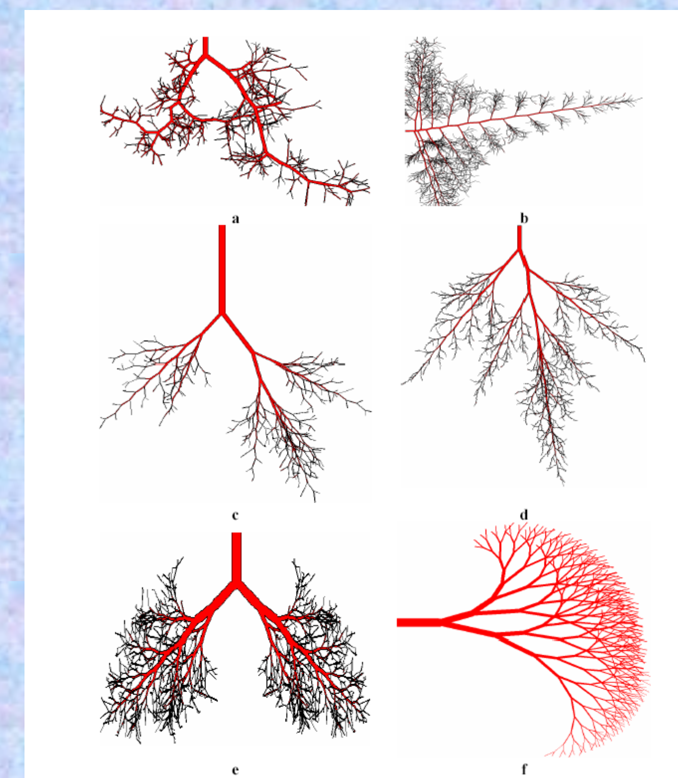
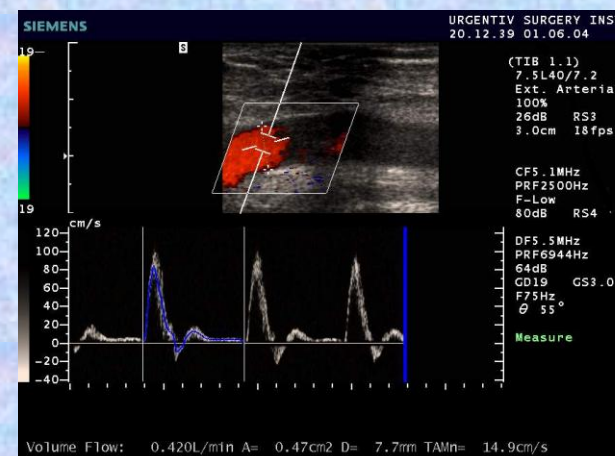
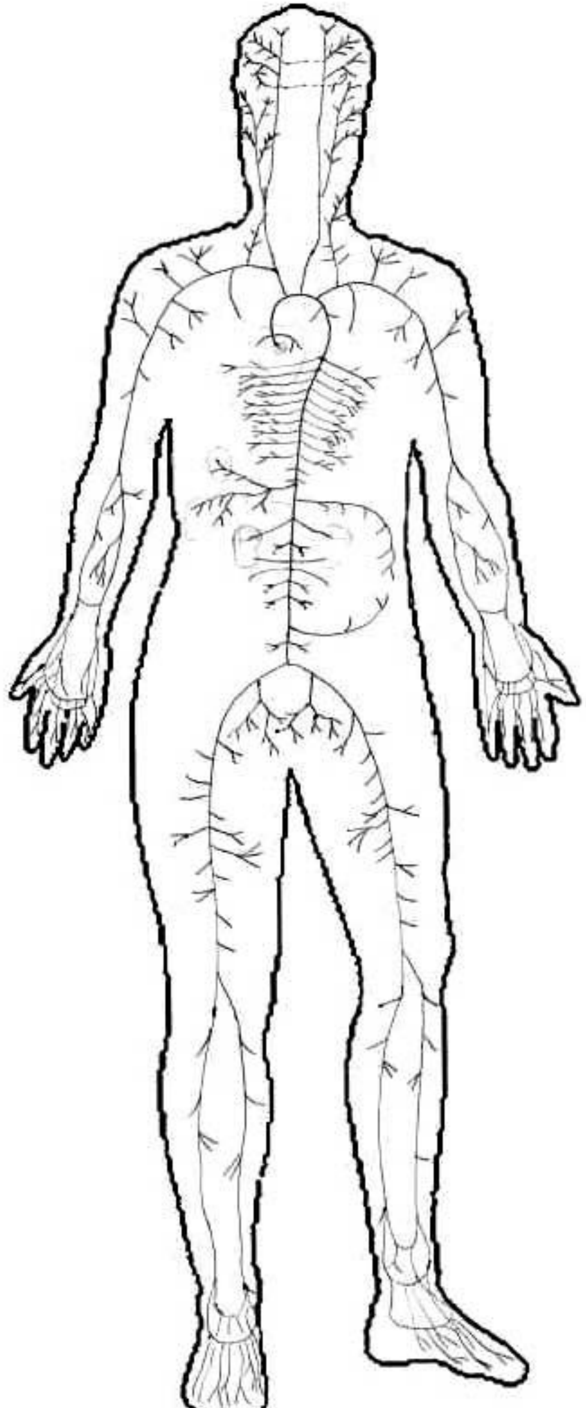
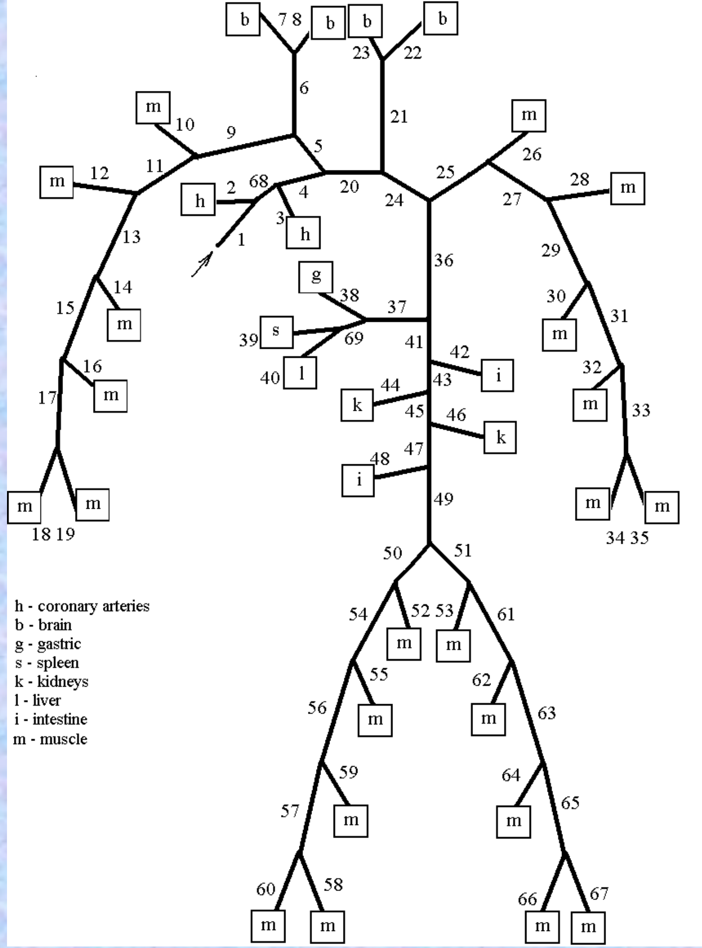


Fig. 4. The restored tree of the right coronary artery (a) and the generated models of brain artery (b), vasculatures of the liver (c), muscle (d), kidneys (e) and a regularly bifurcating tree with given asymmetry coefficient $\xi = 0.7$ (f).

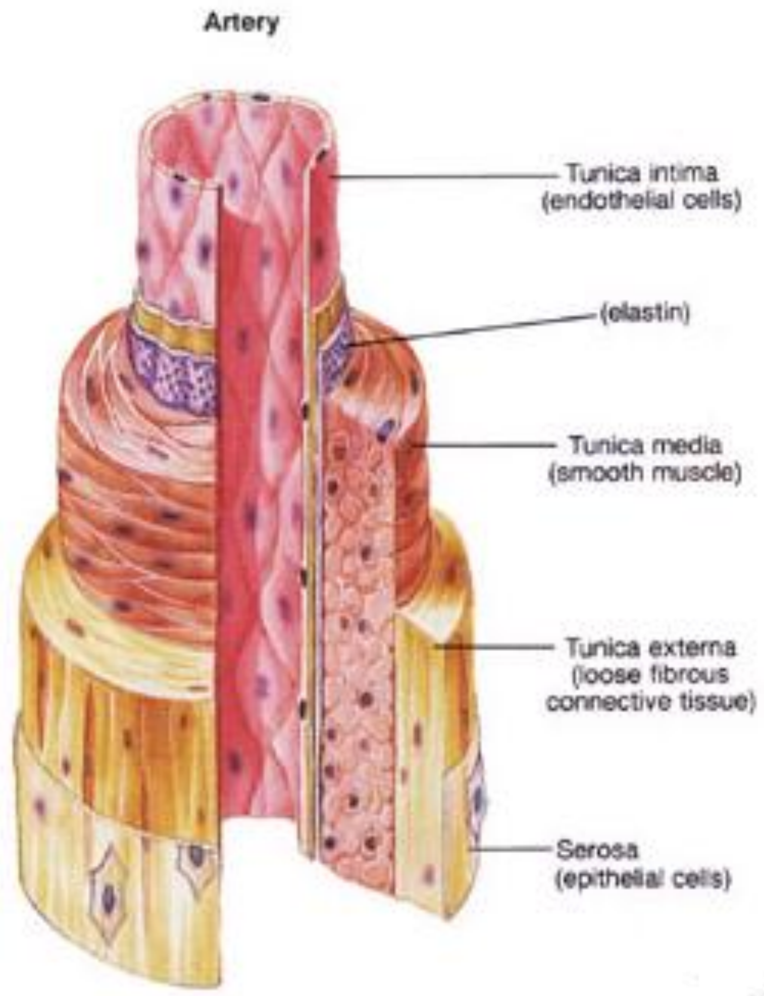




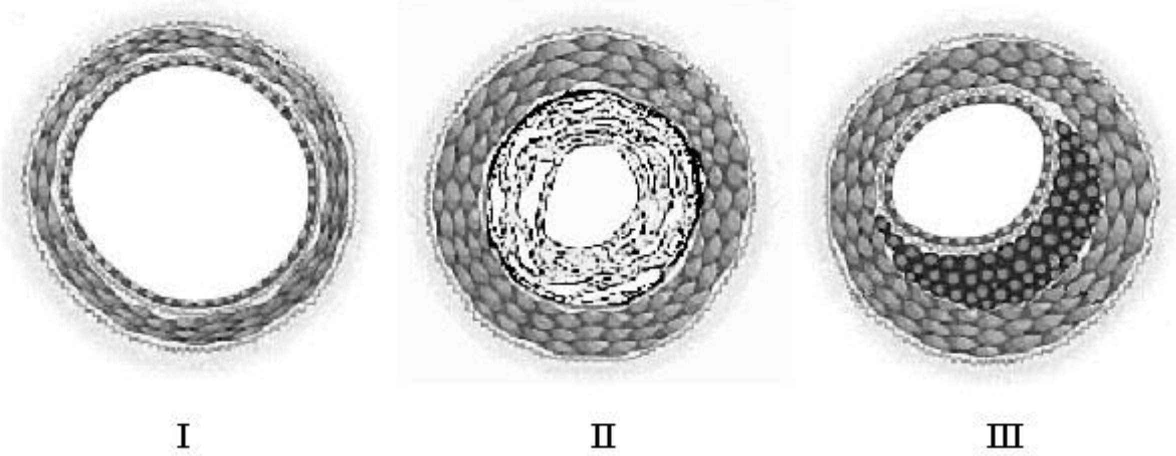


1000-tube in vitro model
+ 2d axisymmetric model of the pulse wave propagation and reflection
[Kizilova, Zenin, Philippova 2008-2009]

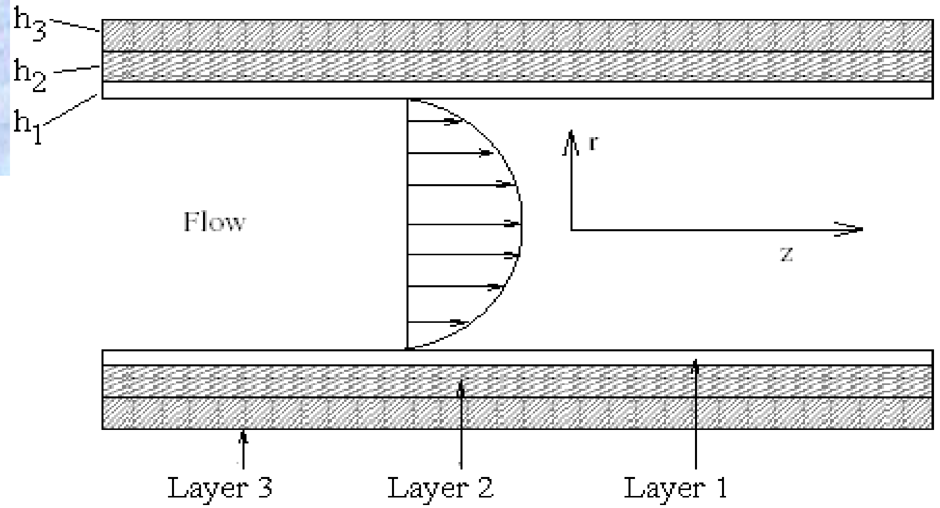
Natalya Kizilova *Optimal transport networks in nature*. World Scientific Publishers. 2009. 204p.



Realistic 3-layer structure of the blood vessel wall



Problem formulation
(2d linearized model)



$$div(\vec{v}) = 0\,,\ \rho_f\,\frac{d\vec{v}}{dt} = -\nabla p + \mu_f\,\Delta\vec{v}$$

$$div(\vec{u}^{(j)}) = 0\,,\ \rho_s^{(j)}\,\frac{\partial^2 \vec{u}^{(j)}}{\partial t^2} = -\nabla p_s^{(j)} + div(\sigma^{(j)})$$

$$\left(\tau_s^{(j)}\frac{D}{Dt} + 1\right)\sigma_i^{(j)} = A_{ik}^{(j)}\varepsilon_k^{(j)} + \mu_s^{(j)}\frac{D}{Dt}\varepsilon_i^{(j)}$$

$$\left(A_{ik}^{(j)}\right)^{-1} = \begin{pmatrix} (E_1^{(j)})^{-1} & -\nu_{21}^{(j)}(E_2^{(j)})^{-1} & -\nu_{31}^{(j)}(E_3^{(j)})^{-1} & 0 & 0 & 0 \\ -\nu_{12}^{(j)}(E_1^{(j)})^{-1} & (E_2^{(j)})^{-1} & -\nu_{32}^{(j)}(E_3^{(j)})^{-1} & 0 & 0 & 0 \\ -\nu_{13}^{(j)}(E_1^{(j)})^{-1} & -\nu_{23}^{(j)}(E_2^{(j)})^{-1} & (E_3^{(j)})^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (G_1^{(j)})^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & (G_2^{(j)})^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & (G_3^{(j)})^{-1} \end{pmatrix}$$

$$r = R:\quad \vec{v} = \partial\vec{u}^{(1)}/\partial t,\quad \sigma_n^{(1)} = \sigma_n$$

$$r = R + h_1:\quad \partial\vec{u}^{(1)}/\partial t = \partial\vec{u}^{(2)}/\partial t,\quad \sigma_n^{(1)} = \sigma_n^{(2)}$$

$$r = R + h_1 + h_2:\quad \partial\vec{u}^{(2)}/\partial t = \partial\vec{u}^{(3)}/\partial t,\quad \sigma_n^{(2)} = \sigma_n^{(3)}$$

$$r = R + h:\quad \vec{u}^{(3)} = 0 \ \text{ or } \ \sigma_n^{(3)} = 0$$

Hamadiche M., Kizilova N., Gad-el-Hak M. Suppression of Absolute Instabilities in the Flow inside a Compliant Tube. //Comm. Numer. Meth. in Eng. 2009. 25:505-531.

Problem formulation (1d nonlinear model)

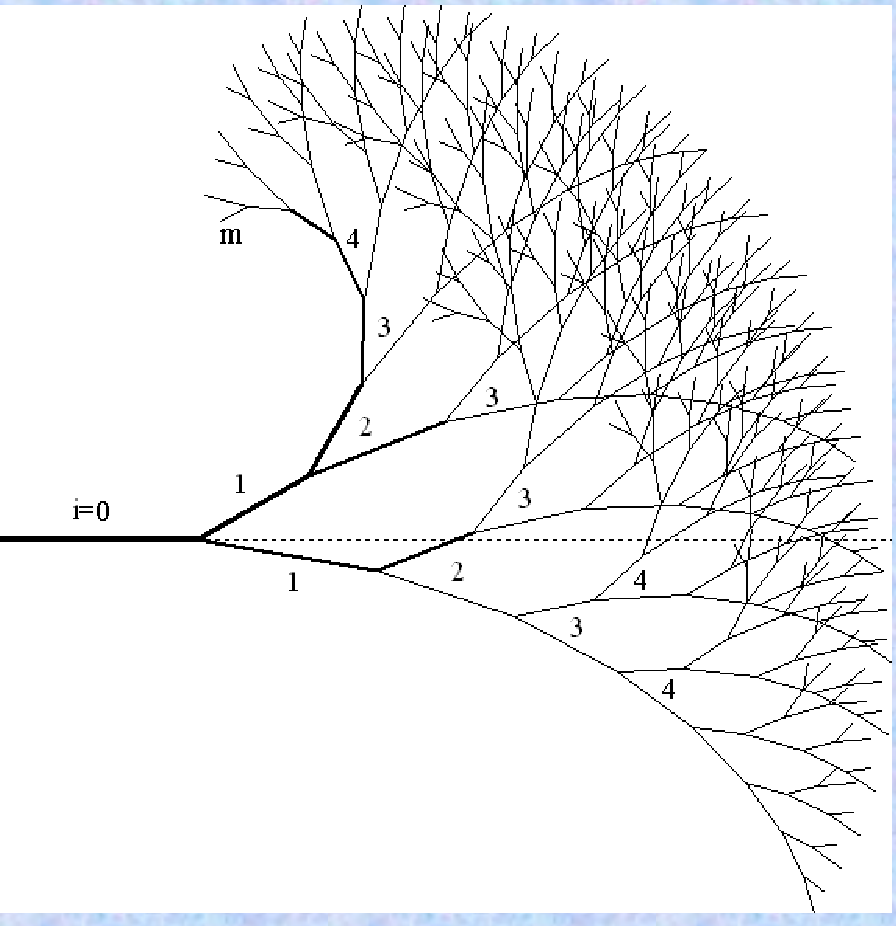
$$\frac{\partial}{\partial t}\begin{bmatrix} A \\ U \end{bmatrix} + \frac{\partial}{\partial x}\begin{bmatrix} AU \\ \frac{U^2}{2} + \frac{P}{\rho} \end{bmatrix} = \begin{bmatrix} 0 \\ -kU \end{bmatrix}$$

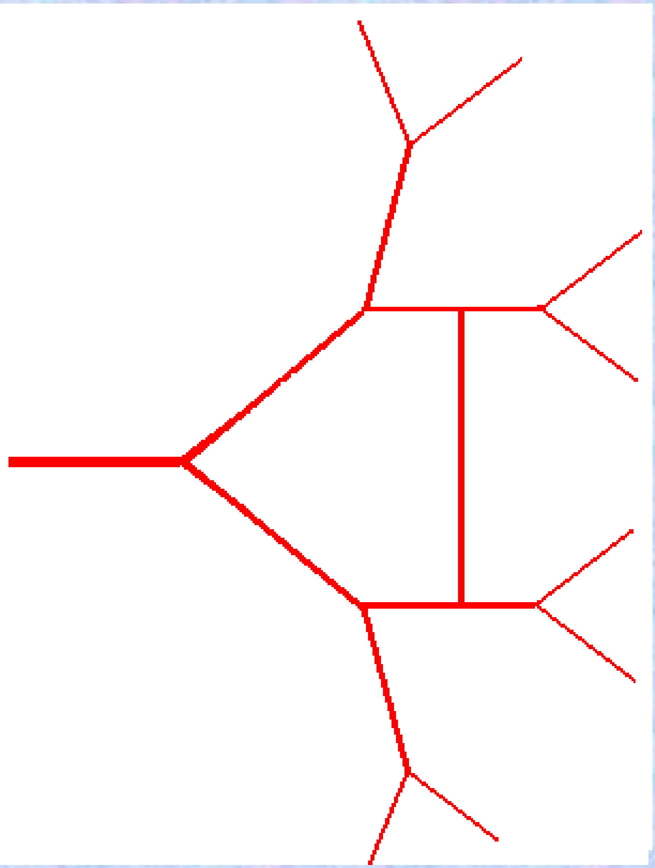
$$P = P_{ext} + \beta\left(\sqrt{A} - \sqrt{A_0}\right)$$

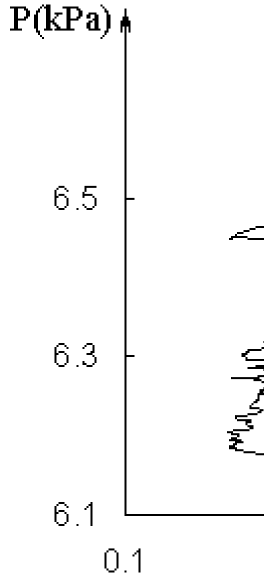
$$\beta = \frac{Eh\sqrt{\pi}}{(1-\nu^2)A_0}$$

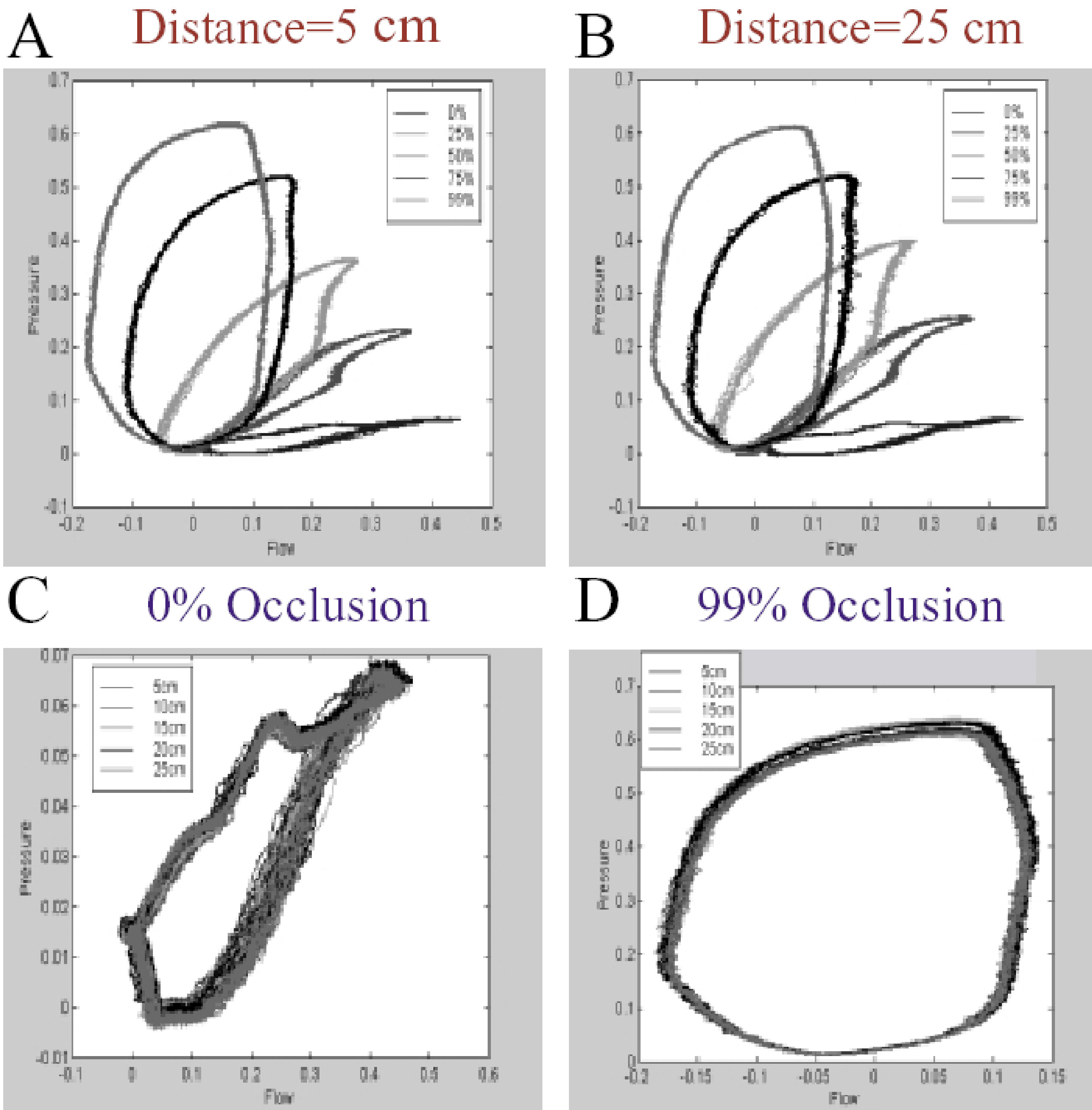
Sherwin S.J., Franke F., Piero J., Parker K.H. One-dimensional modelling of a vascular network in space-time variables. J.Eng.Math. 2003. 47:217-250.

Tree-like systems of distensible tubes



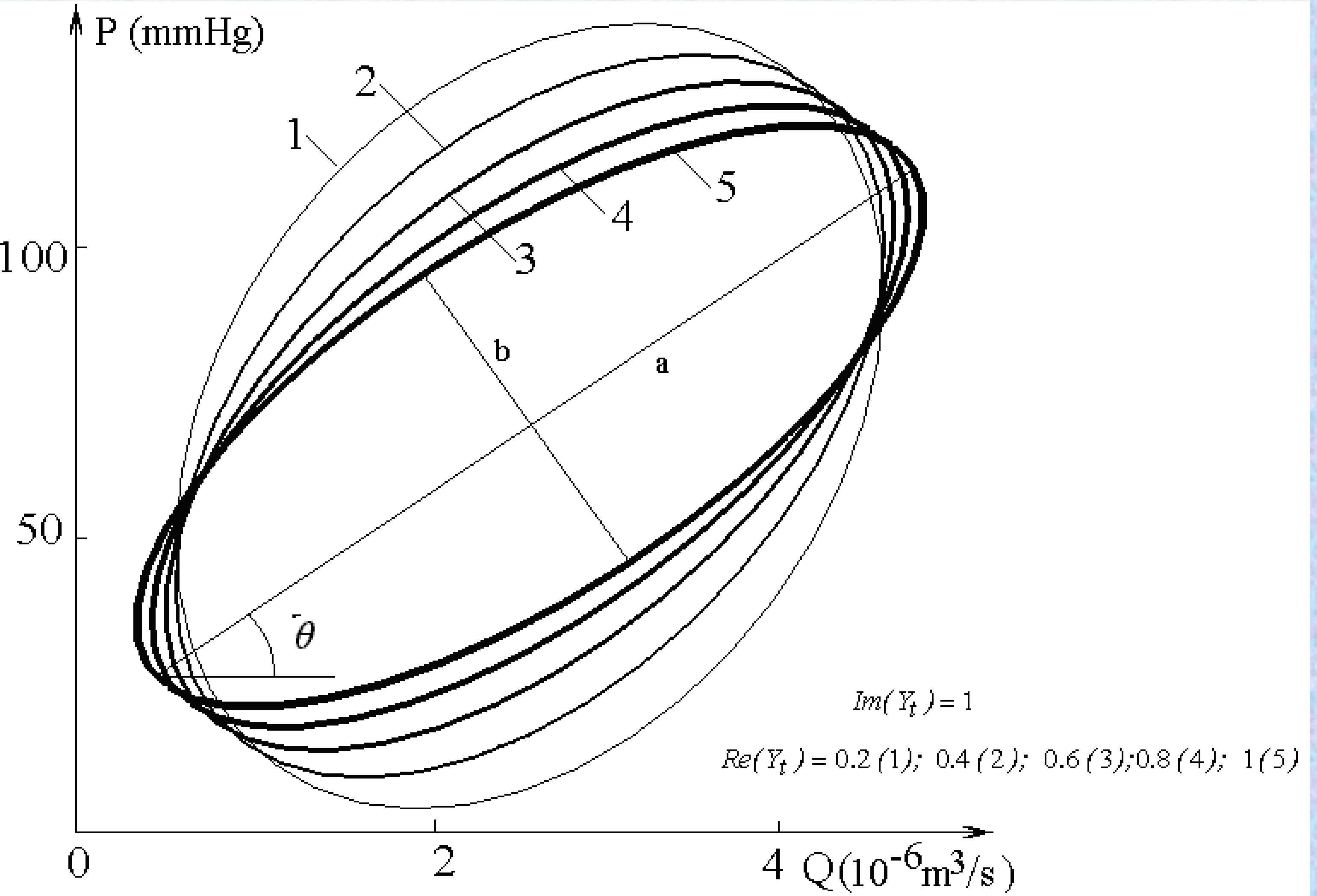



P(kPa)
6.5
6.3
6.1
0.1

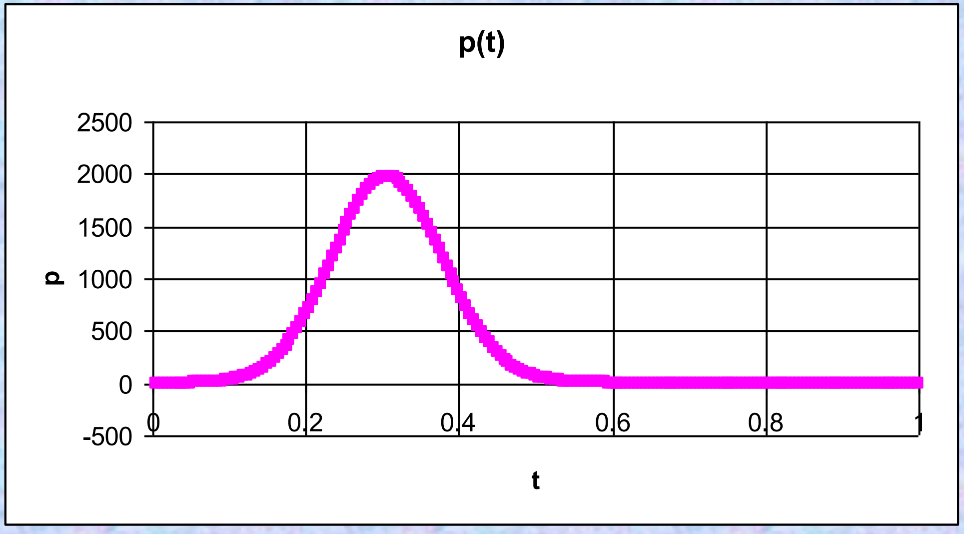

A
Distance=5 cm
B
Distance=25 cm
0%
25%
50%
75%
99%
Pressure
Flow
C
0% Occlusion
D
99% Occlusion
5cm
10cm
15cm
20cm
25cm

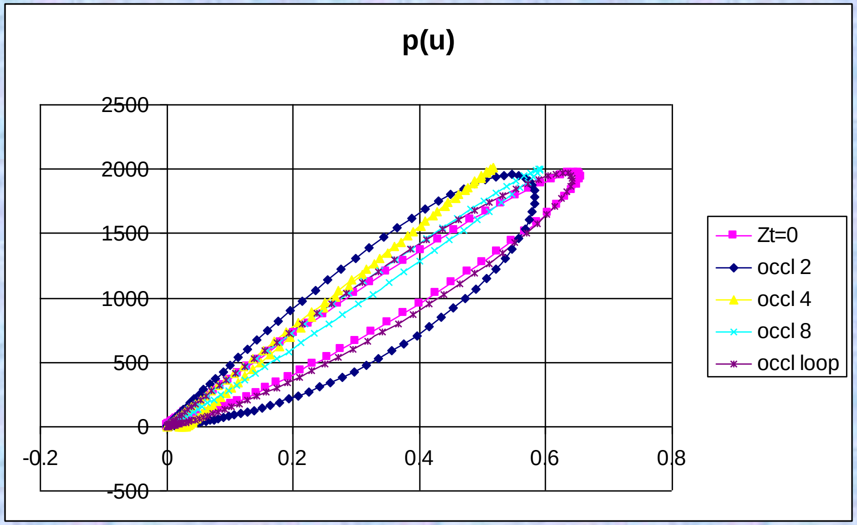

U

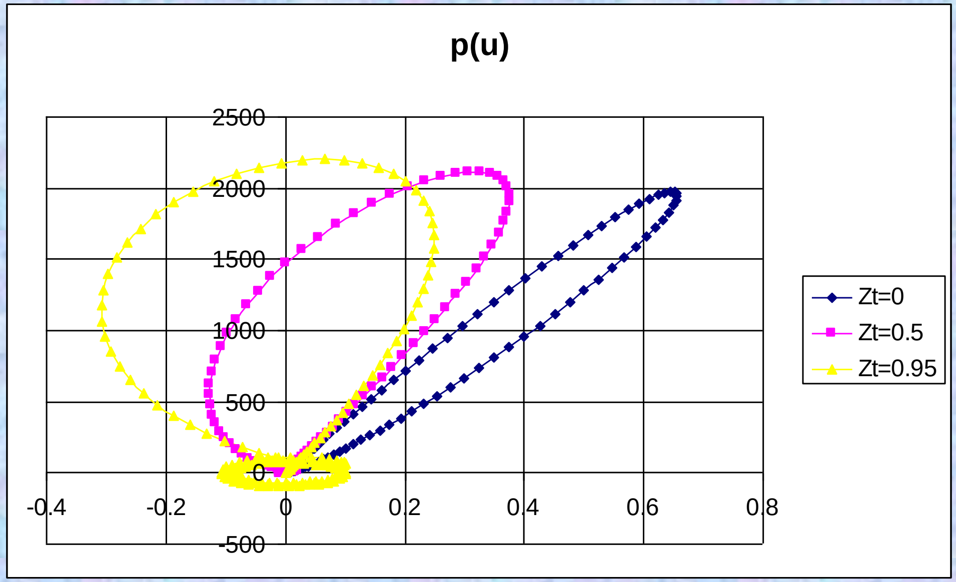
2d linearized model: P(Q) curves

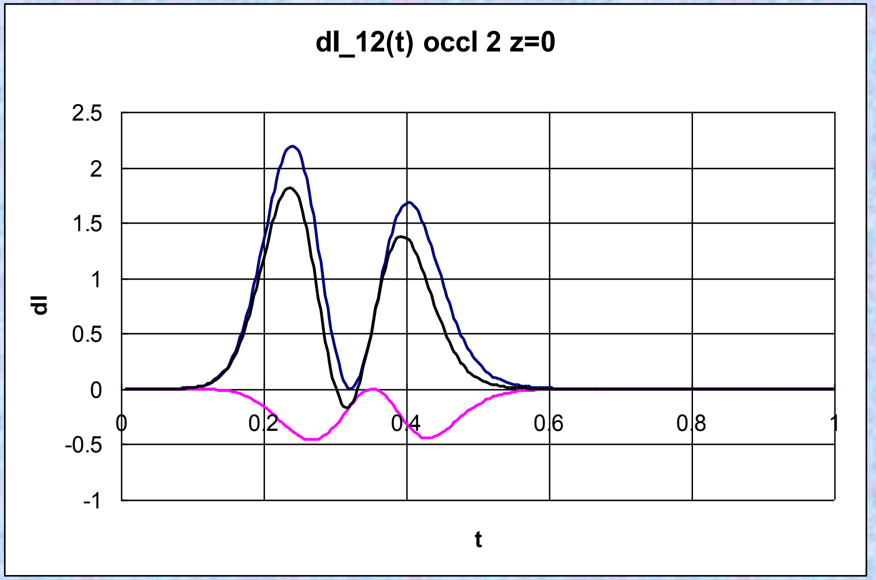


1d nonlinear model

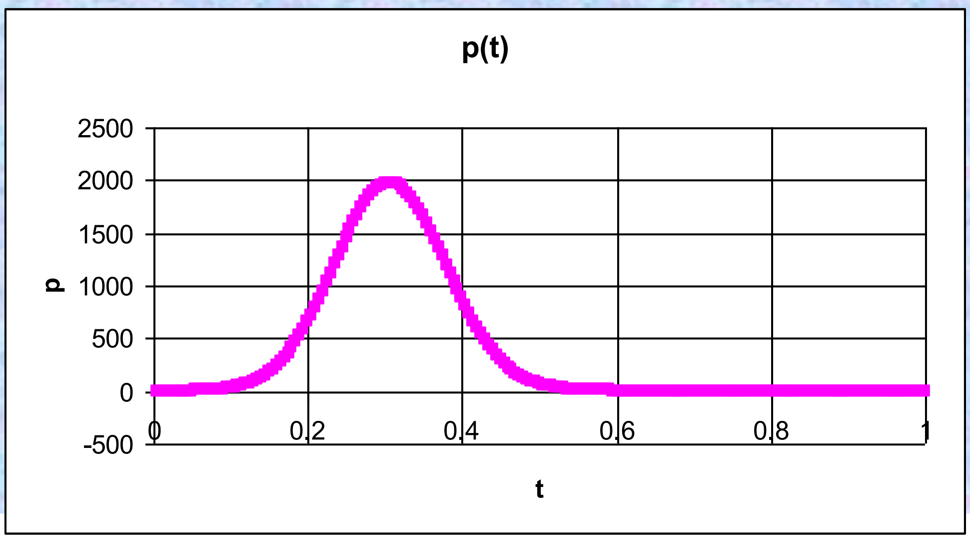


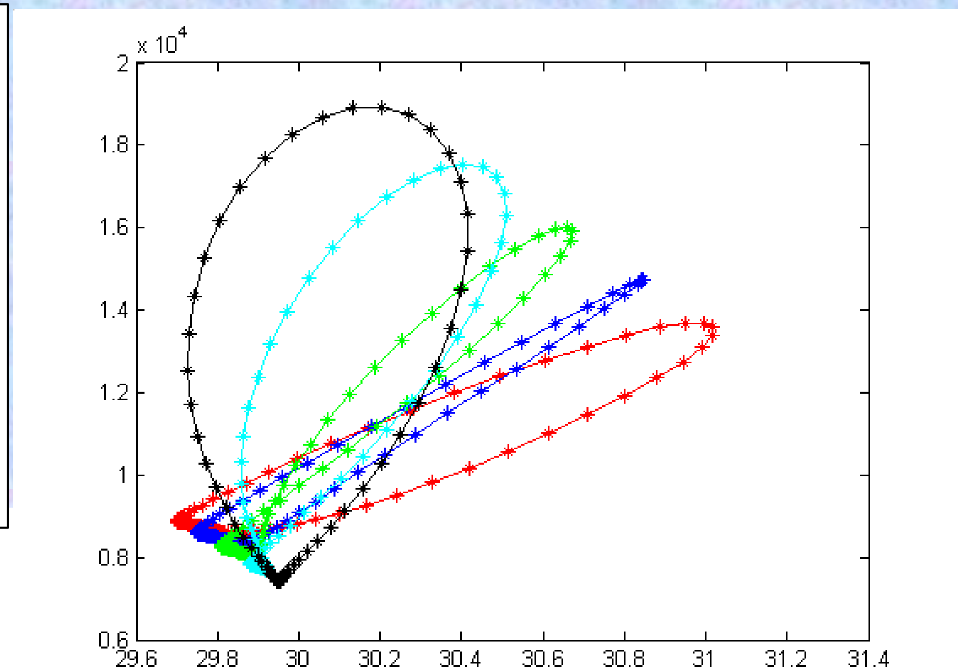


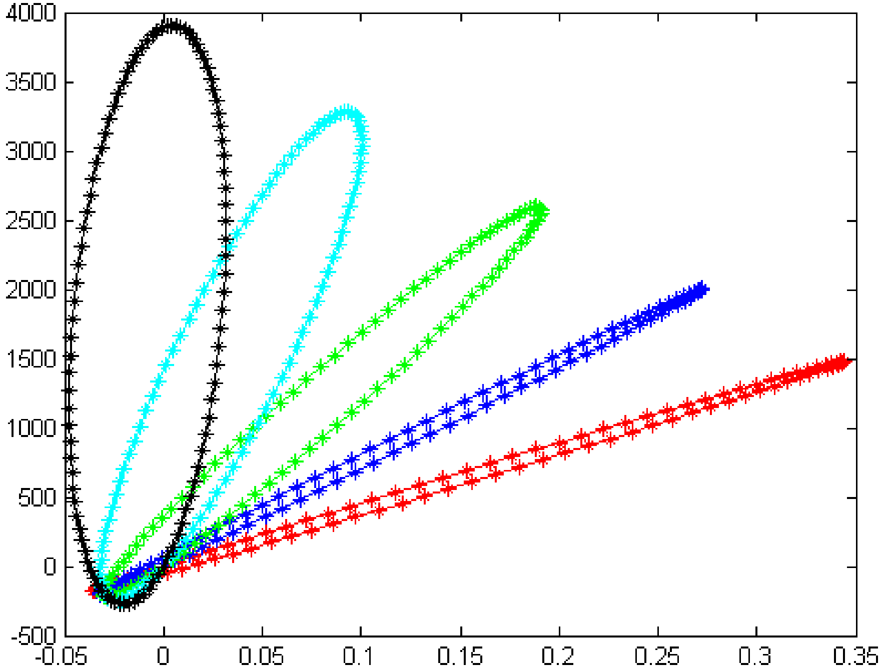


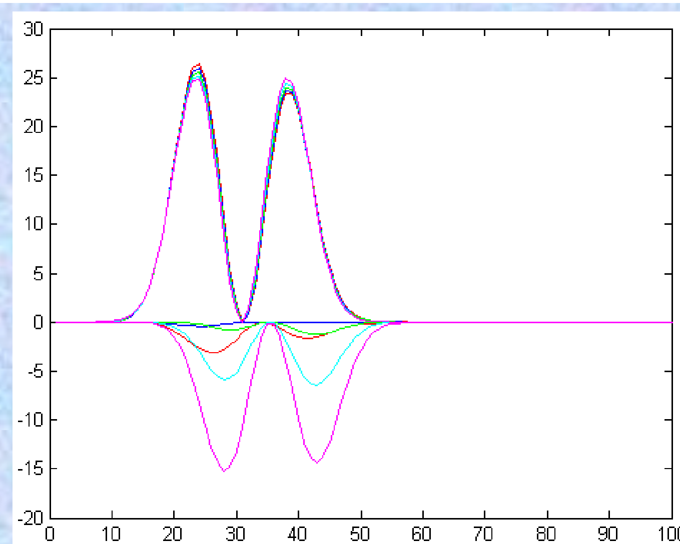


2d linearized model









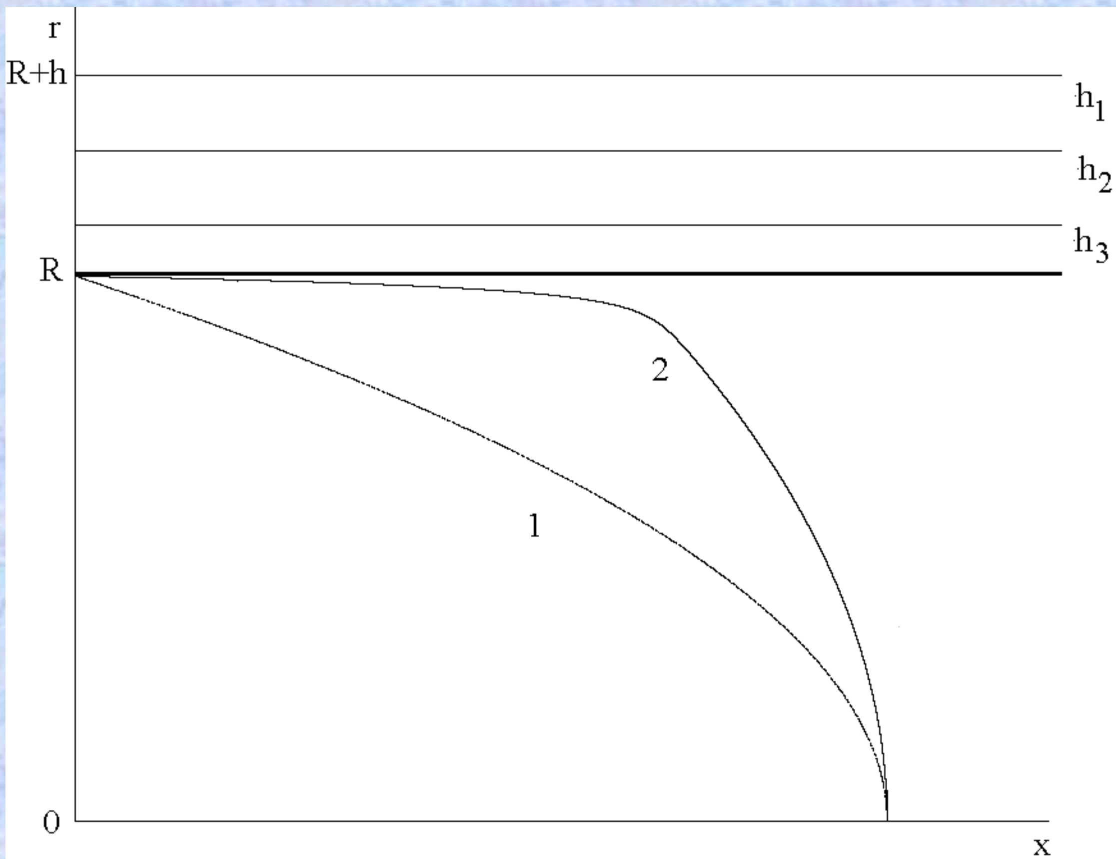
Conclusions:

- 2d linearized model: wave dispersion; wave superposition only; 2-element terminal element – describes very well wave propagation and reflection in normal arteries (maybe except for the late diastole)
- 1d nonlinear model: no dispersion; unrealistic shear stress; pure resistive terminal reflections – describes very well wave intensities in both normal and pathological arteries (maybe except for the late diastole)

SOLUTION OF THE PROBLEM IN THE FORM OF NORMAL MODE

$$\{v, p\} = \{v^b, p^b\} + \{v^*, p^*\}exp(st + in\theta + ikx)$$

$$\{u^{(j)}, p_s^{(j)}\} = \{u^{b(j)}, p_s^{b(j)}\} + \{u^{*(j)}, p^{*(j)}\}exp(st + in\theta + ikx)$$



Governing equations for amplitudes of the disturbances in fluid

$$\frac{d\tilde{v}_r}{dr} = \Lambda \quad ; \quad \frac{d\tilde{v}_\theta}{dr} = \tilde{\xi}_\theta \quad ; \quad \frac{d\tilde{v}_z}{dr} = \tilde{\xi}_z$$

$$\frac{d\tilde{p}_f}{dr} = -ik\tilde{V}_f\tilde{v}_r - s\tilde{v}_r + \frac{d\tilde{\sigma}_{rr}}{dr} + \frac{\tilde{\sigma}_{rr}}{r} + \frac{in}{r}\tilde{\sigma}_{r\theta} + ik\tilde{\sigma}_{rz} - \frac{\tilde{\sigma}_{\theta\theta}}{r}$$

$$\frac{d\tilde{\xi}_\theta}{dr} = -\frac{in}{r}\Lambda + \frac{in\tilde{v}_r}{r^2} - \frac{\tilde{v}_\theta}{r^2} + \frac{1}{r}\tilde{\xi}_\theta + Re\Gamma^{-1}[ik\tilde{V}_f\tilde{v}_\theta + s\tilde{u}_\theta + \frac{in}{r}\tilde{p}_f - \frac{in}{r}\tilde{\sigma}_{\theta\theta} - ik\tilde{\sigma}_{\theta z} - \frac{\tilde{\sigma}_{r\theta} + \tilde{\sigma}_{\theta r}}{r}]$$

$$\frac{d\tilde{\xi}_z}{dr} = -ik\Lambda + Re\Gamma^{-1}[ik\tilde{V}_f\tilde{v}_z + \tilde{v}_r\frac{d\tilde{V}_f}{dr} + s\tilde{v}_z + ik\tilde{p}_f - \frac{\tilde{\sigma}_{zr}}{r} - \frac{in}{r}\tilde{\sigma}_{z\theta} - ik\tilde{\sigma}_{zz}]$$

$$\Lambda = -\frac{\tilde{v}_r}{r} - in\frac{\tilde{v}_\theta}{r} - ik\tilde{v}_z$$

$$\begin{pmatrix} \tilde{\sigma}_{rr} & \tilde{\sigma}_{r\theta} & \tilde{\sigma}_{rz} \\ \tilde{\sigma}_{\theta r} & \tilde{\sigma}_{\theta\theta} & \tilde{\sigma}_{\theta z} \\ \tilde{\sigma}_{zr} & \tilde{\sigma}_{z\theta} & \tilde{\sigma}_{zz} \end{pmatrix} = \frac{\Gamma}{Re}\begin{pmatrix} 2\Lambda & \tilde{\xi}_\theta + \frac{in\tilde{v}_r}{r} - \frac{\tilde{v}_\theta}{r} & \tilde{\xi}_z + ik\tilde{v}_r \\ \tilde{\xi}_\theta + \frac{in\tilde{v}_r}{r} - \frac{\tilde{v}_\theta}{r} & 2(\frac{in\tilde{v}_\theta}{r} + \frac{\tilde{v}_r}{r}) & \frac{in\tilde{v}_z}{r} + ik\tilde{v}_\theta \\ \tilde{\xi}_z + ik\tilde{v}_r & \frac{in\tilde{v}_z}{r} + ik\tilde{v}_\theta & 2ik\tilde{v}_z \end{pmatrix}$$

Governing equations for solid media

$$\frac{d\tilde{u}_r}{dr} = \Upsilon_1 \quad ; \quad \frac{d\tilde{u}_\theta}{dr} = \tilde{\zeta}_\theta \quad ; \quad \frac{d\tilde{u}_z}{dr} = \tilde{\zeta}_z$$

$$\frac{d\tilde{p}_s}{dr} = -\rho_r s^2 \tilde{u}_r + \frac{d\tilde{T}_{rr}}{dr} + \frac{\tilde{T}_{rr}}{r} + \frac{in}{r}\tilde{T}_{r\theta} + ik\tilde{T}_{rz} - \frac{\tilde{T}_{\theta\theta}}{r} + \\ + \frac{d\tilde{D}_{rr}}{dr} + \frac{\tilde{D}_{rr}}{r} + \frac{in}{r}\tilde{D}_{r\theta} + ik\tilde{D}_{rz} - \frac{\tilde{D}_{\theta\theta}}{r}$$

$$\frac{d\tilde{\zeta}_\theta}{dr} = -\frac{in}{r}\Upsilon_1 + \frac{in\tilde{u}_r}{r^2} - \frac{\tilde{u}_\theta}{r^2} + \frac{1}{r}\tilde{\zeta}_\theta + (\Theta + Re^{-1}\Gamma s \mu_r)^{-1}[\rho_r s^2 \tilde{u}_\theta + \\ + \frac{in}{r}\tilde{p}_s - \frac{in}{r}\tilde{T}_{\theta\theta} - ik\tilde{T}_{\theta z} - \frac{\tilde{T}_{r\theta} + \tilde{T}_{\theta r}}{r} - \frac{in}{r}\tilde{D}_{\theta\theta} - ik\tilde{D}_{\theta z} - \frac{\tilde{D}_{r\theta} + \tilde{D}_{\theta r}}{r}]$$

$$\frac{d\tilde{\zeta}_z}{dr} = -ik\Upsilon_1 + (\Theta + Re^{-1}\Gamma s \mu_r)^{-1}[\rho_r s^2 \tilde{u}_z + ik\tilde{p}_s - \frac{\tilde{T}_{zr}}{r} - \\ - \frac{in}{r}\tilde{T}_{z\theta} - ik\tilde{T}_{zz} - \frac{\tilde{D}_{zr}}{r} - \frac{in}{r}\tilde{D}_{z\theta} - ik\tilde{D}_{zz}]$$

$$\Upsilon_1 = -\frac{\tilde{u}_r}{r} - in\frac{\tilde{u}_\theta}{r} - ik\tilde{u}_z \quad ; \quad A = \frac{\Gamma s \mu_r}{Re}$$

$$\begin{pmatrix} \tilde{D}_{rr} & \tilde{D}_{r\theta} & \tilde{D}_{rz} \\ \tilde{D}_{\theta r} & \tilde{D}_{\theta\theta} & \tilde{D}_{\theta z} \\ \tilde{D}_{zr} & \tilde{D}_{z\theta} & \tilde{D}_{zz} \end{pmatrix} = A \begin{pmatrix} 2\Upsilon_1 & \tilde{\zeta}_\theta + r^{-1}(in\tilde{u}_r - \tilde{u}_\theta) & \tilde{\zeta}_z + ik\tilde{u}_r \\ \tilde{\zeta}_\theta + r^{-1}(in\tilde{u}_r - \tilde{u}_\theta) & 2r^{-1}(in\tilde{u}_\theta + \tilde{u}_r) & inr^{-1}\tilde{u}_z + ik\tilde{u}_\theta \\ \tilde{\zeta}_z + ik\tilde{u}_r & inr^{-1}\tilde{u}_z + ik\tilde{u}_\theta & 2ik\tilde{u}_z \end{pmatrix}$$

$$\begin{pmatrix} \Upsilon_1 \\ r^{-1}(in\tilde{u}_\theta + \tilde{u}_r) \\ ik\tilde{u}_z \\ \tilde{\zeta}_\theta + r^{-1}(in\tilde{u}_r - \tilde{u}_\theta) \\ inr^{-1}\tilde{u}_z + ik\tilde{u}_\theta \\ \tilde{\zeta}_z + ik\tilde{u}_r \end{pmatrix} = \begin{pmatrix} \Xi_2^{-1} & -\nu_2\Xi_2^{-1} & -\nu_2\Xi_2^{-1} & 0 & 0 & 0 \\ -\nu_2\Xi_2^{-1} & \Xi_1^{-1} & -\nu_1\Xi_1^{-1} & 0 & 0 & 0 \\ -\nu_2\Xi_2^{-1} & -\nu_1\Xi_1^{-1} & \Xi_1^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \Theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \Theta & 0 \\ 0 & 0 & 0 & 0 & 0 & \Theta \end{pmatrix} \begin{pmatrix} \tilde{T}_{rr} \\ \tilde{T}_{\theta\theta} \\ \tilde{T}_{zz} \\ \tilde{T}_{r\theta} \\ \tilde{T}_{\theta z} \\ \tilde{T}_{rz} \end{pmatrix}$$

Boundary conditions

$$r = 0: \quad \tilde{v}_r = 0 \quad ; \quad \frac{d\tilde{v}_z}{dr} = 0$$

$$r = 1: \quad \tilde{v}_r = s\tilde{u}_r^1 \quad ; \quad \tilde{v}_z + \frac{dV_f(r)}{dr}\tilde{u}_r^1 = s\tilde{u}_z^1$$

$$-\tilde{p}_s^1 + \tilde{T}_{rr}^1 + \tilde{D}_{rr}^1 = -\tilde{p}_f + \tilde{\sigma}_{rr} \quad ; \quad \tilde{T}_{rz}^1 + \tilde{D}_{rz}^1 = \tilde{\sigma}_{rz}$$

$$r = 1 + H_j: \quad \tilde{u}_r^j = \tilde{u}_r^{j+1} \quad ; \quad \tilde{u}_z^j = \tilde{u}_z^{j+1}$$

$$-\tilde{p}_s^j + \tilde{T}_{rr}^j + \tilde{D}_{rr}^j = -\tilde{p}_s^{j+1} + \tilde{T}_{rr}^{j+1} + \tilde{D}_{rr}^{j+1}$$

$$\tilde{T}_{rz}^j + \tilde{D}_{rz}^j = \tilde{T}_{rz}^{j+1} + \tilde{D}_{rz}^{j+1}$$

$$r = 1 + H: \quad -\tilde{p}_s^3 + \tilde{T}_{rr}^3 + \tilde{D}_{rr}^3 = 0 \; ; \; \tilde{T}_{rz}^3 + \tilde{D}_{rz}^3 = 0 \quad \Big| \quad \tilde{u}_r^3 = 0 \; ; \; \tilde{u}_z^3 = 0$$