NUMERICAL STABILITY OF MASS LUMPING SCHEMES FOR HIGHER ORDER ELEMENTS

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- Finite element dispersion error
  - bilinear versus serendipity elements
  - numerical examples
- Effect of time integration
  - explicit and implicit schemes
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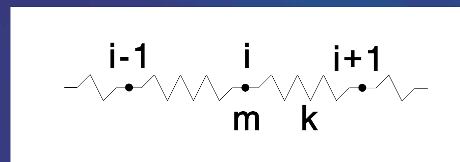
- Conclusions

Dispersion curves

After Newton, Kelvin, Born ...

$$m\ddot{u}_i = k(u_{i-1} - 2u_i + u_{i+1})$$



solution form

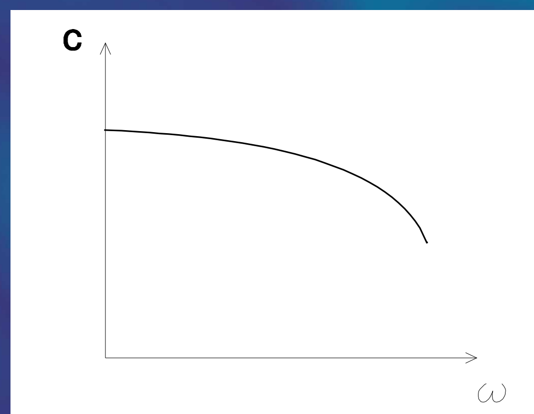
$$u_i = \hat{u} \sin K(x_i - ct)$$

wave number

$$K = \frac{2\pi}{\Lambda} = \frac{\omega}{c}$$

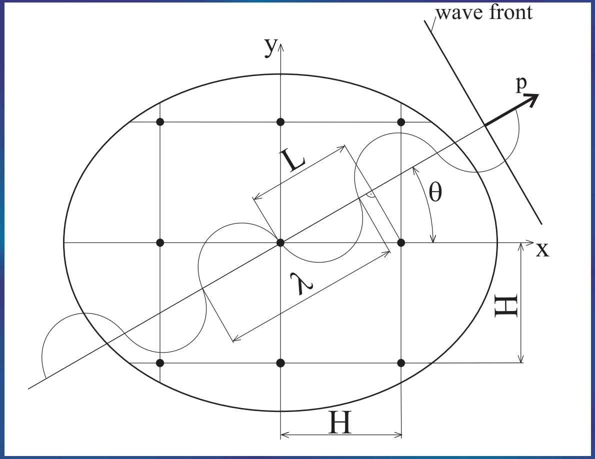
solvability condition

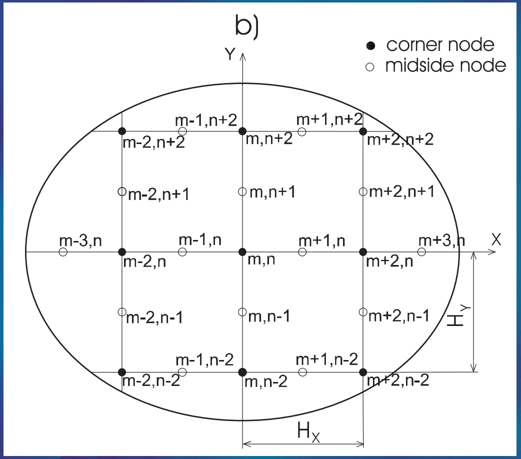
$c = \text{function}(\omega)$



Finite element method

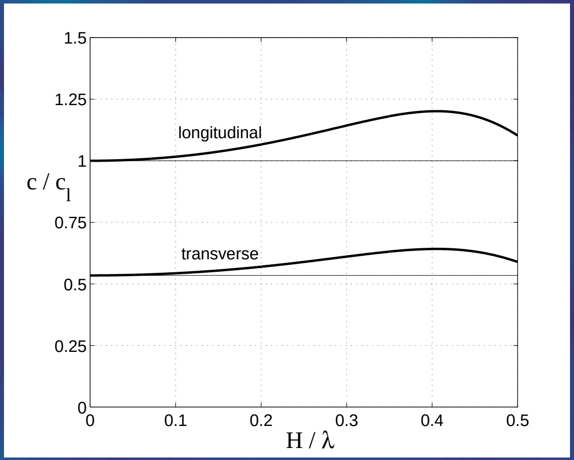
- Belytschko, T., Mullen, R.: On dispersive properties of finite element solutions, In: *Modern Problems in Elastic Wave Propagation.* Wiley 1978.
- Abboud, N.N., Pinsky, P.M.: Finite element dispersion analysis for the three-dimensional second-order scalar wave equation. Int. J. Num. Meth. Engrg., **35**, pp. 1183–1218, 1992.



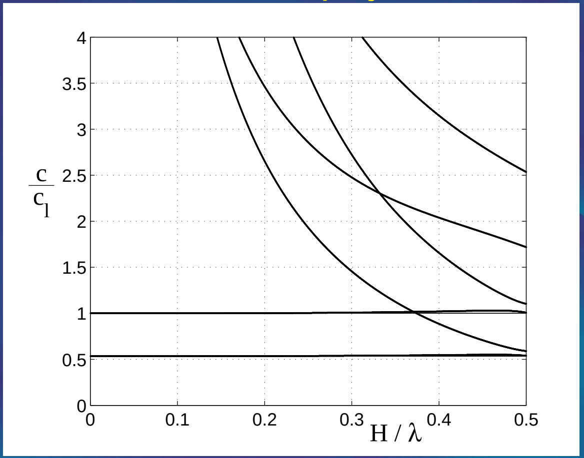


Linear versus quadratic elements

linear



serendipity



Accuracy of quadratic finite elements is by far better.

Numerical test

- Plane strain square domain 100×100 serendipity finite elements
- Unit material properties

  Young's modulus $E = 1$

  Poisson's ratio $\nu = 0.3$

  density $\rho = 1$
- Pointwise harmonic loading in the horizontal direction

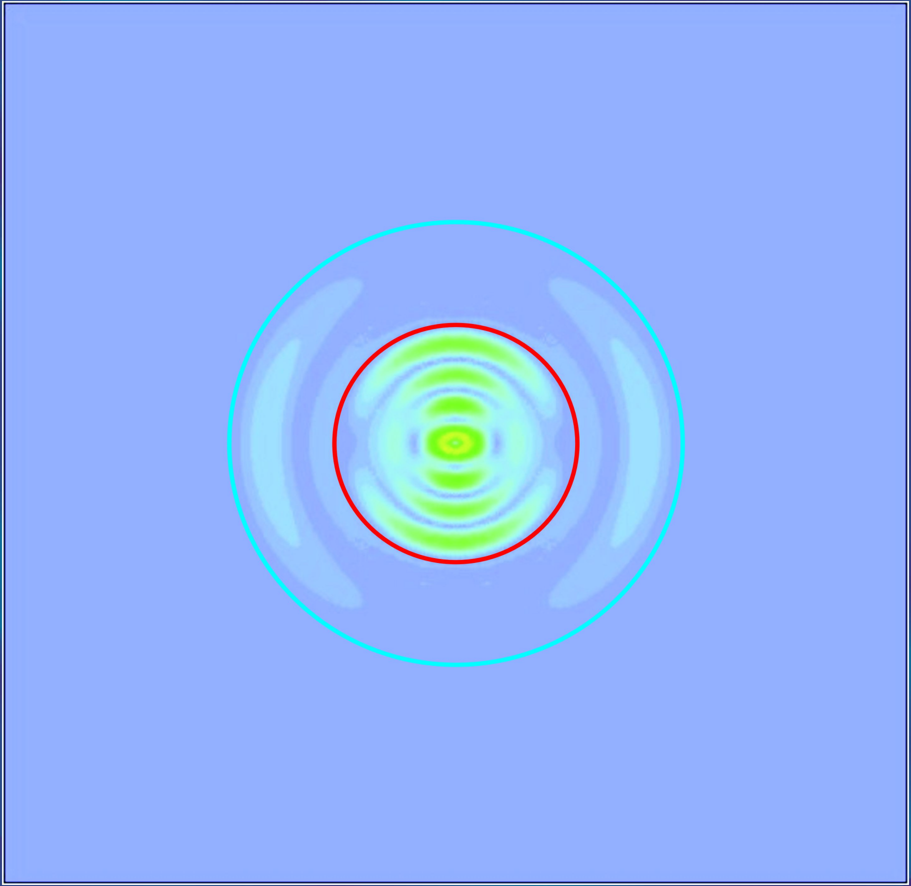
  $F_x = \hat{F}_x \sin \omega t$

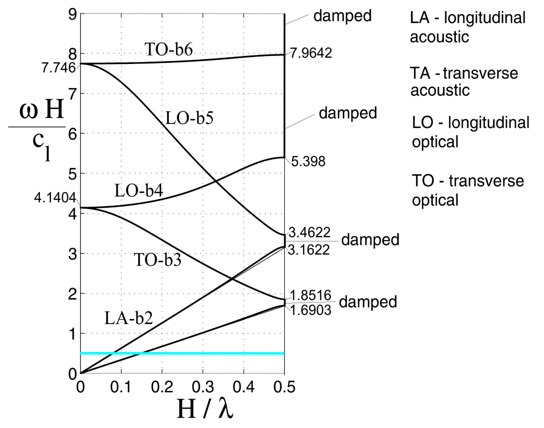
  $\omega H / c_1$ stepped by 0.1 increment
- Newmark's method with small Courant number

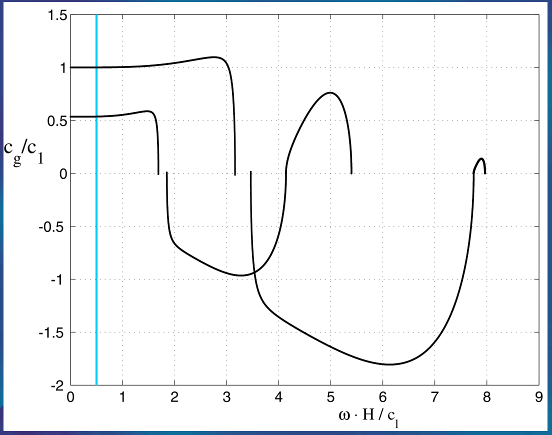
  time integration effect disabled

  Co $= c_1 \Delta t / H = 0.01$

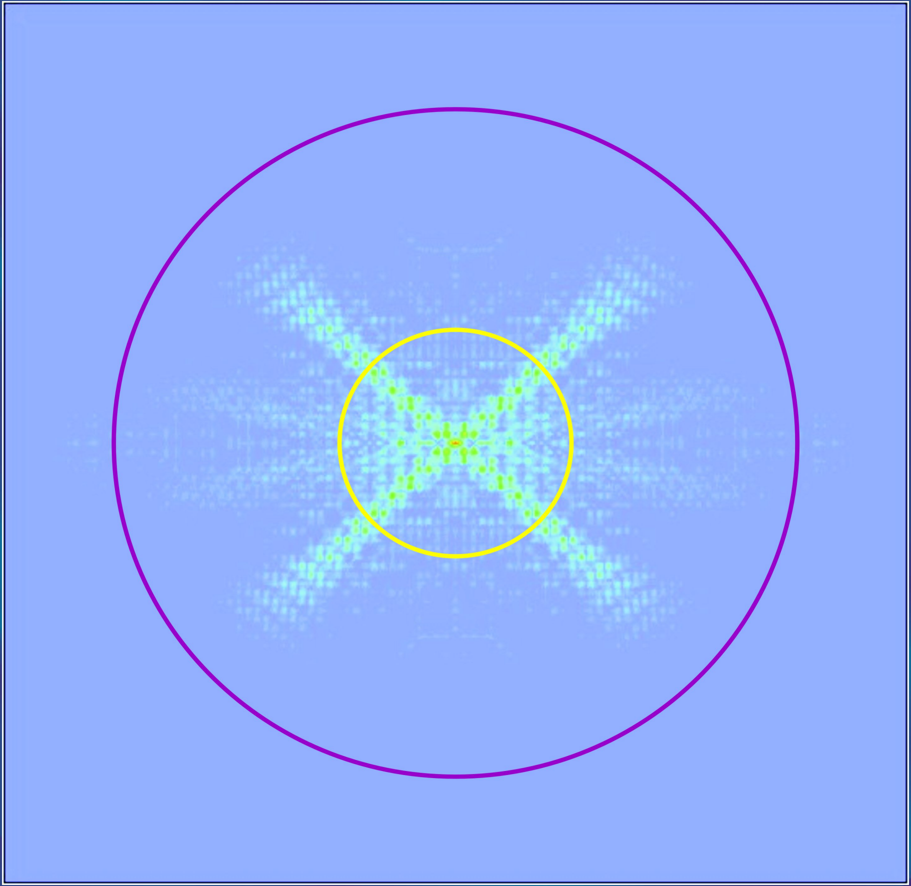
frequency $\omega H/c_1$=0.5

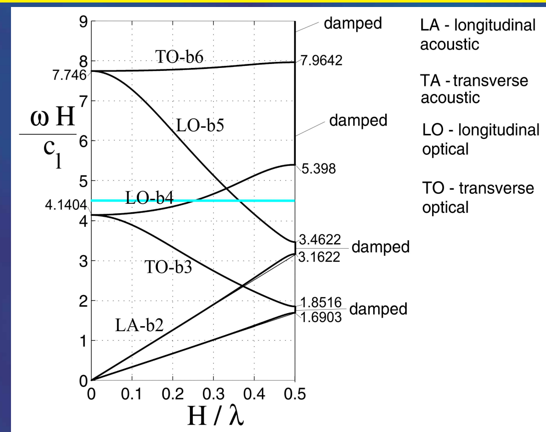


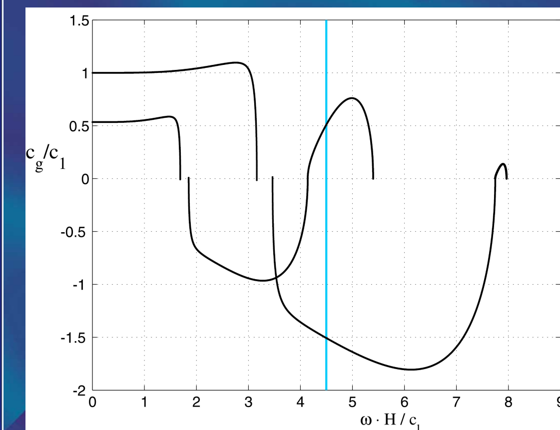




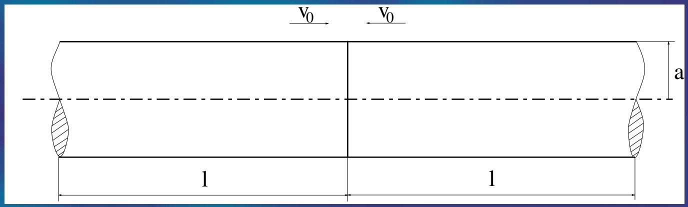
frequency $\omega H/c_1$=4.5







Contact-impact problem of two cylinders



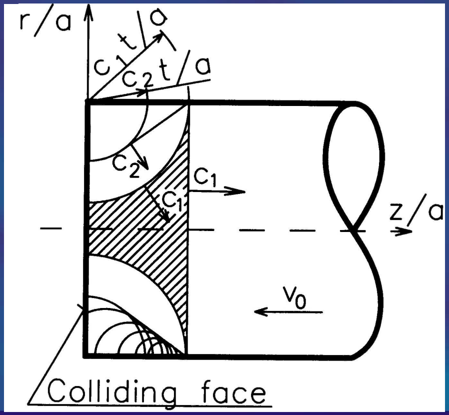
Geometry: $a = 2.5$ mm, $l = 6.25$ mm

Material parameters:
$E = 2.1 \times 10^5$ MPa, $\nu = 0.3$, $\rho = 7800\,\mathrm{kg/m^3}$

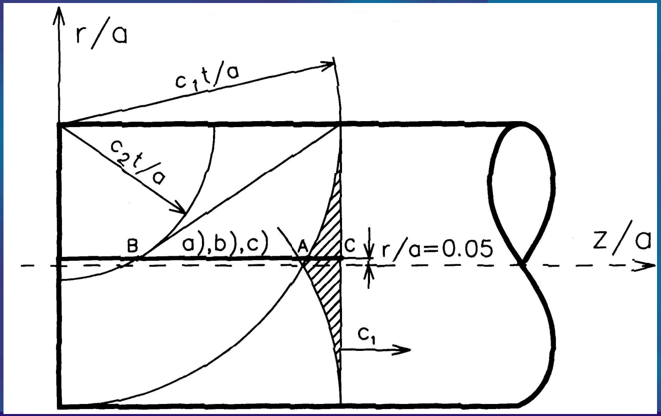
Initial condition: $v_0 = 5$ [m/s]

Theoretical position of wave fronts in colliding cylinders

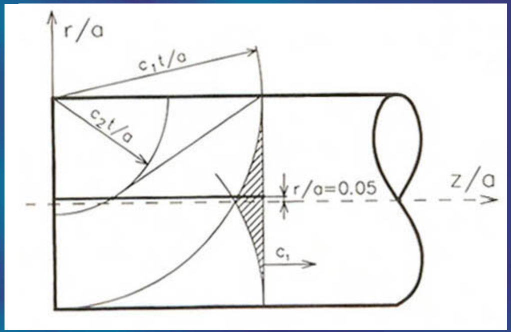
$c_1 t/a = 0.8$



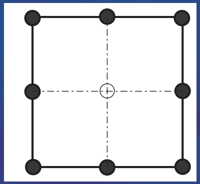
$c_1 t/a = 2$



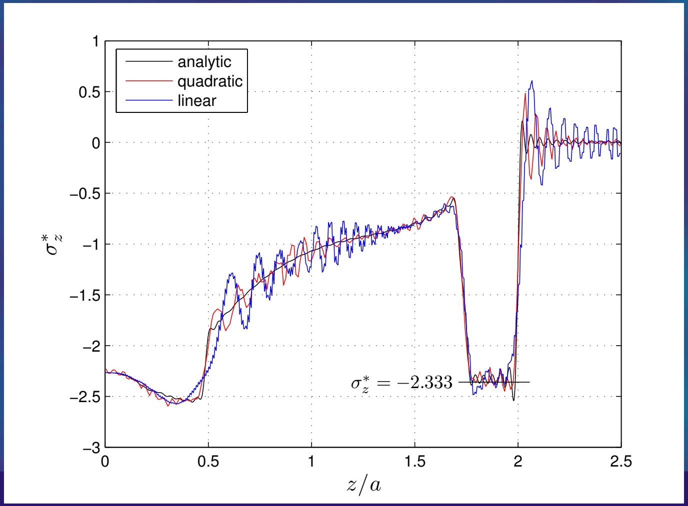
Discretization error



Equivalent meshes



axial stress distribution



Contents

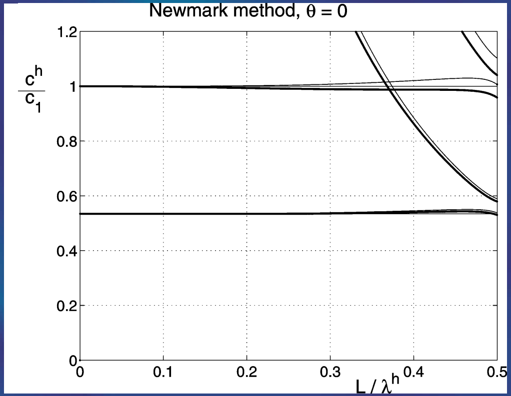
Newmark method

Discrete operator

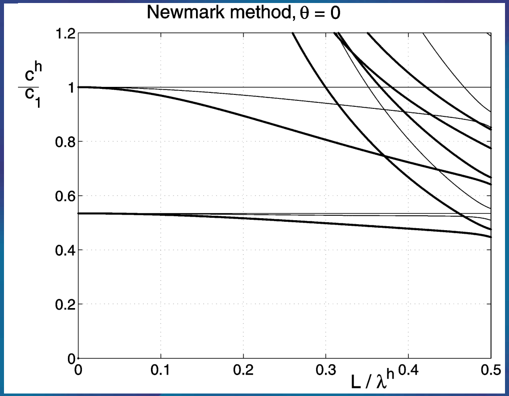
$$\left(\mathbf{K} + \frac{4}{\Delta t^2}\mathbf{M}\right)\mathbf{u}^{t+\Delta t} = \mathbf{R}^{t+\Delta t} + \mathbf{M}\left(\frac{4}{\Delta t^2}\mathbf{u}^t + \frac{2}{\Delta t}\dot{\mathbf{u}}^t + \frac{4}{\Delta t}\ddot{\mathbf{u}}^t\right)$$

Dispersion curves

Co = 0.00–0.25



Co = 0.25–1.00



Insensitive to time step for Co $\leq$ 0.25.

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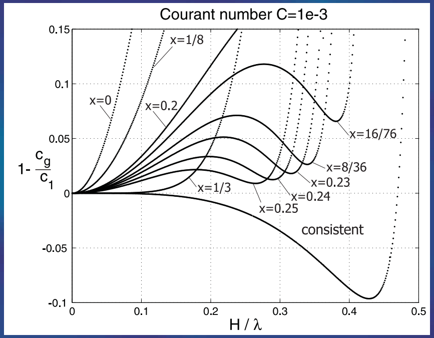
Central difference method

Discrete operator

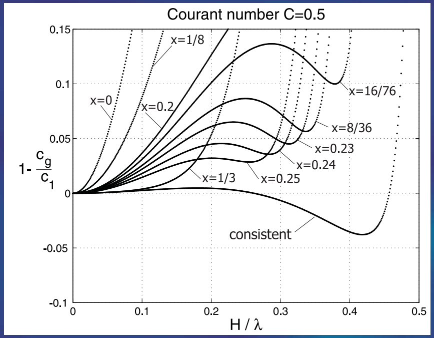
$$\frac{1}{\Delta t^2}\mathbf{M}\mathbf{u}^{t+\Delta t} = \mathbf{R}^t - \left(\mathbf{K} - \frac{2}{\Delta t^2}\mathbf{M}\right)\mathbf{u}^t - \frac{1}{\Delta t^2}\mathbf{M}\mathbf{u}^{t-\Delta t}$$

Dispersion curves

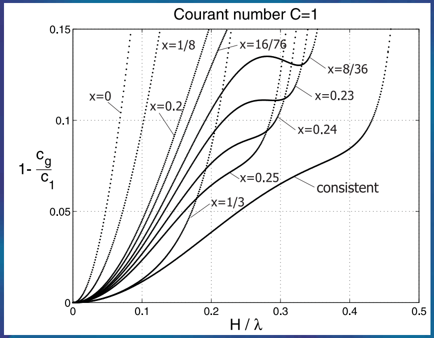
Co $=$ 0.001



Co $=$ 0.5



Co $=$ 1

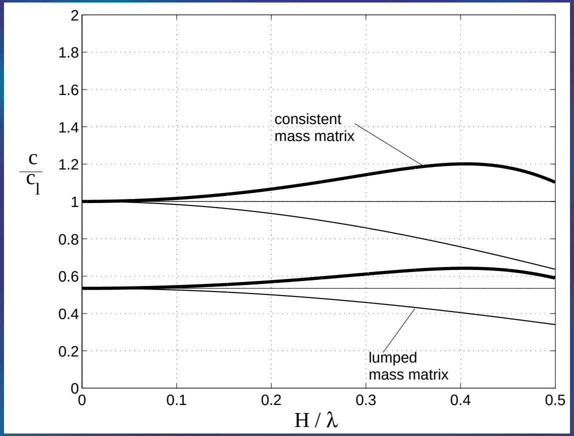


Insensitive to time step for Co $\leq$ 0.5.

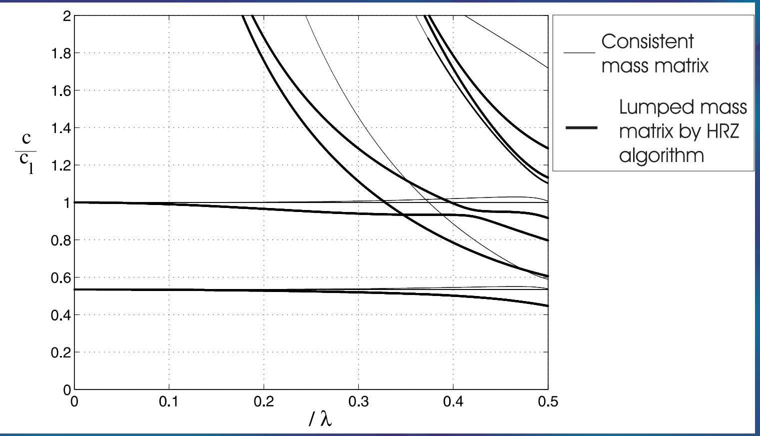
Mass matrix lumping

Row sum and Hinton-Rock-Zienkiewicz methods used.

linear

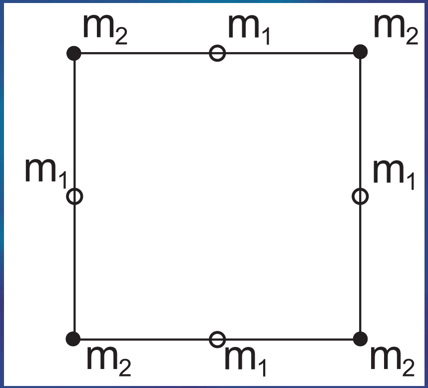


quadratic



Similar performance—advantage lost.

General lumping scheme



$$
\begin{aligned}
m &= 4m_1 + 4m_2 > 0 \\
m_1 &= xm > 0 \\
m_2 &= (0.25 - x)m > 0
\end{aligned}
$$

$$x \in (0; 0.25)$$

Examples:

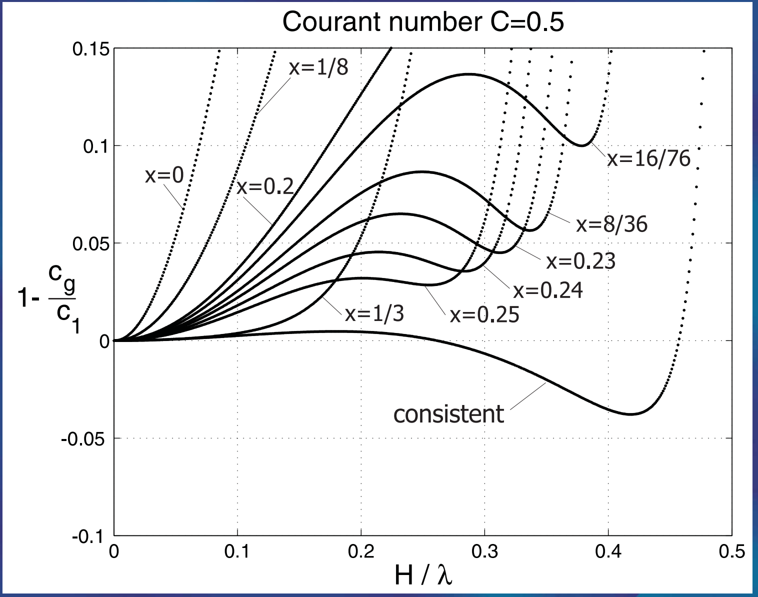
$x = 16/76 = 0.21$ HRZ (3×3 rule)

$x = 8/36 = 0.22$ HRZ (2×2 rule)

$x = 1/3 = 0.33$ row sum method—out of the interval!

Optimum mass distribution

(central difference method, Co = 0.5)



Dispersion suppressed as $x \to 0.25$.

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  - **numerical stability**
- Conclusions

Numerical stability

- Fried, I.: Discretization and round-off errors in the finite element analysis of elliptic boundary value problems and eigenproblems. Ph.D thesis, MIT, 1971.
- Dokainish, M.A., Subbaraj, K.: A survey of direct time-integration methods in computational structural dynamics - I. Explicit methods. Computers and Structures, **32**(6), pp 1371–1386, 1989.

Stability condition

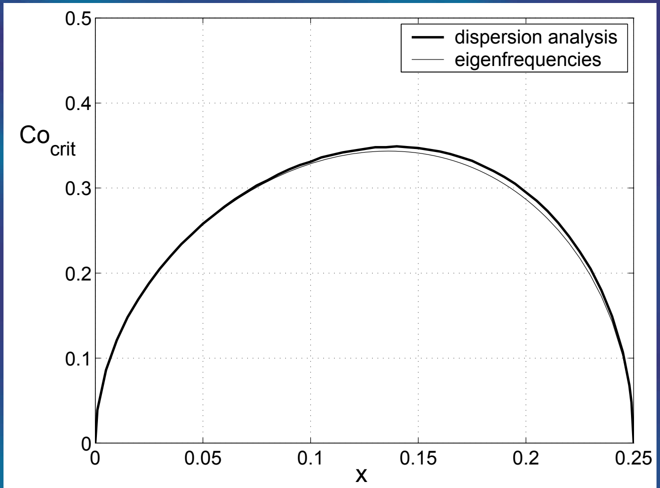
$$\Delta t \leq \Delta t_{cr} = \frac{2}{\omega_{max}}$$

Estimation of the maximum eigenvalue

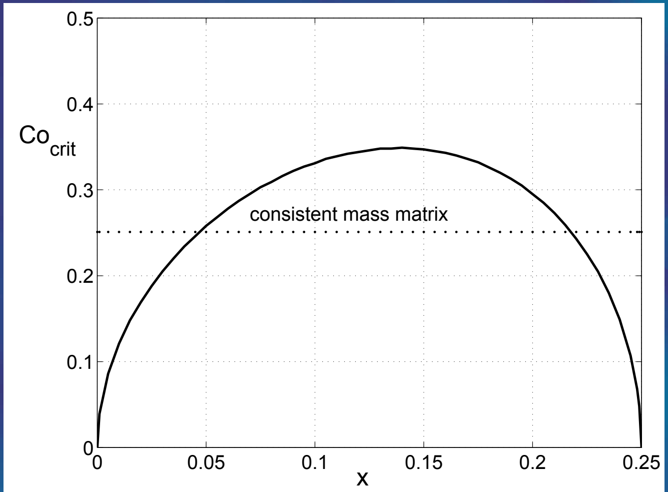
$$\sup_{H/\lambda \in (0;0.5)} \omega(H/\lambda) \leq \omega_{max} \leq \max_{m=1,\text{nelem}} \omega_{max}^{(m)}$$

Critical Courant number

bounding eienvalues



lumping optimization



Remark 1: Critical Courant number for the consistent mass matrix is 0.25.

Remark 2: Critical Courant number for $x = 0.23$ is 0.21.

Conclusions

- Threshold values of the time step for bilinear elements:

  Co = 0.5 (Newmark, consistent, best accuracy)

  Co = 1.0 (CDM, row sum lumping, stability)

- The same for serendipity elements

  Co = 0.25 (Newmark, consistent, best accuracy)

  Co = 0.25 (CDM, HRZ lumping, stability)

  Co = 0.20 (CDM, x=0.23 lumping, stability)