WAVE TURBULENCE IN SHALLOW WATER: THEORY AND NUMERICAL SIMULATIONS

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Operational wave forecasting is based on the numerical integration of the water wave kinetic equation (Hasselmann equation, 1962)

$$\frac{\partial N_1}{\partial t} + \nabla \cdot (\vec{C}_g N_1) = S_{NL} + S_{diss} + S_{In}$$

where

$$S_{NL} = \int \left| T_{1,2,3,4} \right|^2 \delta(\Delta \boldsymbol{k}) \delta(\Delta \omega) [N_3 N_4 (N_1 + N_2) - N_1 N_2 (N_3 + N_4)] d\boldsymbol{k}_{2,3,4}$$

$$S_{diss} = -\gamma_D (\mathbf{k}) N(\mathbf{k})$$

$$S_{In} = \beta(\mathbf{k}) N(\mathbf{k})$$

MOTIVATION

WAVE MODELING IN SHALLOW WATER

Common thinking in the wave forecasting community:

a) FOUR-WAVE RESONANT INTERACTIONS IN DEEP /INTERMEDIATE WATER

b) THREE -WAVE INTERACTIONS (NEVER RESONANT) IN SHALLOW WATER

i) deterministic modeling (Boussinesq or higher order)
ii) evolution equations for spectra and bi-spectra based on Boussinesq-like equations

i) Are the four-wave interactions relevant in shallow water?
ii) Does an irreversible transfer of energy occur in flat-bottom shallow water waves?

METHODOLOGY

Theoretical analysis of the oldest, flat-bottom shallow water model:

THE BOUSSINESQ EQUATIONS

$$\eta_t + \nabla \cdot \left[(\eta + h)\mathbf{u} \right] = 0$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla \eta - \frac{h^2}{3} \nabla \nabla \cdot \mathbf{u}_t = 0$$

SUMMARY OF THE RESULTS

i) A FOUR-WAVE HASSELMANN-LIKE EQUATION (BOUSSINESQ KINETC EQUATION) HAS BEEN DERIVED FROM THE BOUSSINESQ EQUATIONS

ii) BOUSSINESQ KINETC EQUATION CORRESPONDS TO THE ARBITRARY DEPTH HASSELMANN EQUATION WHEN THE LIMIT OF SMALL DEPTH IS PROPERLY TAKEN

iii) THE ESTIMATED TIME SCALE OF THE FOUR-WAVE NONLINEAR INTERACTIONS IN SHALLOW WATER IS

$$\tau_{\mathrm{NL}} \sim \frac{1}{\omega}\left(\frac{kh}{\varepsilon}\right)^4 \sim \frac{1}{\omega}\left(\frac{1}{a/h}\right)^4$$

iv) BASED ON DIMENSIONAL ARGUMENTS, DIRECT ENERGY CASCADE SOLUTION IS FOUND TO BE OF THE FORM OF $E(k) \sim k^{-4/3}$

v) RESULTS ON DIRECT CASCADE ARE CONFIRMED BY DIRECT NUMERICAL COMPUTATIONS OF THE BOUSSINESQ EQUATIONS.

vi) EXPERIMENTAL INDICATION OF CASCADE IN SHALLOW WATER (see also Smith and Vincent 2003, Kaihatu et al. 2007 for shoaling waves)

References:

Janssen and M.O., J. Phys. Oceanography (2007)
M.O. et al JFM (2009)
Zakharov, Europ. J. Mechanics B/Fluids (1999)
Zakharov, in Nonlinear Waves and Weak Turbulence (1998)

Derivation of the Boussinesq kinetic equations

$$\eta_t + \nabla \cdot \left[(\eta + h)\mathbf{u} \right] = 0$$

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + g \nabla \eta - \frac{h^2}{3} \nabla \nabla \cdot \mathbf{u}_t = 0$$

i) WRITE THE EQUATION IN FOURIER SPACE AND INTRODUCE THE COMPLEX AMPLITUDE

$$a(\mathbf{k},t) = \left(\frac{g}{2\omega_k} \right)^{1/2} \eta(\mathbf{k},t) + i\gamma_k \left(\frac{\omega_k}{2g} \right)^{1/2} \phi(\mathbf{k},t)$$

with $\omega(k) = \sqrt{gh} \frac{k}{\left(1 + k^2 h^2 / 3\right)^{1/2}}$ $\quad \gamma(k) = 1 + k^2 h^2 / 3$

ii) USE OF THE MULTIPLE SCALE EXPANSION:

- **Introduce a slow time scale:** $\tau = \varepsilon^2 t$

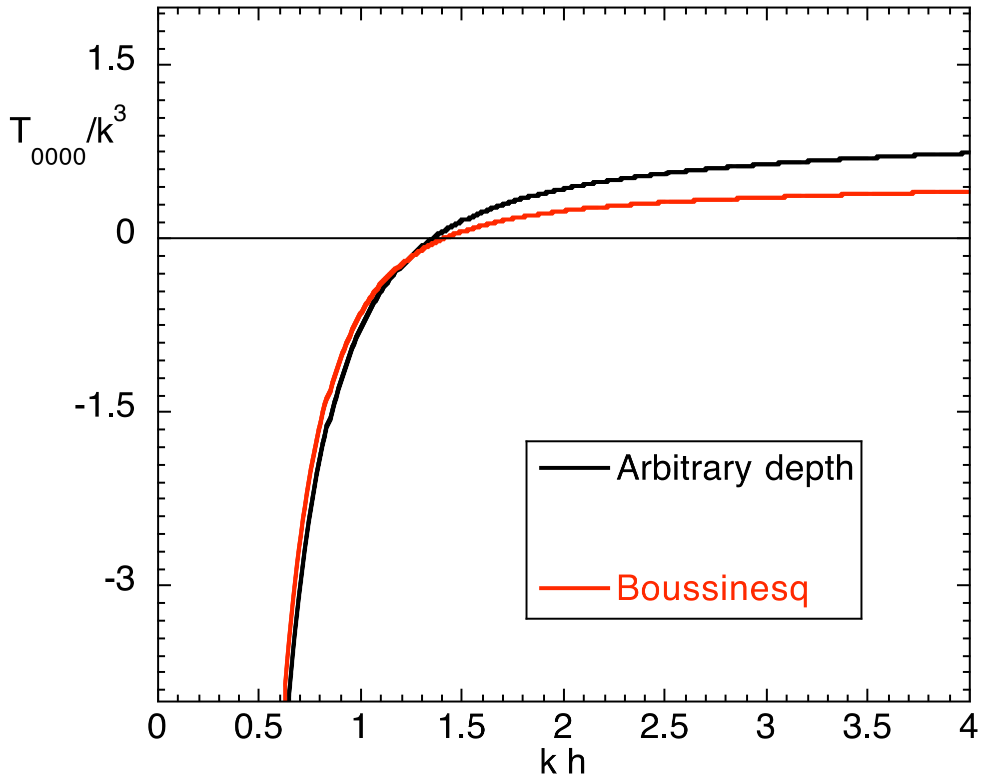
- **Look for a solution of the form:**

$$a_k(t,\tau) = b_k(t,\tau) + \varepsilon b_k^{(1)}(t,\tau) + \varepsilon^2 b_k^{(2)}(t,\tau) + \ldots$$

$$\frac{\partial b_0}{\partial t} + i\omega_0 b_0 = -i\varepsilon^2 \int T_{0123}^{(B)} b_1^* b_2 b_3 \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}$$

Relation with the Hasselmann equation

$$T_{0123}^{(B)} = \lim_{kh \to 0} T_{0123}^{(ad)}$$



Narrow band limits of the Boussinesq-Zakharov equation

- **Long crested case (one dimensional propagation):**

  Defocousing Nonlinear Schroedinger equation in shallow water

- **Weakly two-dimensional case:**

  Shallow water limit of the Davey-Stewartson

BOTH EQUATIONS ARE INTEGRABLE!!

NO ENERGY TRANSFER!!

iii) STATISTICAL DESCRIPTION OF THE SHALLOW WATER ZAKHAROV EQUATION

Hypothesis:

i) Homogeneity

ii) Quasi-Gaussian approximation

$$\frac{\partial N_0}{\partial t} = \int \left(T_{0123}^{(B)}\right)^2 N_0 N_1 N_2 N_3 \left(\frac{1}{N_0} + \frac{1}{N_1} - \frac{1}{N_2} - \frac{1}{N_3}\right) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}$$

where

$$< a(\mathbf{k}_i, t) a^*(\mathbf{k}_j, t) >= N(\mathbf{k}_i, t) \delta(\mathbf{k}_i - \mathbf{k}_j)$$

NONLINEAR TIME SCALE

$$\frac{\partial N_0}{\partial t} = \int \left(T_{0123}^{(B)}\right)^2 N_0 N_1 N_2 N_3 \left(\frac{1}{N_0} + \frac{1}{N_1} - \frac{1}{N_2} - \frac{1}{N_3}\right) \delta(\omega_0 + \omega_1 - \omega_2 - \omega_3) \delta(\mathbf{k}_0 + \mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_{123}$$

$$1/\tau_{NL} \sim (T^{(B)})^2 \omega^{-1} N^2 k^4$$

with $T^{(B)} \sim k^2/h$

$$\tau_{\mathrm{NL}} \sim \frac{1}{\omega}\left(\frac{kh}{\varepsilon}\right)^4 \sim \frac{1}{\omega}\left(\frac{1}{a/h}\right)^4$$

Power law solutions of the Boussinesq kinetic equation

Hypothesis: existence of an equilibrium range in Fourier space where there is a constant flux of energy, independent of *k*

Dimensional analysis of the kinetic equation

$$\Delta N/\Delta t \sim T^{(B)2}\omega^{-1}N^3k^4 \text{ with } T^{(B)} \sim k^2/h$$

Flux of energy *Π*:

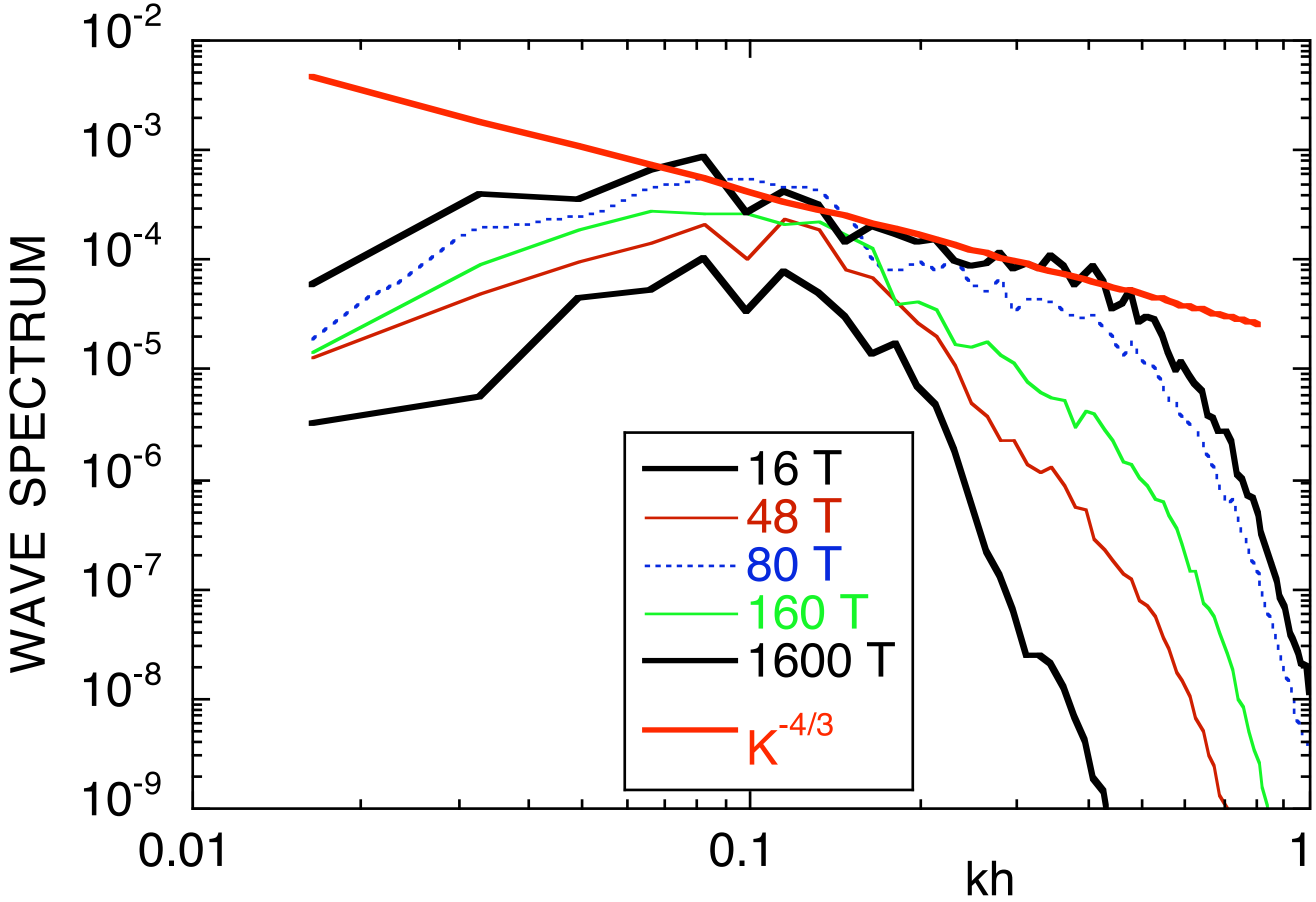
$$\Pi \sim k^2 \Delta E/\Delta t \sim N^3 k^{10}$$

must be independent of k, therefore

$$E \sim k^{-4/3}$$

(in agreement with Zakharov 1999)

Numerical simulations of Boussinesq equations



EFFECTS OF REGULAR DISCTRETIZATION: LACK OF EXACT RESONANCES

Example:

$h=1\ m,\ k_p=0.2\ m^{-1}$

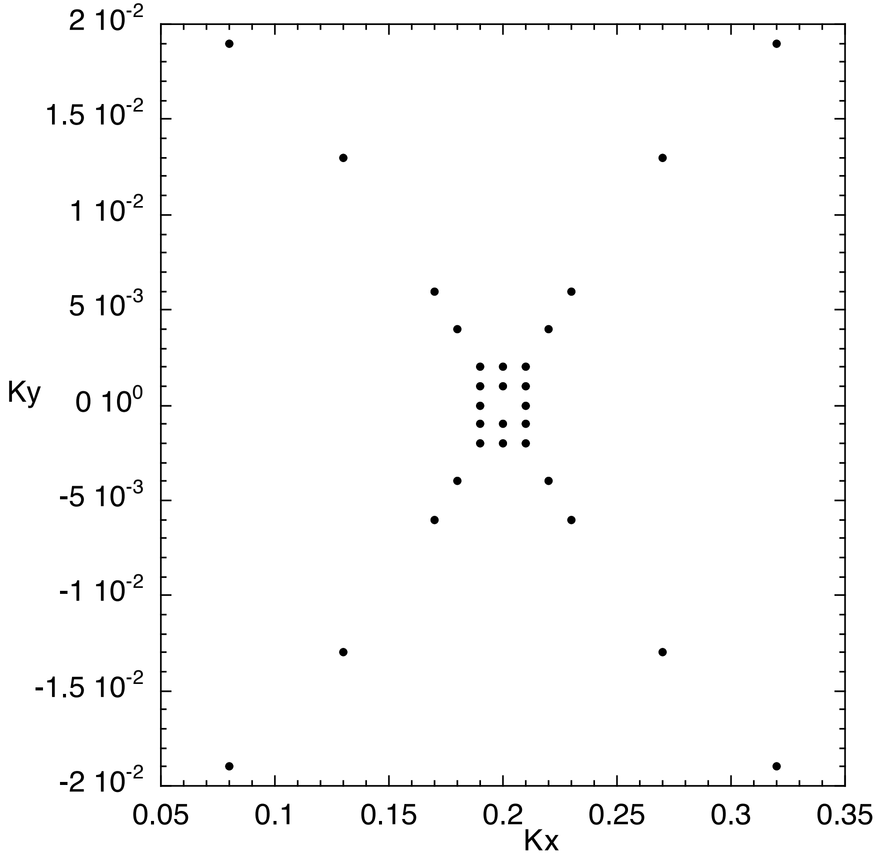
$\Delta k_x=0.01\ m^{-1}$

$\Delta k_y=0.001\ m^{-1}$

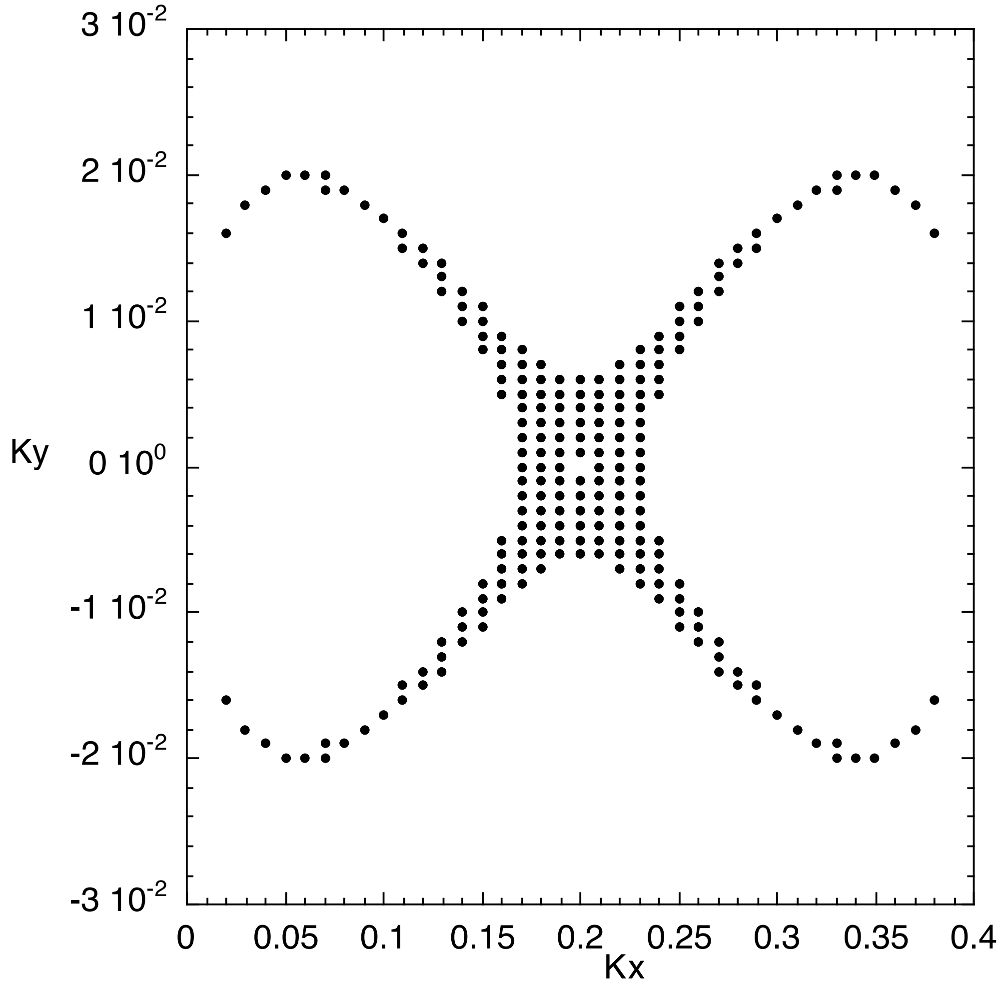
$k_1= k_p,\ k_2=k_p$

IN A 256x256 GRID THERE ARE NOT EXACT RESONANCES!

$$|\omega_1+\omega_2-\omega_3-\omega_4|<10^{-4}\omega_1$$



$$|\omega_1+\omega_2-\omega_3-\omega_4|<10^{-3}\omega_1$$



Some experimental indication of shallow water cascade

SMITH AND VINCENT: EQUILIBRIUM RANGES IN SURF ZONE SPECTRA

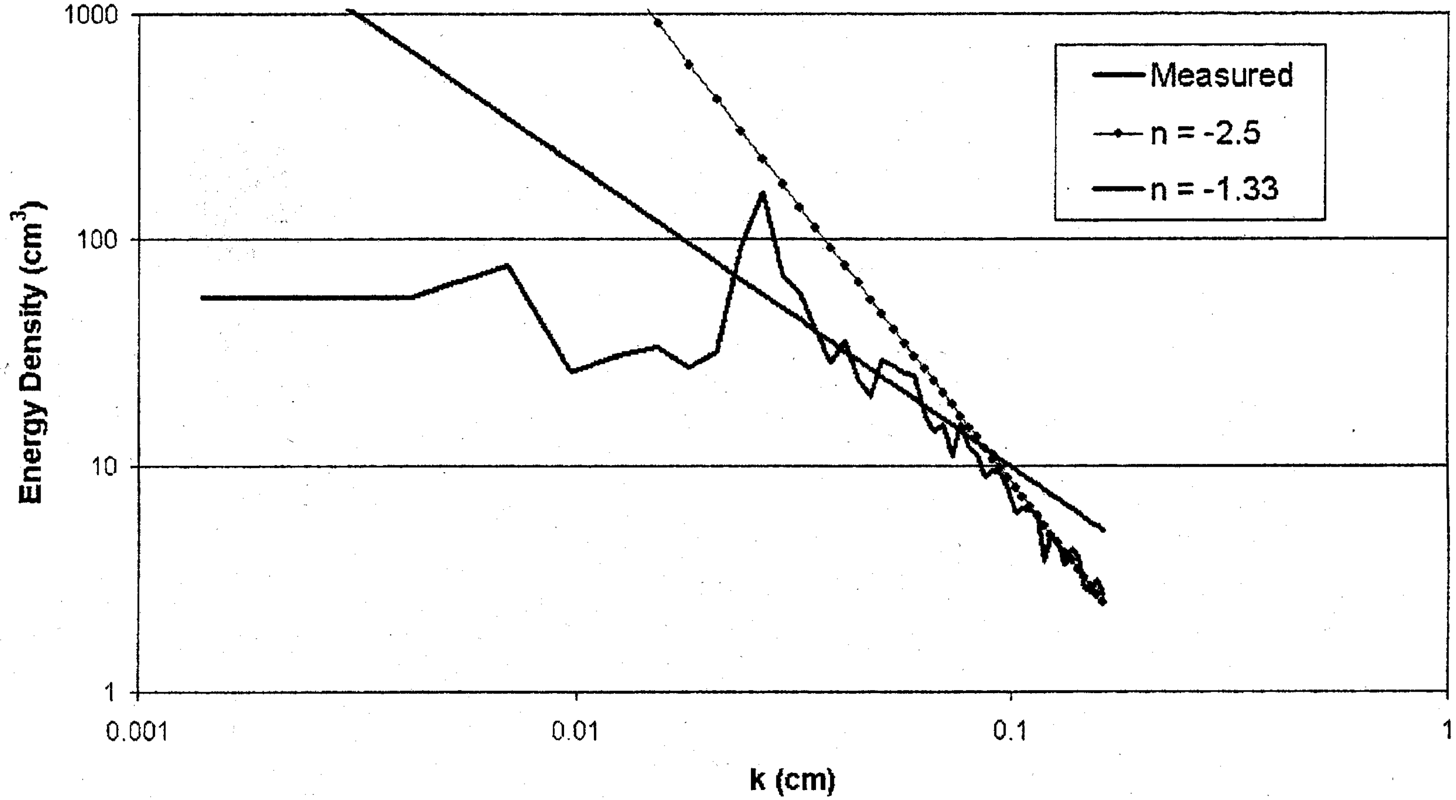
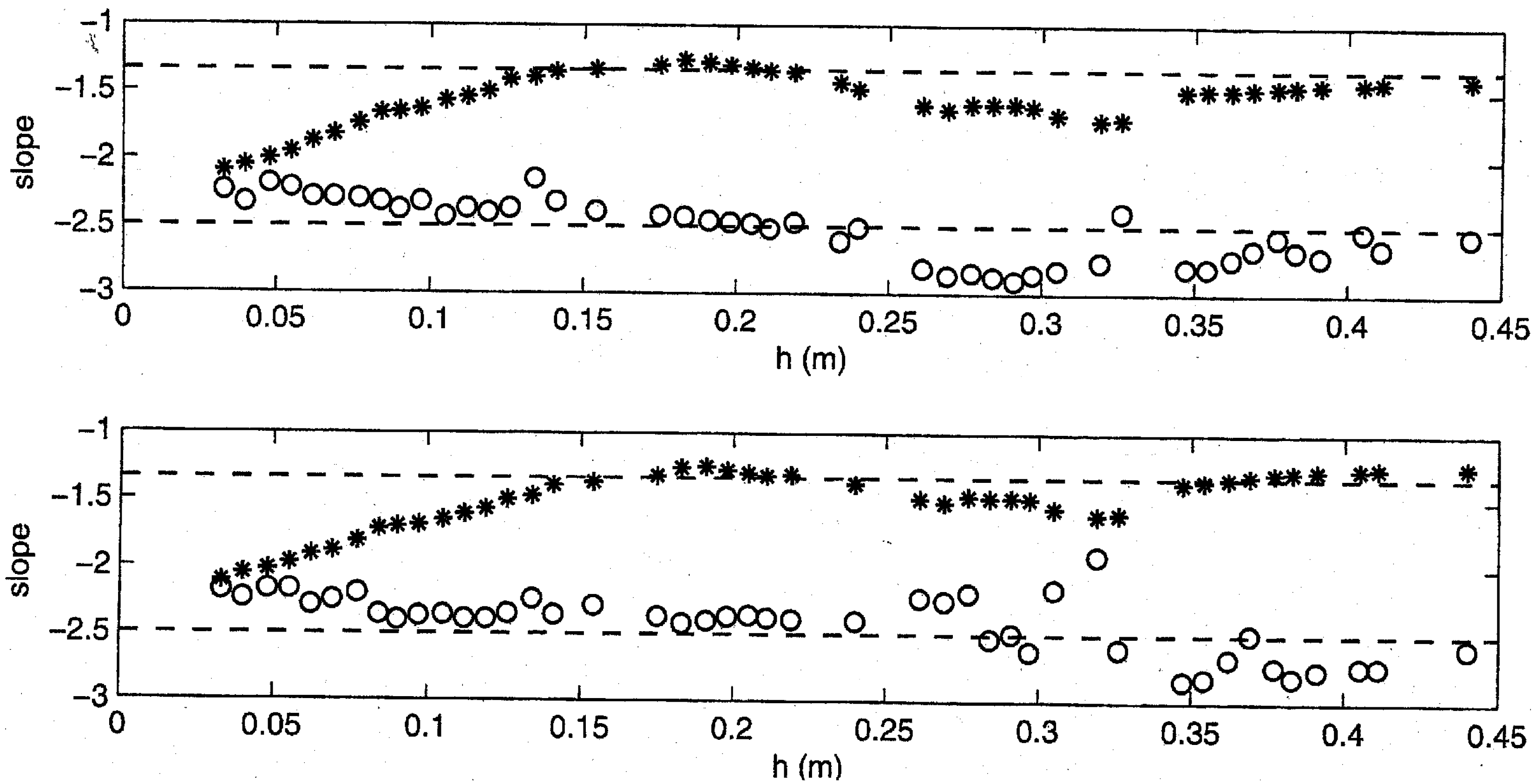


Figure 3. Example of Toba equilibrium range ($n = -5/2$) fit to lab data ($d = 18.3$ cm).

(from Smith and Vincent JGR 2003)

Some indication of shallow water cascade



(from Kaihatu et al. JGR 2007)

From Resio and Long (JGR 2006)

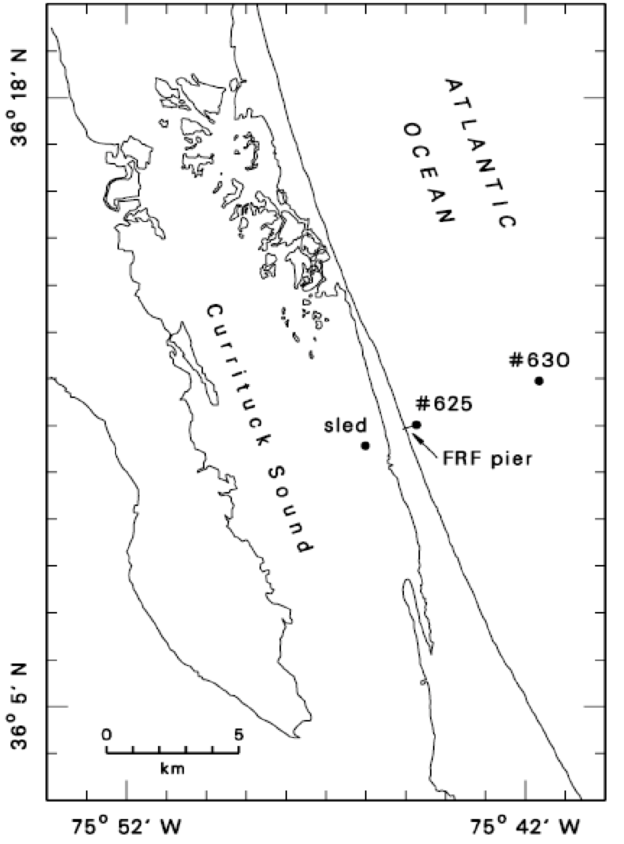
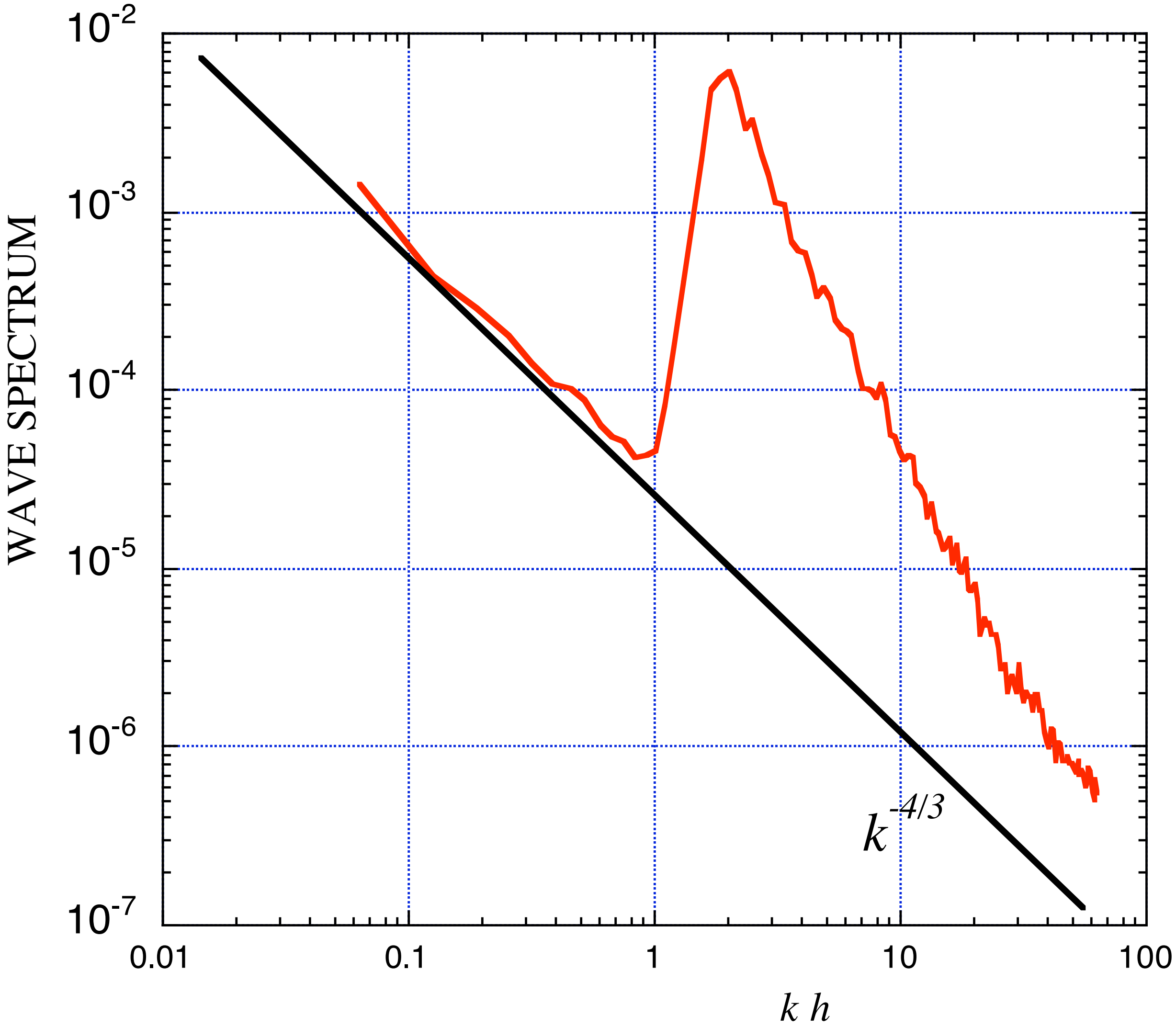


Figure 2. Site map showing the boundaries of Currituck Sound nearby the instrument platform (solid circle labeled "sled"). The sound is isolated from the Atlantic Ocean to the east by the barrier island of North Carolina's Outer Banks. At the nearest points, the sled is about 1 km from the east side of the sound and about 6 km from the west side. A broad channel of about 2.5-m depth runs along the long axis of the sound in the vicinity of the sled. This channel narrows considerably to the north. The opening to the south, representing an azimuthal arc of about 20 deg from the sled location, leads to the larger, deeper Albemarle Sound.



WAVE SPECTRUM
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
0.01
0.1
1
10
100
$k\ h$
$k^{-4/3}$

THREE WAVE INTERACTIONS CAN BE ASYMPTOTYCALLY RESONANT

$$k_3 = k_1 - k_2$$

$$\omega_3 \sim \omega_1 - \omega_2$$

If k_1 *is close to* k_2 *then*

$$\omega_3 + \omega_2 - \omega_1 \sim c_0 (k_2 - k_1)(kh)^2 \sim 0$$

CONCLUSIONS

- **The Boussinesq kinetic equation is equivalent to the shallow water limit of the Hasselmann equation**

- **Four-wave resonant interaction can be relevant in shallow water**

- **Power law solutions of the Boussinesq kinetic have been confirmed by numerical simulations**

- **There is some indication that this solution can be observed in the surf-zone**

- **Infragravity waves in Currituck Sound show also power law** $k^{-4/3}$