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Hierarchy of Nonlinear Waves in Complex Microstructured Solids

Franco Pastrone
Department of Mathematics
University of Torino

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The cornerstones for describing dynamical processes of microstructured materials at intensive and high-speed deformations are the following:

- non-classical theory of continua able to account for internal scales;
- hierarchical structure of waves due to the scales in materials;
- nonlinearities caused by large deformation and character of stress-strain relations.

In this short talk three examples will be considered:

1. 1D elastic body with two different scales of microsctructures

   ✦Engelbrecht, J. and Pastrone, F., Braun, M., Berezovski, A., Hierarchy of waves in nonclassical materials, Universality of Nonclassical Nonlinearity, P.P. Delsanto ed., Springer, 29-48, (2007)
   ✦Casasso, A., Pastrone, F., Wave propagation in solids with vectorial microstructure. (to appear)

2. Non dissipative plane granular media

   ✦Casasso A., Pastrone F., Non linear waves in in plane granular media, Proceed. of the *Intern. Seminar "Days on Diffraction-2005"*, Saint-Petersburg, Russia, 30-39, (2005).

3. 2D microstructured media

   ✦Porubov A.V., Pastrone F., Maugin G.A., Selection of two-dimensional nonlinear strain waves in micro.structured media, C. R. Acad. Sc, Paris, Ser. I 337 , 513-518, (2004).

1 - **1D solids with two microscales**

In this model the body is a one-dimensional manifold so we consider the material coordinates x and t.

We deal with three different scalar functions:

$v=v(x,t)$ for the macrostructure,

$\varphi=\varphi(x,t)$ and $\psi=\psi(x,t)$ for the microstructure, respectively for the first and the second scale level.

The macro body is supposed to be elastic. The first and second level microstructures satisfy the same generalized elasticity hypotesis as well, such that we can assume the existence of an internal strain energy.

Strain energy function for elastic solids with microstructures is assumed to be a function of the vector fields and their gradients

$$W = W(v, v_x, \varphi, \varphi_x, \psi, \psi_x, x).$$

The kinetic energy is a quadratic form in v_t, φ_t, ψ_t:

$$K = \frac{1}{2}(\rho v_t^2 + I_1 \varphi_t^2 + I_2 \psi_t^2)$$

The field equations:

$$\begin{cases} \rho v_{tt} = \left(\dfrac{\partial W}{\partial v_x}\right)_x - \dfrac{\partial W}{\partial v} \\ I_1 \varphi_{tt} = \left(\dfrac{\partial W}{\partial \varphi_x}\right)_x - \dfrac{\partial W}{\partial \varphi} \\ I_2 \psi_{tt} = \left(\dfrac{\partial W}{\partial \psi_x}\right)_x - \dfrac{\partial W}{\partial \psi} \end{cases} \tag{1}$$

The particular choice of the strain energy function *W* gives rises to different nonlinear models; we consider the following form:

$$W = \frac{1}{2}\alpha v_x^2 + \frac{1}{3}\beta v_x^3 - A_1 \varphi v_x + \frac{1}{2}B_1\varphi^2 + \frac{1}{2}C_1\varphi_x^2 - A_2\varphi_x\psi + \frac{1}{2}B_2\psi^2 + \frac{1}{2}C_2\psi_x^2.$$

To obtain the governing equation in dimensionless form, it is necessary to introduce some parameters and constants.
For the first and second level of microstructure we pose:

$$C_1 = C_1^* l_1^2 \qquad I_1 = \rho l_1^2 I_1^*$$

$$C_2 = C_2^* l_2^2 \qquad I_2 = \rho l_2^2 I_2^* \qquad A_2 = l_2 A_2^*$$

Then we introduce different parameters

$$\delta_1 = \left(\frac{l_1}{L}\right)^2 \qquad \delta_2 = \left(\frac{l_2}{L}\right)^2$$

characterizing the ratio between the microscructure and the wave length

$$\epsilon = \frac{v_0}{L}$$

accounting for elastic strain

Field equations can be written as:

$$\begin{cases} \rho v_{tt} = \alpha v_{xx} + \left(\beta v_x^2\right)_x - A_1\varphi_x \\ I_1\varphi_{tt} = C_1\varphi_{xx} + A_1 v_x - B_1\varphi - A_2\psi_x \\ I_2\psi_{tt} = C_2\psi_{xx} + A_2\varphi_x - B_2\psi \end{cases} \quad (2)$$

Introducing the macrostrain $\nu = v_x$

and the dimensionless variables $u = \dfrac{\nu}{v_0}$ $X = \dfrac{x}{L}$ $T = \dfrac{c_0}{L}t$

we get the dimensionless equations:

$$\begin{cases} u_{TT} = \dfrac{\alpha}{\rho c_0^2} u_{XX} + \dfrac{\beta \epsilon L}{\rho c_0^2}\left(u_X^2\right)_X - \dfrac{A_1}{\epsilon L \rho c_0^2}\varphi_{XX} \\ \varphi = \dfrac{A_1 v_0}{B_1} u - \dfrac{A_2^* \sqrt{\delta_2}}{B_1}\psi_X + \dfrac{\delta_1}{B_1}\left[C_1^*\varphi_{XX} - \rho I_1^* c_0^2 \varphi_{TT}\right] \\ \psi = \dfrac{A_2^* \sqrt{\delta_2}}{B_2}\varphi_X + \dfrac{\delta_2}{B_2}\left[C_2^*\psi_{XX} - \rho I_2^* c_0^2 \psi_{TT}\right] \end{cases} \quad (3)$$

In the first step, from $(3)_3$ we obtain the expansion

$$\psi = \frac{A_2^*\sqrt{\delta_2}}{B_2}\varphi_X + \frac{A_2^*\sqrt{\delta_2}\delta_2}{B_2^2}\left[C_2^*\varphi_{XXX} - \rho I_2^* c_0^2 \varphi_{XTT}\right]$$

It is inserted into $(3)_2$, which also is expanded and yields

$$\varphi = \frac{A_1 v_0}{B_1}u + \frac{\delta_1 A_1 v_0}{B_1^2}\left[C_1^* u_{XX} - \rho I_1^* c_0^2 u_{TT}\right] - \frac{\delta_2^2 A_1 (A_2^*)^2 v_0}{B_1^2 B_2^2}\left[C_2^* u_{XXXX} - \rho I_2^* c_0^2 u_{XXTT}\right]$$

This expression is inserted $(3)_1$ resulting in the partial differential equation

$$u_{TT} = \left(\frac{\alpha\epsilon L B_1 + A_1^2 v_0}{\epsilon L \rho c_0^2 B_1}\right) u_{XX} + \frac{\beta\epsilon L}{\rho c_0^2}(u_X^2)_X + \frac{\delta_1 A_1^2 v_0}{\epsilon L \rho c_0^2 B_1^2}\left[\rho I_1^* c_0^2 u_{TT} - C_1^* u_{XX}\right]_{XX} + \\ - \frac{\delta_2^2 A_1^2 (A_2^*)^2 v_0}{\epsilon L \rho c_0^2 B_1^2 B_2^2}\left[\rho I_2^* c_0^2 u_{TT} - C_2^* u_{XX}\right]_{XXXX} \qquad (4)$$

Equation (4) can be rewritten as

$$u_{TT} + \alpha_1 u_{XX} + \alpha_2 (u^2)_{XX} + (\alpha_3 u_{XX} + \alpha_4 u_{TT})_{XX} + (\alpha_5 u_{XX} + \alpha_6 u_{TT})_{XXXX} = 0$$

where we have defined

$$\alpha_1 = -\frac{\alpha\epsilon L B_1 + A_1^2 v_0}{\epsilon L \rho c_0^2 B_1} \qquad \alpha_2 = -\frac{\beta\epsilon L}{\rho c_0^2}$$

$$\alpha_3 = \frac{\delta_1 A_1^2 v_0 C_1^*}{\epsilon L \rho c_0^2 B_1^2} \qquad \alpha_4 = -\frac{\delta_1 A_1^2 v_0 I_1^*}{\epsilon L B_1^2}$$

$$\alpha_5 = -\frac{\delta_2^2 A_1 (A_2^*)^2 v_0 C_2^*}{\epsilon L \rho c_0^2 B_1^2 B_2^2} \qquad \alpha_6 = \frac{\delta_2^2 A_1^2 (A_2^*)^2 v_0 I_2^*}{\epsilon L B_1^2 B_2^2}$$

We will find some exact solutions to an ODE corresponding to the PDE mentioned, when eq. (4) is reformulated in terms of the phase variable *z=x ± Vt*, where *V* is the velocity of propagation.

$$(V^2+\alpha_1)u^{(\mathrm{II})}+\alpha_2(u^2)^{(\mathrm{II})}+\left(\alpha_3+V^2\alpha_4\right)u^{(\mathrm{IV})}+\left(\alpha_5+V^2\alpha_6\right)u^{(\mathrm{VI})}=0$$

If the nonlinearity is neglected (4) is equivalent to eq. (3.57) of [Eng.Pas.Br.Be.]

$$u_{tt} = \left(c_0^2-c_A^2\right)u_{xx}+c_A^2\,p_1^2\left(u_{tt}-\left(c_1^2-c_B^2\right)u_{xx}\right)_{xx}-c_A^2\,p_2^2\,q^2\left(u_{tt}-c_2^2\,u_{xx}\right)_{xxxx}$$

We can assume non linearity also in the microstructures, hence add in the strain energy function *W* two terms as:

$$B_3\varphi_x^3 \qquad C_3\psi_x^3$$

The governing equations will contain now terms

$$B_3\varphi_x\varphi_{xx} \qquad C_3\psi_x\psi_{xx}$$

2 - **Nonlinear waves in nondissipative plane granular media**

We consider the vector $\boldsymbol{r}=\boldsymbol{r}(x,y,t)=u(x,y,t)\mathbf{e}_1+v(x,y,t)\mathbf{e}_2$ for the macrostructure and, for the microstructure, the function $\theta = \theta(x,y,t)$ that represents the angle of rotation of the particle with respect to the fixed basis.

The kinetic energy density reads:

$$T = \frac{1}{2}\left[\rho\left(u_t^2 + v_t^2\right) + I\theta_t^2\right]$$

The strain energy density is choosen in the form:

$$W = \frac{1}{2}\alpha\left(u_x^2 + v_x^2\right) + \frac{1}{2}\beta\left(u_y^2 + v_y^2\right) + \frac{1}{6}\gamma(u_x^3 + u_y^3 + v_x^3 + v_y^3)$$
$$+\frac{1}{2}\gamma(u_x^2 v_x + u_y^2 v_y + v_x^2 u_x + v_y^2 u_y) - A\theta\left(u_x + u_y + v_x + v_y\right)$$
$$+\frac{1}{2}B\theta^2 + \frac{1}{2}C\left(\theta_x^2 + \theta_y^2\right) + \frac{1}{3}D\left(\theta_x^3 + \theta_y^3\right)$$

We assume that the dissipation is negligeable, and we calculate the Lagrange equations:

$$\begin{cases} \rho u_{tt} = \alpha u_{xx} + \beta u_{yy} - A(\theta_x + \theta_y) + \frac{1}{2}\gamma \left[\left[(u_x + v_x)^2 \right]_x + \left[(u_y + v_y)^2 \right]_y \right] \\ \rho v_{tt} = \alpha v_{xx} + \beta v_{yy} - A(\theta_x + \theta_y) + \frac{1}{2}\gamma \left[\left[(u_x + v_x)^2 \right]_x + \left[(u_y + v_y)^2 \right]_y \right] \\ I\theta_{tt} = C(\theta_{xx} + \theta_{yy}) + D \left[(\theta_x^2)_x + (\theta_y^2)_y \right] + A(u_x + u_y + v_x + v_y) - B\theta \end{cases}$$

We introduce a new variable *w=u+v*.
We add the first two equations of the previous system and get:

$$\begin{cases} \rho w_{tt} = \alpha w_{xx} + \beta w_{yy} - 2A(\theta_x + \theta_y) + \gamma \left[(w_x^2)_x + (w_y^2)_y \right] \\ I\theta_{tt} = C(\theta_{xx} + \theta_{yy}) + D \left[(\theta_x^2)_x + (\theta_y^2)_y \right] + A(w_x + w_y) - B\theta \end{cases} \quad (5)$$

The dimensionless variables are introduced

$$W = \frac{w}{W_0}, \quad X = \frac{x}{L}, \quad Y = \frac{y}{L}, \quad T = \frac{c_0}{L}t$$

We need a scale *l* for the microstructure and two dimensionless parameters:

$$\begin{cases} \delta \sim \left(\dfrac{l}{L}\right)^2 \\ \varepsilon \sim \left(\dfrac{W_0}{L}\right) \end{cases}$$

characterizing the ratio between the microstructure and the wave lenght, with the relevant meaning of a characteristic lenght

accounting for elastic strain

Following [Por.Pas.] we suppose $I=\rho l^2 I^*$, $C=l^2C^*$, $D=l^2D^*$ where I^* is dimensionless and C^* and D^* have the dimension of the stress.
Then (5) yields:

$$\begin{cases} W_{TT} = \dfrac{1}{\rho c_0^2}(\alpha W_{XX} + \beta W_{YY}) - \dfrac{2A}{\rho c_0^2 \varepsilon}(\theta_X + \theta_Y) + \dfrac{\varepsilon^2\gamma}{L}\left[(W_X^2)_X + (W_Y^2)_Y\right] \\ \theta_{TT} = \dfrac{1}{\rho c_0^2 I^*}\left[\dfrac{\varepsilon A}{B}(W_X + W_Y) + \dfrac{\delta}{B}\left[C^*(\theta_{XX} + \theta_{YY}) + \dfrac{D^*}{L}\left[(\theta_X^2)_X + (\theta_Y^2)_Y\right]\right] - \theta\right] \end{cases} \quad (6)$$

Consider the expansion in terms δ: $\theta=\theta_0+\delta\theta_1+\ldots$ and equalize the coefficients of the power of δ. The system obtained is:

$$\theta_0 = \varepsilon\,\tilde{\theta}_0 = \frac{\varepsilon A}{B}(W_X + W_Y)$$

$$\theta_1 = \varepsilon\,\tilde{\theta}_1 + \varepsilon^2\,\tilde{\theta}_2 = \frac{\varepsilon A}{B^2}\left[C^*(W_{XXX}+W_{XXY}+W_{XYY}+W_{YYY}) - \rho c_0^2 I^*(W_{XTT}+W_{YTT})\right] + \\ + \frac{\varepsilon^2 A^2 D^*}{B^3 L}\left\{\left[(W_{XX}+W_{XY})^2\right]_X + \left[(W_{XY}+W_{YY})^2\right]_Y\right\}$$

Let us consider the following approximation for θ:

$$\theta \simeq \frac{\varepsilon A}{B}(W_X+W_Y) + \frac{\delta\varepsilon A}{B^2}\left[C^*(W_{XXX}+W_{XXY}+W_{XYY}+W_{YYY}) \right.\\ \left. -\rho c_0^2 I^*(W_{XTT}+W_{YTT})\right]$$

namely $\theta \cong \varepsilon(\tilde{\theta}_0 + \tilde{\theta}_1)$

Equation $(6)_1$ becomes:

$$W_{TT} = \frac{1}{\rho c_0^2}\left(\alpha - \frac{2A^2}{B}\right)W_{XX} + \frac{1}{\rho c_0^2}\left(\beta - \frac{2A^2}{B}\right)W_{YY} +$$
$$-\frac{4A^2}{\rho c_0^2 B}W_{XY} + \frac{2\delta A^2 I^*}{B^2}(W_{XX} + 2W_{XY} + W_{YY})_{TT} +$$
$$-\frac{2\delta A^2 C^*}{\rho c_0^2 B^2}[(W_{XX} + 2W_{XY} + W_{YY})_{XX} + (W_{XX} + 2W_{XY} + W_{YY})_{YY}]$$

which describes longitudinal wave propagation only if the movement is provided along the *x*-axis.

It accounts for both the longitudinal and the shear horizontal waves.

As shown in [Por], in the 1D case the wave evolution is described by the "Double Dispersion Equation":

$$W_{TT} = \alpha_1 W_{XX} + \alpha_2 (W_X^2)_X + \alpha_3 W_{XXXX} + \alpha_4 W_{XXTT}$$

where $$\alpha_1 = \frac{1}{\rho c_0^2}\left(\alpha - \frac{2A^2}{B}\right),\ \alpha_2 = \frac{\varepsilon^2 \gamma}{L},\ \alpha_3 = -\frac{2\delta A^2 C^*}{\rho c_0^2 B^2},\ \alpha_4 = \frac{2\delta A^2 I^*}{B^2}$$

3 - **2D Microstructured media**

$\mathbf{u} = U(x,y,t)\mathbf{i} + V(x,y,t)\mathbf{j}$	displacement
$U(x,y,t)$, $V(x,y,t)$	displacements along x and y axis
$v=U_x$, $w=V_y$	strains
W	scale for longitudinal strains $v=U_x$, W<<1 is natural for the Murnaghan materials.
κW	scale for the strain $w=V_y$
L/c_0	scale for time t
$c_0^2=(\lambda+2\mu)/\rho$	characteristic velocity
λ, μ	Lamè coefficients
ρ	macro-density
p	typical size of a microstructure element
d	dissipation parameter (dimension of a length)

Positive dimensionless parameters

$\varepsilon=W<<1$	accounting for elastic strains
$\delta=p^2/L^2<<1$	characterizing the ratio between the microstructure size and the wavelength
$\gamma=d/L$	characterizing the influence of the dissipation

In [Por.Pas.Maug.] the interest was in weak transverse variation.
If we are interested in longitudinal waves, we can assume $w \approx 0$, hence in the governing equations in [Por.Pas.Maug.] the terms in w disappear and, if we assume that the macrobody is linearly elastic and dissipation is not taken into account, such equations read:

$$v_{xx} - v_{tt} + \delta\alpha v_{xxxx} - \delta\alpha v_{xxtt} = 0 \qquad (7)$$

For simmetry reasons $\alpha_3=\alpha_4=\alpha$, $\alpha_5=\alpha_2=\beta$.
To obtain transverse waves it must be $\alpha_3 - \alpha_4 > 0$, but the coefficients can be equal for longitudinal waves.

The introduction of a micro-dissipation acts to modify (7) as follows:

$$v_{xx} - v_{tt} + \delta\alpha(v_{xx} - v_{tt})_{xx} + \gamma\delta\beta(v_{ttt} - v_{xxt})_{xx} = 0 \qquad (8)$$

In (8) we can introduce the macro-nonlinearity, adding the term $\varepsilon\, \alpha_1 (v^2)_{xx}$.

We can use an asymptotic approximation: $v = v_0(x,t) + \gamma v_1(x,t) + \dots$ valid for a small dissipation, and obtain two equations

$$v_{0,xx} - v_{0,tt} + \delta\alpha(v_{0,xx} - v_{0,tt})_{xx} + \alpha_1(v_0^2)_{xx} = 0 \quad (9)$$

$$v_{1,xx} - v_{1,tt} + \delta\alpha(v_{1,xx} - v_{1,tt})_{xx} - \alpha_1(v_0 v_1)_{xx} + \gamma\delta\beta(v_{0,tt} - v_{0,xx})_{txx} = 0 \quad (10)$$

The sostitution: *θ*=*x*-*t* allow us to write (9) as

$$2v_{0.\theta\theta} + 2\delta\alpha(v_{0,\theta t})_{tt} - \alpha_1(v_0^2)_{\theta\theta} = 0$$

and (10) as $$2v_{1_\theta\theta} + 2\delta\alpha\beta v_{0,\theta\theta\theta\theta\theta\theta} = 0$$

namely a fourth and sixth order differential equation, respectively.