Nonlinear counterpropagating waves in inhomogeneous materials

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Purpose

To use the data about complex phenomena of nonlinear wave-wave, wave-material and wave-prestress interaction for ultrasonic nondestructive characterization of inhomogeneous materials

Problems to discuss

- Superposition or interaction
  - wave versus wave
  - wave versus prestress
  - wave versus material properties
- Wave interaction for nondestructive testing (NDT)
  - characterization of inhomogeneous prestress in nonlinear elastic material
  - characterization of inhomogeneous physical properties of functionally graded materials
- Conclusions

Theory

Basic assumptions

- material is elastic with quadratic nonlinearity
- physical and geometrical nonlinearity is considered
- deformations are small but finite

Equation of motion

$$[T_{KL}(X_J,t)(\delta_{kL} + \delta_{kM} U_{M,L}(X_J,t))]_{,K} - \rho_0 \delta_{kM} U_{M,tt}(X_j,t) = 0$$

T_{KL} - second Piola-Kirchhoff stress tensor

U_k - displacement vektor

X_K - Lagrangian rectangular coordinates

x_K - Eulerian rectangular coordinates

t - time

ρ_0 - density of the material

δ_{kL} - Euclidean shifter

k,K,L,M - 1,2,3

Superposition or interaction

$$U^*_K = U_K + U^0_K$$

U^*_K - displacement at the present state

U_K - displacement evoked by exitation # one (wave)

U^0_K - displacement evoked by exitation # two (prestress)

$$2E_{KL} = U_{K,L} + U_{L,K} + U_{M,K} U_{M,L}$$

E_{KL} - Green-Lagrange strain tensor

$$T_{KL} = T_{KL}(E_{KK} \; E_{KL}E_{LK} \; E_{KL}E_{ML}E_{KM})$$

$K,L,M = 1,2,3$

Superposition or interaction

$$T_{KL} = (\lambda I_1 + 3\nu_1 I_1^2 + \nu_2 I_2)\partial I_1/\partial E_{KL}$$

$$+(\mu + \nu_2 I_1)\,\partial I_2/\partial E_{KL} + \nu_3 \partial I_3/\partial E_{KL}$$

$I_1 = E_{KK}$, $I_2 = E_{KL} E_{LK}$, $I_3 = E_{KL} E_{ML} E_{KM}$, I_K - Green-Lagrange strain tensor invariants

$k, K, L, M = 1,2,3$

Material properties

$\rho 0$ - density

λ, μ - Lamé constants

ν_1, ν_2, ν_3 - third order elastic constants

Superposition or interaction

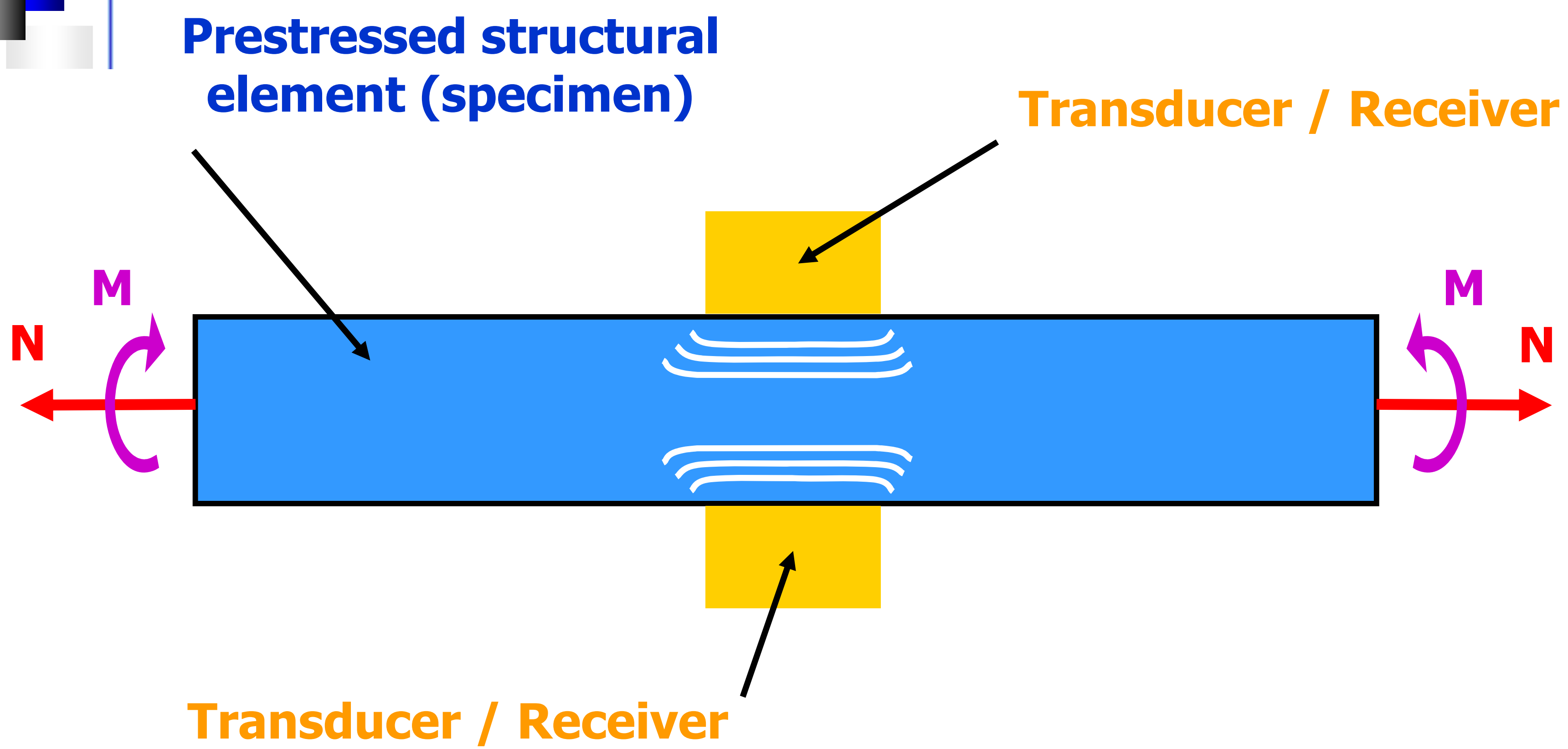
- Wave versus wave (prestress)
  - superposition occurs in linear case
  - interaction occurs by considering
    - physical nonlinearity
    - geometrical nonlinearity
    - simultaneous impact of physical and geometrical nonlinearity

- Wave versus material properties
  - interaction occurs in linear and nonlinear case

Wave interaction for NDT

- Counter-propagation of waves in inhomogeneously prestressed nonlinear elastic material
  - method for qualitative characterization of prestress
  - method for quantitative characterization of two-parametric prestress
- Counterpropagating waves in functionally graded materials
  - method for qualitative characterization of exponentially graded nonlinear elastic material

Prestress diagnostics



Equation of motion

$$[\,1+f_1\,]\,U_{1,11} + f_2\, U_{1,1} + f_3\, U_{1,1} U_{1,11} - c^{-2}\, U_{1,tt} = 0$$

----- linear terms

----- dispersive linear term

----- nonlinear term

$$f_1 = k_1\, U^0_{1,1} + k_2\, U^0_{2,2}$$

$$f_2 = k_1\, U^0_{1,11} + k_3\, U^0_{1,22} + (k_2 + k_4)\, U^0_{2,21}$$

$$f_3 = k_1 \qquad U^0 \equiv U^0(X_1, X_2)$$

Prestress state

Equations of equilibrium

$$(1+ k_1\, U^0{}_{I,I} + k_2\, U^0{}_{J,J}\,)\, U^0{}_{I,II} + (2\, k_3\, U^0{}_{I,J} + 2\, k_4\, U^0{}_{J,I}\,)\, U^0{}_{I,IJ}$$

$$+ (k_7 + k_3\, U^0{}_{I,I} + k_3\, U^0{}_{J,J}\,)\, U^0{}_{I,JJ} + (k_4\, U^0{}_{I,J} + k_3\, U^0{}_{J,I}\,)\, U^0{}_{J,II}$$

$$+ (k_3\, U^0{}_{I,J} + k_4\, U^0{}_{J,I}\,)\, U^0{}_{J,JJ} + (k_6 + k_5\, U^0{}_{I,I} + k_5\, U^0{}_{J,J}\,)\, U^0{}_{J,JI}$$

$$+ \rho_0\, B_I = 0$$

$$k_5 = k\,[\lambda + \mu + 3\,(2\nu_1 + \nu_2 + \nu_3\,/2)]$$

$$k_6 = k\,(\lambda + \mu\,), \quad k_7 = k\,\mu$$

$I = 1$, $J = 2$ - first equation

$I = 2$, $J = 1$ - second equation $\qquad U^0 \equiv U^0\,(X_1, X_2)$

Prestress state

Solution

$$U_1^0(X_1,X_2) = \sum_{n=1}^{\infty} \varepsilon^{(n)} U_1^{0(n)}(X_1,X_2)$$

First approximation

$$U^{0(1)}{}_{I,II} + k_7\, U^{0(1)}{}_{I,JJ} + k_6\, U^{0(1)}{}_{J,JI} = 0$$

$$T^{0(1)}{}_{11}(0,X_2) = T^{0(1)}{}_{11}(h,X_2) = T^{0(1)}{}_{12}(0,X_2) = T^{0(1)}{}_{12}(h,X_2) = 0$$

$$T^{0(1)}{}_{22}(X_1, \pm L/2) = \sum_{n=0}^{5} w_n X_1^n, \qquad T^{0(1)}{}_{21}(X_1, \pm L/2) = 0$$

Prestress state

nth approximation

$$U^{0(n)}{}_{I,II} + k_7\, U^{0(n)}{}_{I,JJ} + k_6\, U^{0(n)}{}_{J,JI} = F\left(U^{0(n-1)}{}_{I}, U^{0(n-1)}{}_{J}\right)$$

$$T^{0(n)}{}_{11}(0,X_2) = T^{0(n)}{}_{11}(h,X_2) = T^{0(n)}{}_{12}(0,X_2) = T^{0(n)}{}_{12}(h,X_2) = 0$$

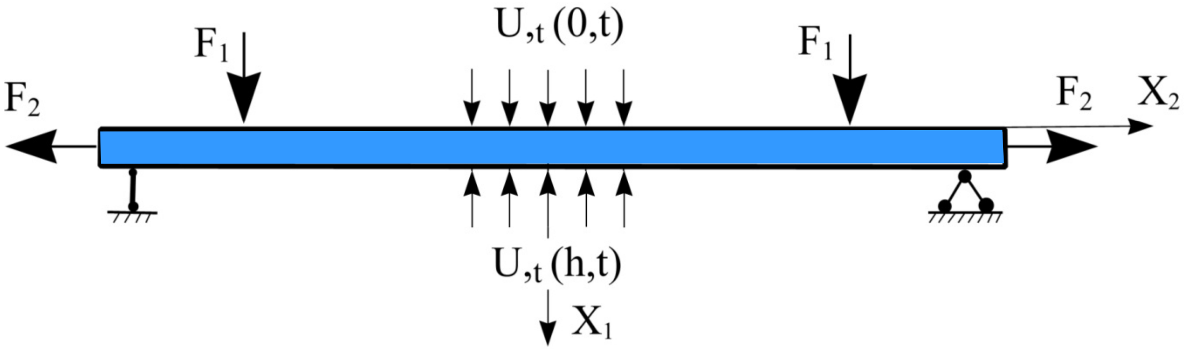
$$T^{0(n)}{}_{22}(X_1, \pm L/2) = T^{0(n)}{}_{21}(X_1, \pm L/2) = 0, \quad n = 2, 3, \ldots$$

Final solution

$$U_1^0(X_1,X_2) = \varepsilon\, P_1^{5.5}(X_1,X_2) + \varepsilon^2 P_2^{7.7}(X_1,X_2)$$

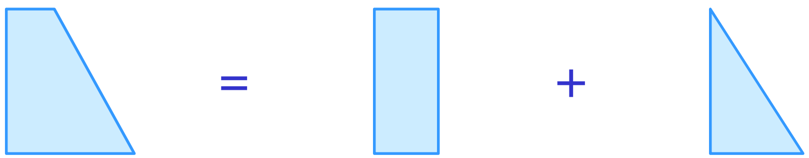
$$U_2^0(X_1,X_2) = \varepsilon\, P_3^{5.5}(X_1,X_2) + \varepsilon^2 P_4^{7.7}(X_1,X_2)$$

Loading scheme



Stress

$T_{22} = a + b\, X_1 \qquad a \qquad b\, X_1$



Equation of motion

$$[1+f_1]\, U_{1,11} + f_2\, U_{1,1} + f_3\, U_{1,1} U_{1,11} - c^{-2}\, U_{1,tt} = 0$$

----- linear terms

----- dispersive linear term

----- nonlinear term

$$f_1 = k_1\, U^0_{1,1} + k_2\, U^0_{2,2}$$
$$f_2 = k_1\, U^0_{1,11} + k_3\, U^0_{1,22} + (k_2 + k_4) U^0_{2,21}$$
$$f_3 = k_1 \qquad U^0 \equiv U^0(X_1, X_2)$$

Equation of motion

Initial and boundary conditions

$$U_{1,t}(X_1,X_2,0) = U_1(X_1,X_2,0) = 0$$

$$U_{1,t}(0,X_2,t) = \varepsilon\, a_0\, \varphi(t)\, H(t)$$

$$U_{1,t}(h,X_2,t) = \varepsilon\, a_h\, \psi(t)\, H(t)$$

$$|\,\varepsilon\,| << 1$$

a_0 , a_h - constants

$max\,|\varphi(t)| = max\,|\psi(t)| = 1$

$H(t)$ - Heaviside function

Perturbative solution

Solution

$$U_1(X_1,t) = \sum_{n=1}^{\infty} \varepsilon^{(n)} U_1^{(n)}(X_1,t)$$

First approximation

$$U_{1,11}^{(1)}(X_1,0) - c^{-2} U_{1,tt}^{(1)}(X_1,0) = 0$$

$$U_1^{(1)}(X_1,0) = U_{1,t}^{(1)}(X_1,0) = 0$$

$$U_{1,t}^{(1)}(0,t) = a_0\, \varphi(t)\, H(t)$$

$$U_{1,t}^{(1)}(h,t) = a_h\, \psi(t)\, H(t)$$

Perturbative solution

n^th approximation

$$U_{1,11}^{(n)}(X_1,0) - c^{-2} U_{1,tt}^{(n)}(X_1,0) = \sum_{j=1}^{m} G_j^{(n)}(X_1)\, F_j^{(n)}(\zeta_j^{(n)})$$

$$\zeta_j^{(n)} = t - g_j^{(n)}(X_1), \quad g_j^{(n)}(X_1) \geq 0$$

$$U_1^{(n)}(X_1,0) = U_{1,t}^{(n)}(X_1,0) = 0$$

$$U_{1,t}^{(n)}(0,t) = U_{1,t}^{(n)}(h,t) = 0, \quad n = 2, 3, \ldots$$

Harmonic waves

$$U_{1,t}(X_1,t) = \sum \varepsilon^{(n)} U_{1,t}^{(n)}, \qquad U_{1,t}^{0}(X_1,X_2) = \sum h^{(n)} U_{1,t}^{0(n)}$$

$$U_{1,t}(X_1,X_2,t) = A_0 + A_1 \sin(\omega\zeta + \theta_1) + A_2 \sin(2\omega\zeta + \theta_2) + A_3 \sin(3\omega\zeta + \theta_3)$$

$\varepsilon \neg h^2$ Weak wave	$\varepsilon U_{1,t}^{(1)}$	$A_1^{(1)}$	-	-
	$\varepsilon^2 U_{1,t}^{(2)}$	$A_1^{(2)}\ \theta_1^{(2)}$	-	-
	$\varepsilon^3 U_{1,t}^{(3)}$	$A_1^{(3)}\ \theta_1^{(3)}$	$A_2^{(3)}$	-
$\varepsilon \neg h$	$\varepsilon U_{1,t}^{(1)}$	$A_1^{(1)}$	-	-
	$\varepsilon^2 U_{1,t}^{(2)}$	$A_1^{(2)}\ \theta_1^{(2)}$	$A_2^{(2)}$	-
	$\varepsilon^3 U_{1,t}^{(3)}$	$A_1^{(3)}\ \theta_1^{(3)}$	$A_2^{(3)}\ \theta_2^{(3)}$	$A_3^{(3)}$
$\varepsilon^2 \neg h$ Strong wave	$\varepsilon U_{1,t}^{(1)}$	$A_1^{(1)}$	-	-
	$\varepsilon^2 U_{1,t}^{(2)}$	-	$A_2^{(2)}$	-
	$\varepsilon^3 U_{1,t}^{(3)}$	$A_1^{(3)}\ \theta_1^{(3)}$	-	$A_3^{(3)}$

Numerical simulation

Input data

$\rho_0 = 2800$ kg/m^3

$\lambda = 50$ GPa

$\mu = 27.6$ GPa

$\nu_1 = -136$ GPa

$\nu_2 = -197$ GPa

$\nu_3 = -38$ GPa

$h = 0.1$ m

$\varepsilon = 1*10^{-4}$

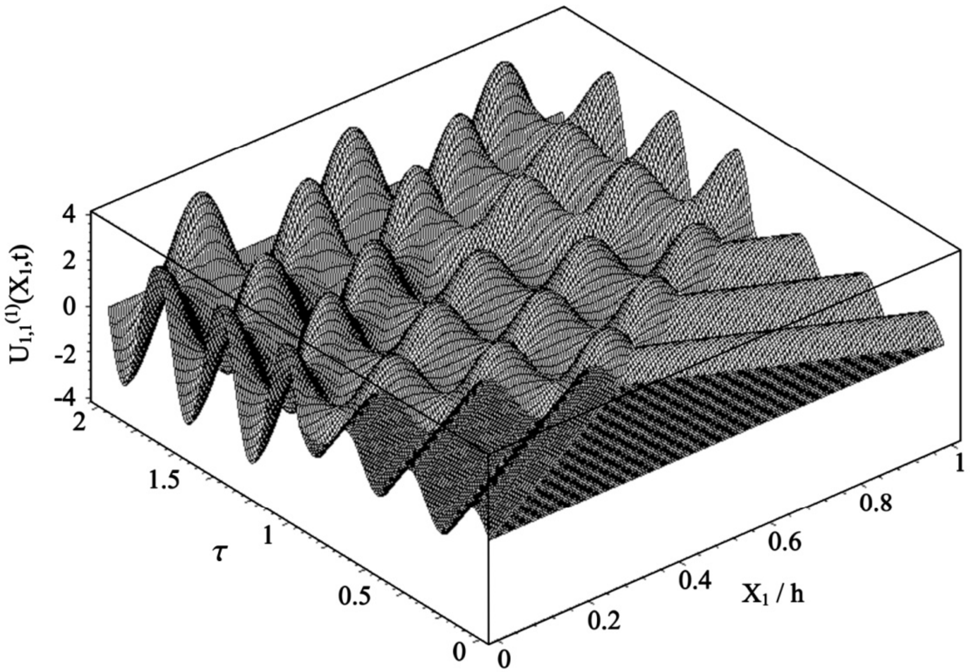
$\omega = 10^6,\ldots,10^7$ rad/s

$a = -60,\ldots,60$ MPa

$b = -1.2,\ldots,1.2$ GPa/m

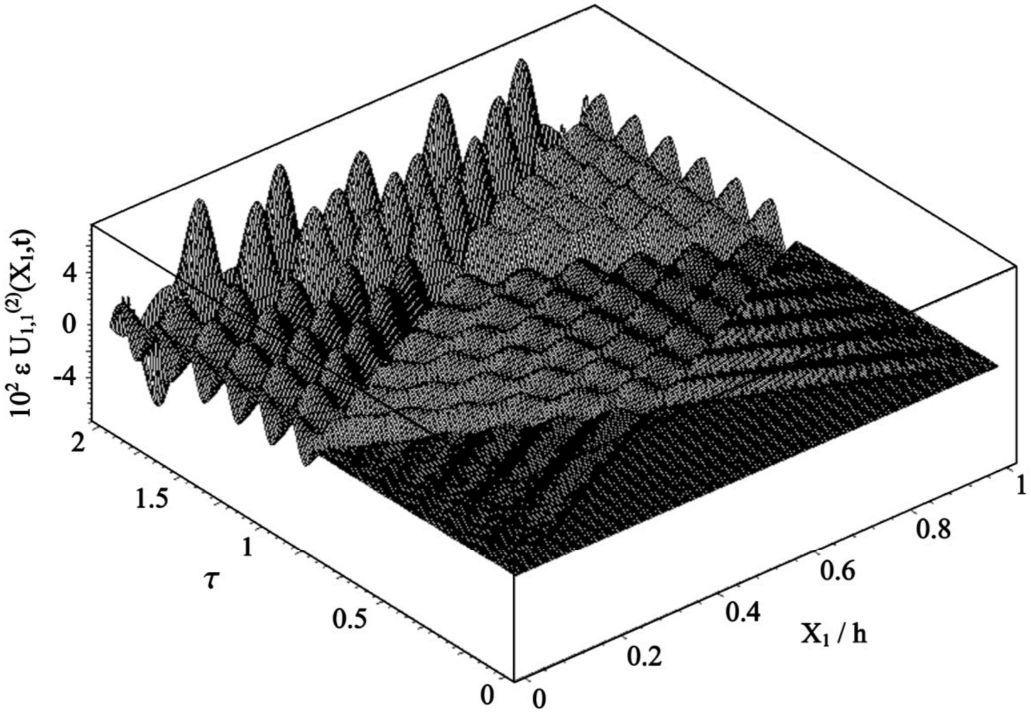
Wave interaction

Sine wave propagation



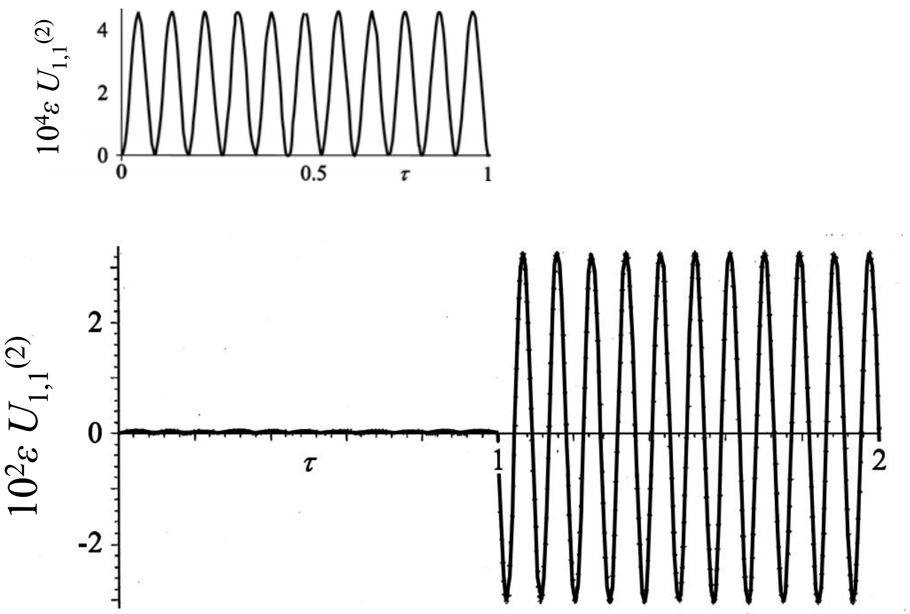
Wave interaction

Nonlinear effects



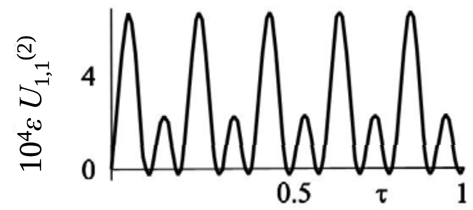
Boundary oscillations

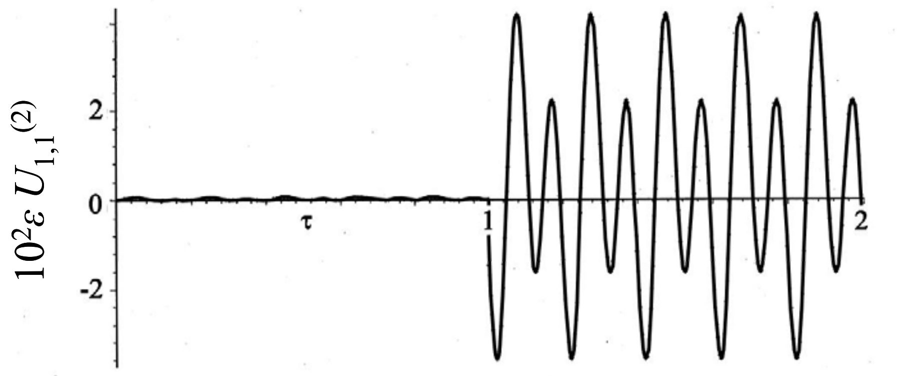
Homogeneous prestress-fee material



Boundary oscillations

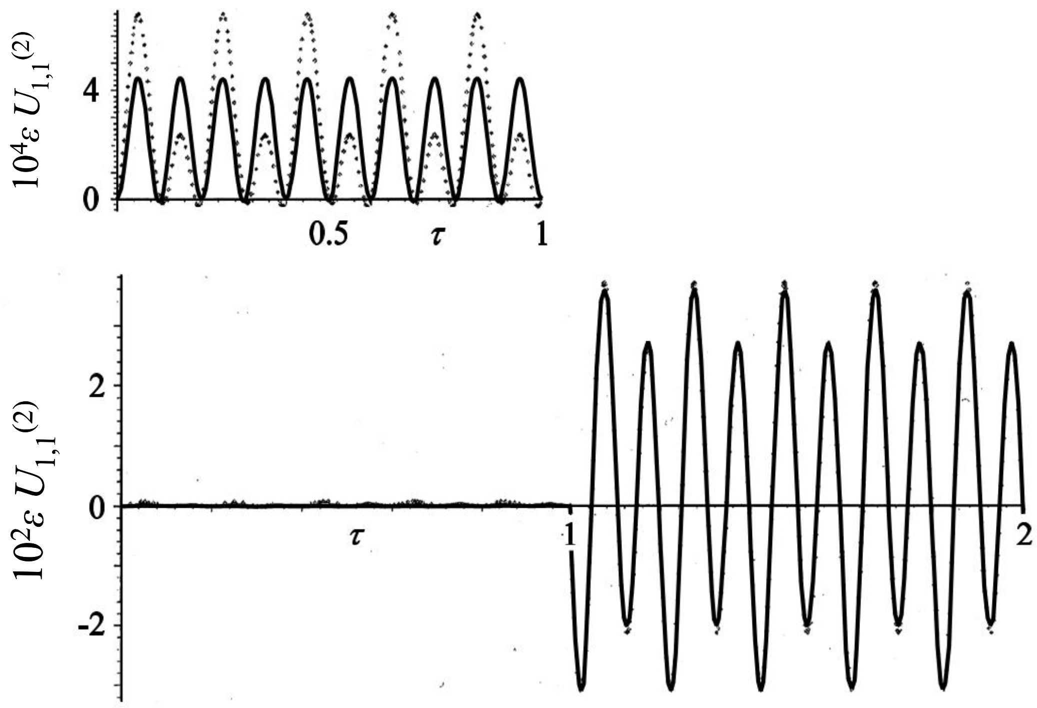
Homogeneously prestressed material





Boundary oscillations

Inhomogeneously prestressed material



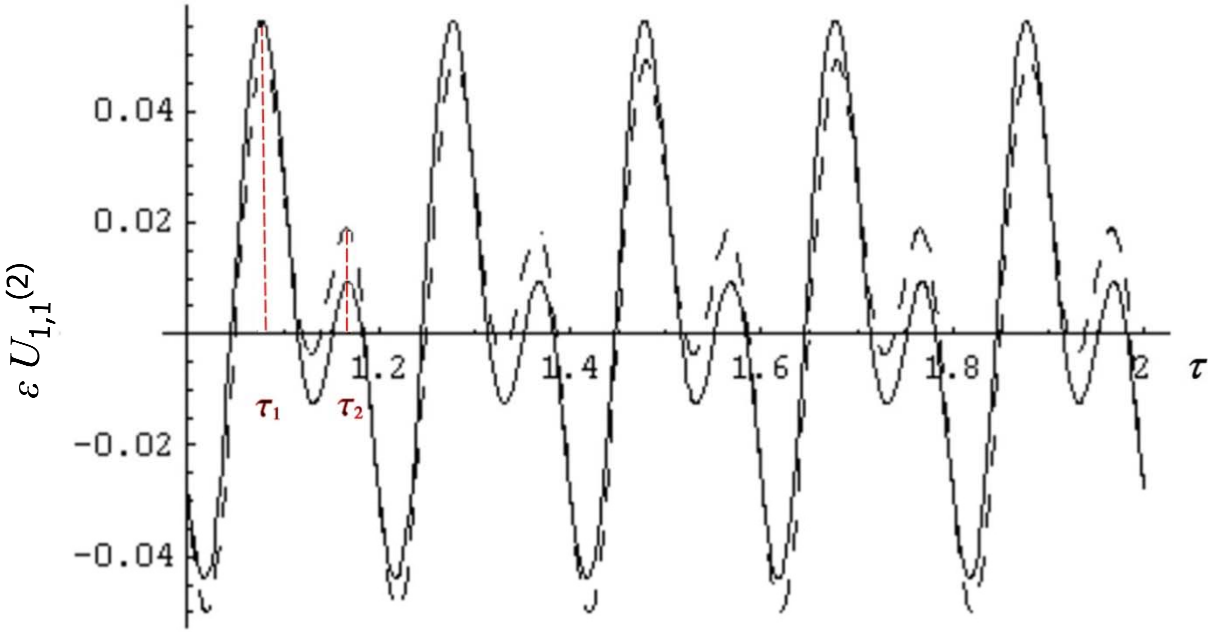
Wave interaction technique

Qualitative prestress characterization

- Boundary oscillations permit to distinguish:
  - prestress-free material
  - homogeneously prestressed material
  - material undergoing pure bending
  - material undergoing pure bending with tension or compression

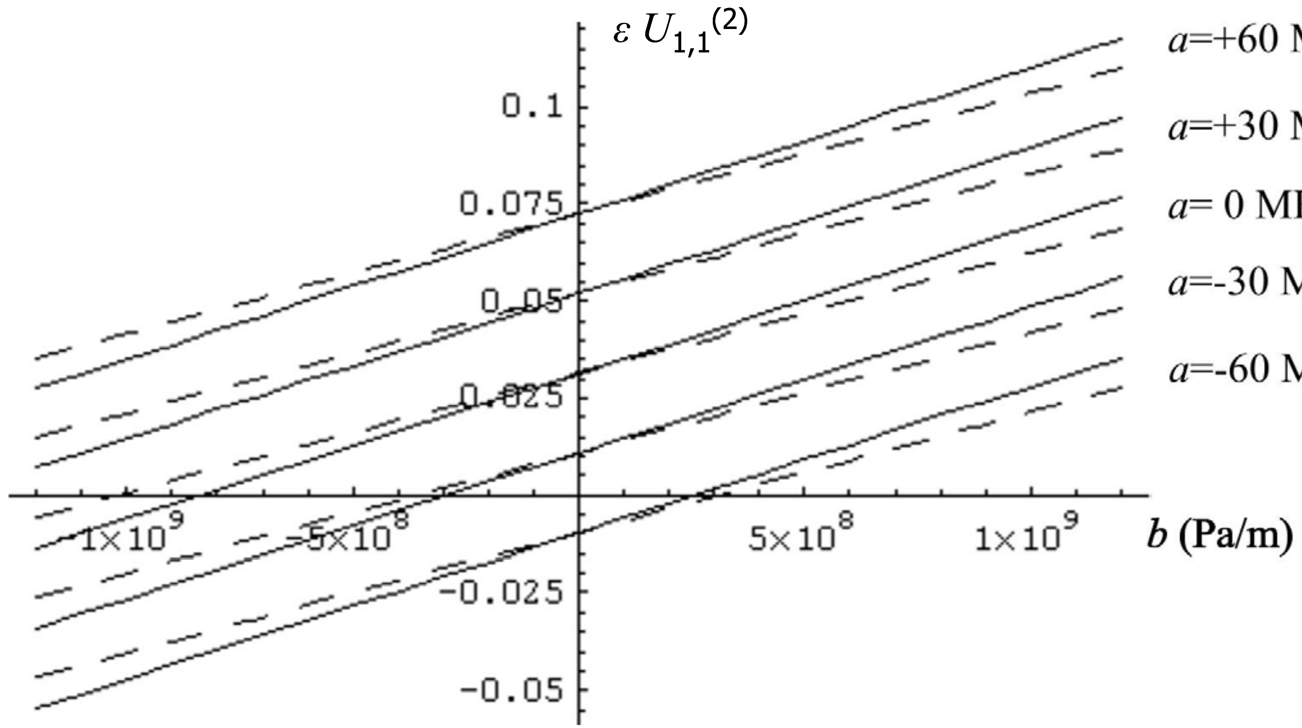
Wave interaction technique

Quantitative NDE



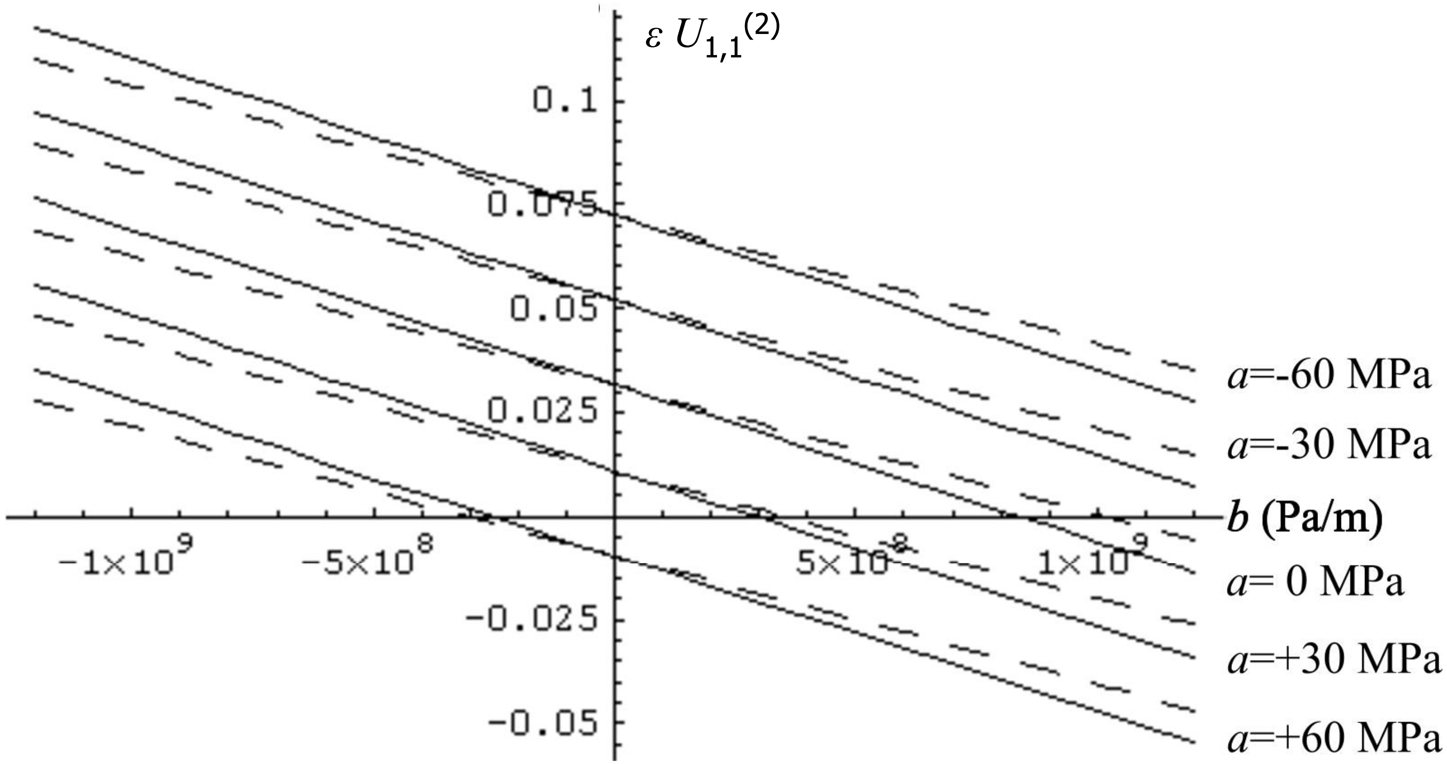
Wave interaction technique

Instant τ_1



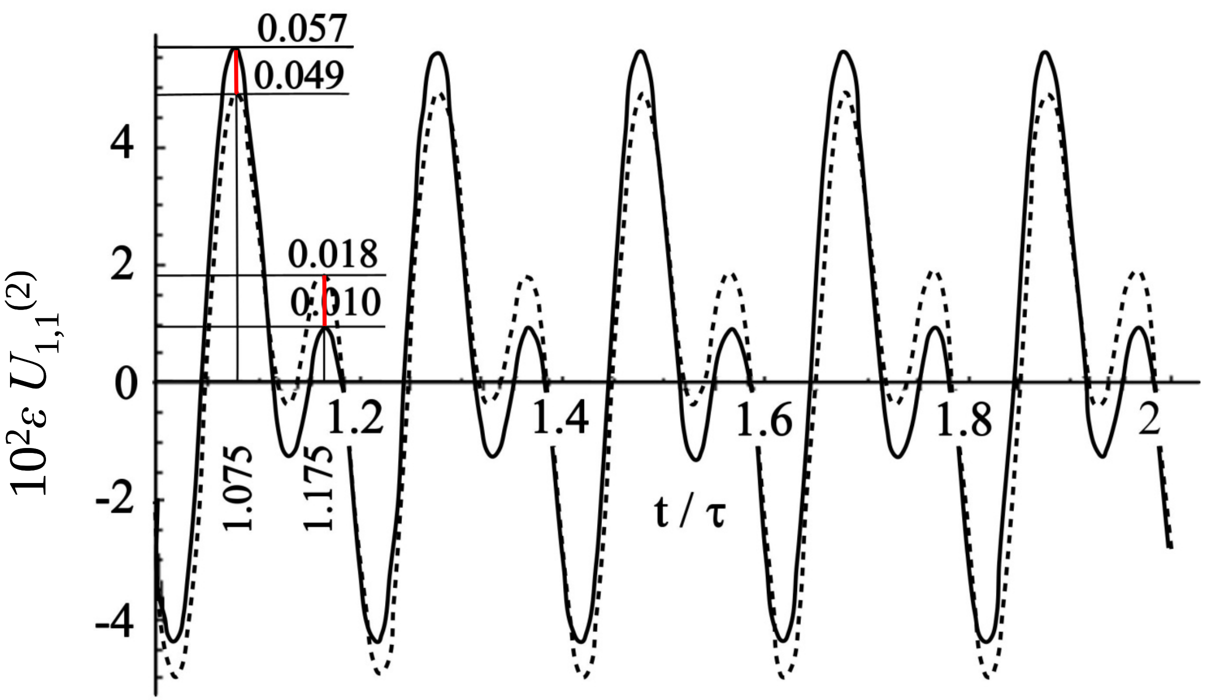
Wave interaction technique

Instant τ_2



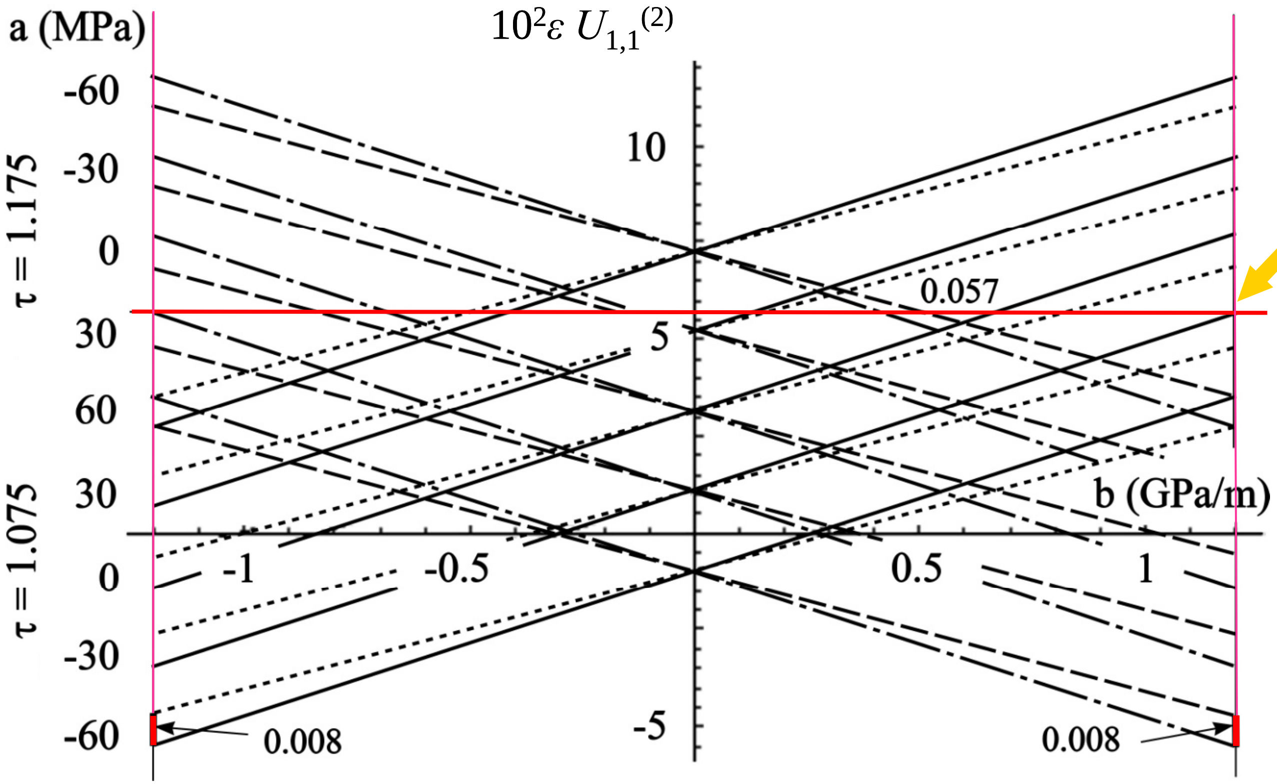
Wave interaction technique

Recoded data



Quantitative NDE

Prestress evaluation, *a* =-30 MPa, *b* =1.2 GPa/m



Functionally graded material

Elastic functionally graded materials with smoothly and arbitrarily variable nonlinear properties

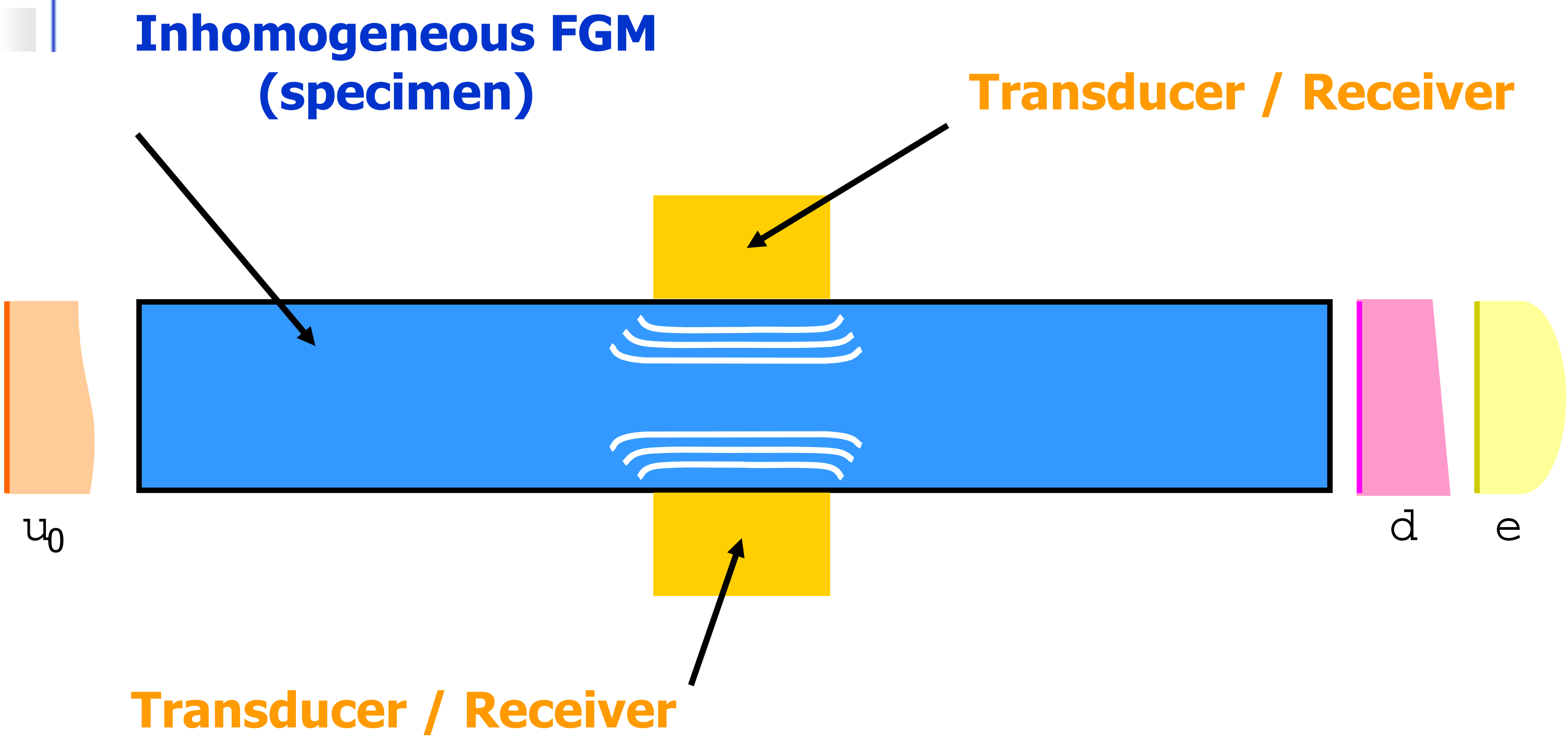
Material properties by 1D

$\rho(X)$

$\alpha(X) = \lambda(X) + 2\,\mu(X)$

$\beta(X) = 2[\nu_1(X) + \nu_2(X) + \nu_3(X)]$

Functionally graded material



Equation of motion

$$U_{,xx} + f_1 U_{,x} + f_2 U_{,x} U_{,xx} + f_3 U_{,x}^2 - c^{-2} U_{,tt} = 0$$

----- linear terms

----- dispersive linear term

----- nonlinear terms

$f_1 = f_1(a, a_{,1})$

$f_2 = f_2(a, \beta)$

$f_3 = f_3(a, a_{,1}, \beta_{,1})$

$c^{-2} = u / a$

Equation of motion

Initial- and boundary conditions

$$U_{,t}(X,0) = U(X,0) = 0$$

$$U_{,x}(0, t) = a_0 \sin(\omega t)\, H(t)$$

$$U_{,x}(h, t) = a_0\, sin(\omega t)\, H(t)$$

a_0 , ω - constants

$H(t)$ - Heaviside function

Exponentially graded material

Exponentially graded material

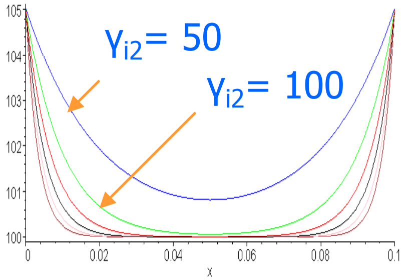
$$\gamma(X) = \gamma_0 \,[1 + \gamma_{11} \exp(-\gamma_{12}X) + \gamma_{21} \exp(\gamma_{22}(X-h))]$$

$$\gamma = \rho,\ \alpha,\ \beta$$

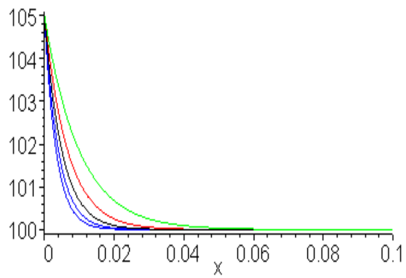
Example

$\gamma_{11} = \gamma_{21} = 0.05$, $\gamma_{12} = \gamma_{22} = 50, 100, 150, 200, 250, 300$, $i = 1, 2$

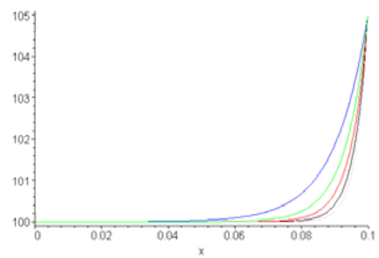
Case A



Case B



Case C



Numerical simulation

Input data

$\rho_0 = 6000$ kg/m^3

$\alpha_0 = 400$ GPa

$\beta_0 = -1000$ GPa

$\gamma_{i1} = 1$

$\gamma_{i2} = 150$ m^{-1}

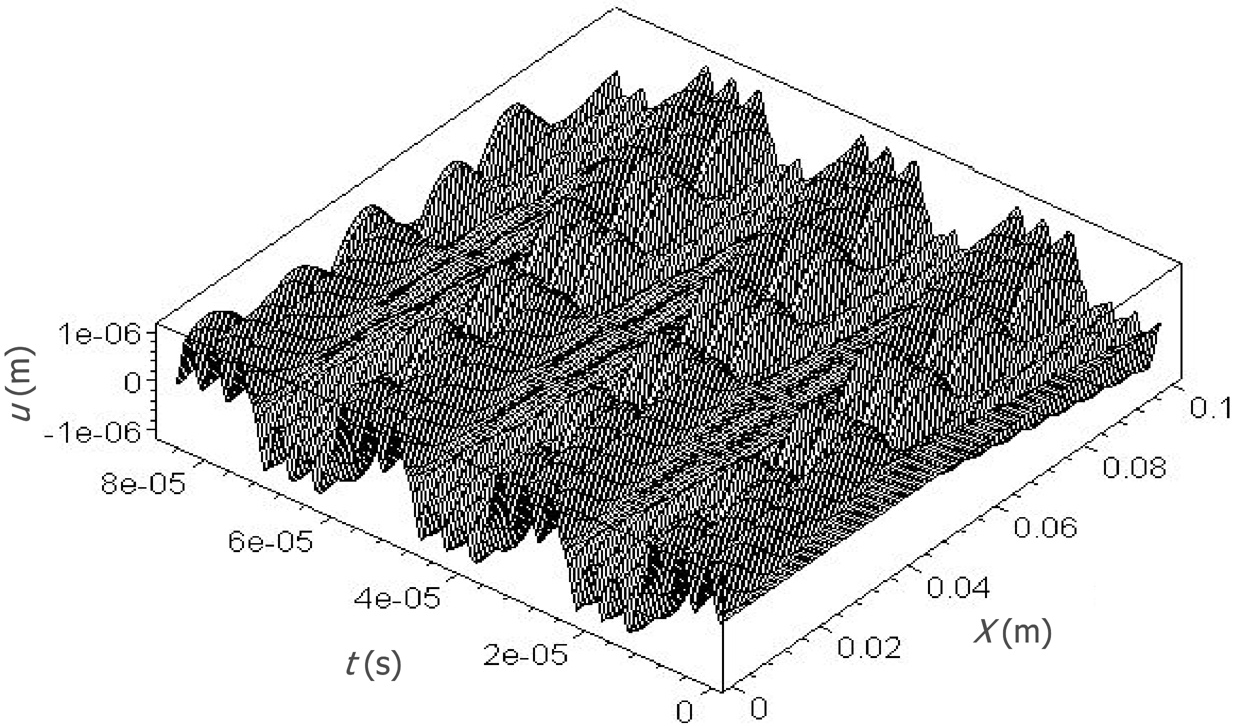
$n = 3$

$\varepsilon = 10^{-4}$

$h = 0.1$ m

Wave interaction

Homogeneous nonlinear elastic material (u [m])

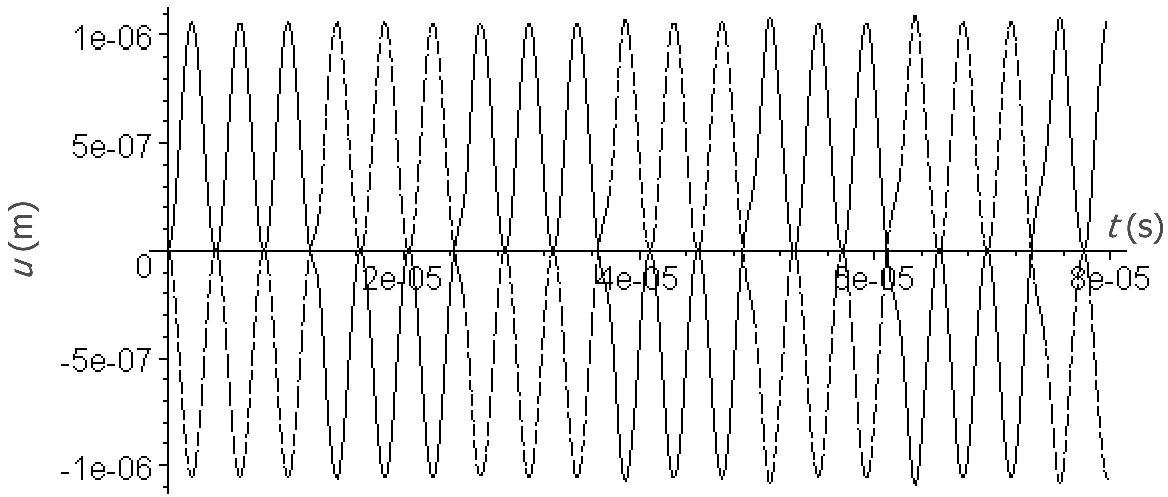


Boundary oscillations

Homogeneous nonlinear elastic material

—— oscillations (u [m]) at X=0

----- oscillations (u [m]) at X=h

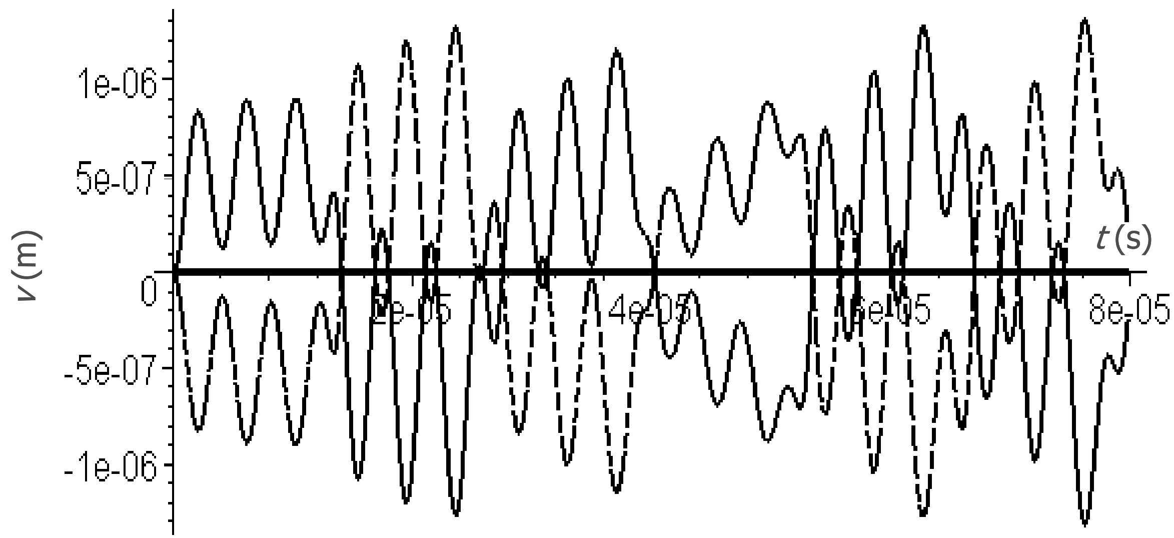


Boundary oscillations

Inhomogeneous nonlinear elastic material

—— oscillations (v [m]) at X=0

----- oscillations (v [m]) at X=h

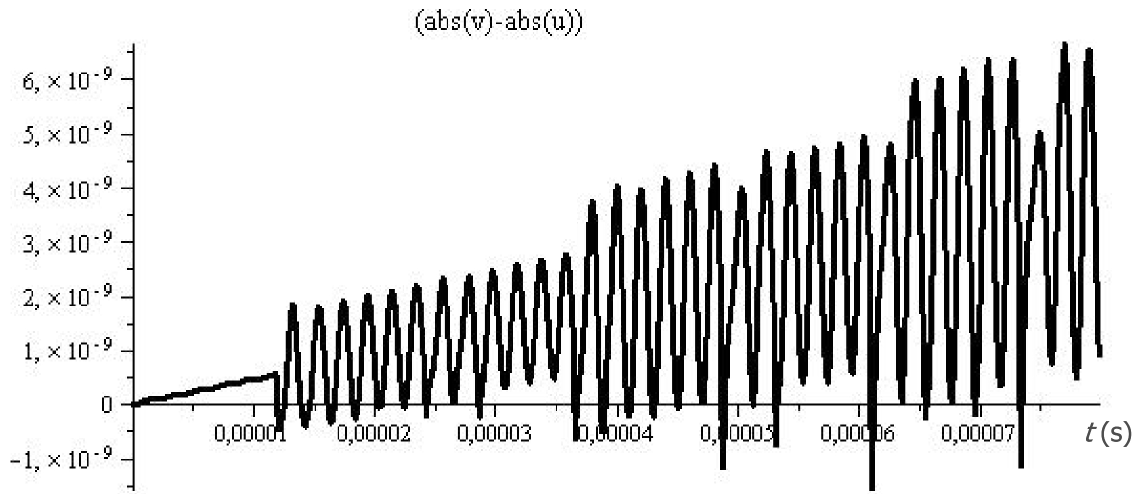


Boundary oscillations

Nonlinear constituent in boundary oscillations of homogeneous nonlinear elastic material at X=0

v - amplitude of nonlinear oscillations (m)

u - amplitude of linear oscillations (m)

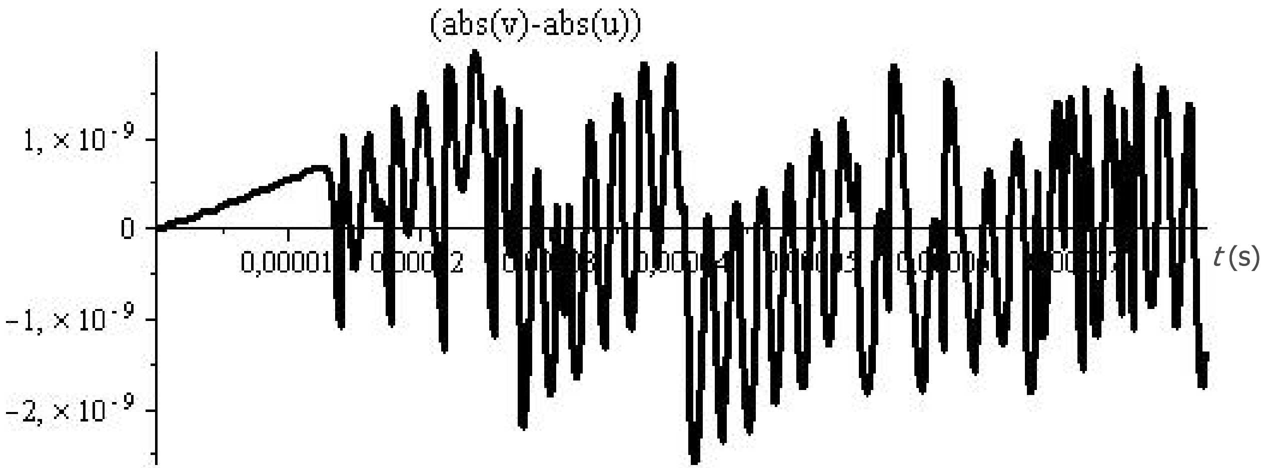


Boundary oscillations

Nonlinear constituent in boundary oscillations of inhomogeneous nonlinear elastic material at X=0

v - amplitude of nonlinear oscillations (m)

u - amplitude of linear oscillations (m)

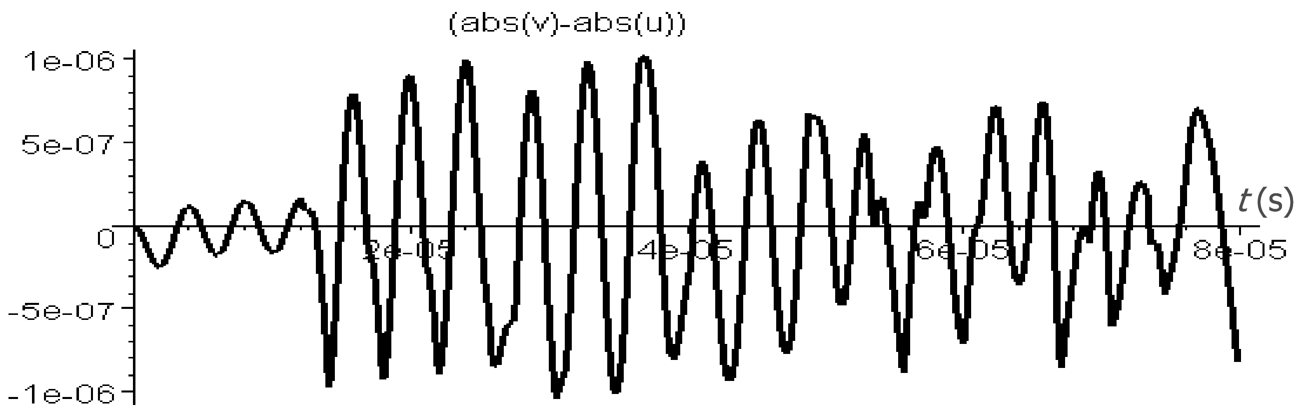


Boundary oscillations

Inhomogeneous constituent in boundary oscillations of nonlinear elastic material at X=0

v - amplitude of nonlinear oscillations in inhomogeneous material (m)

u - amplitude of nonlinear oscillations in homogeneous material (m)



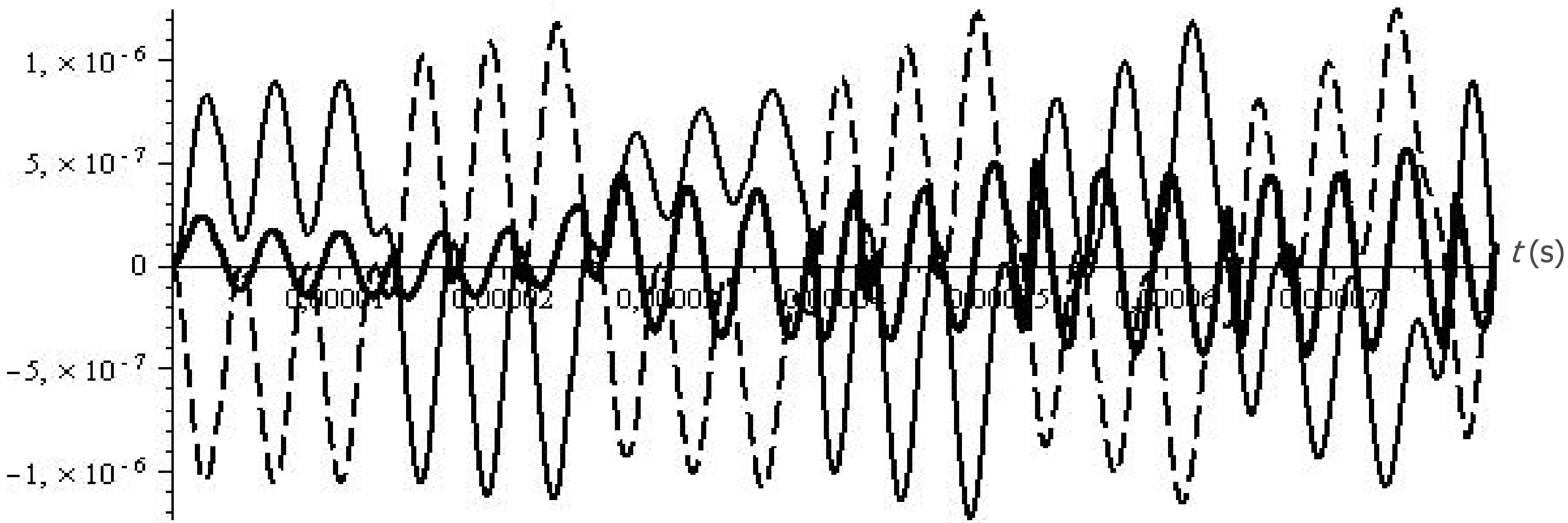
Boundary oscillations

Oscillations in asymmetrically inhomogeneous material

—— oscillations (v [m]) at X=0

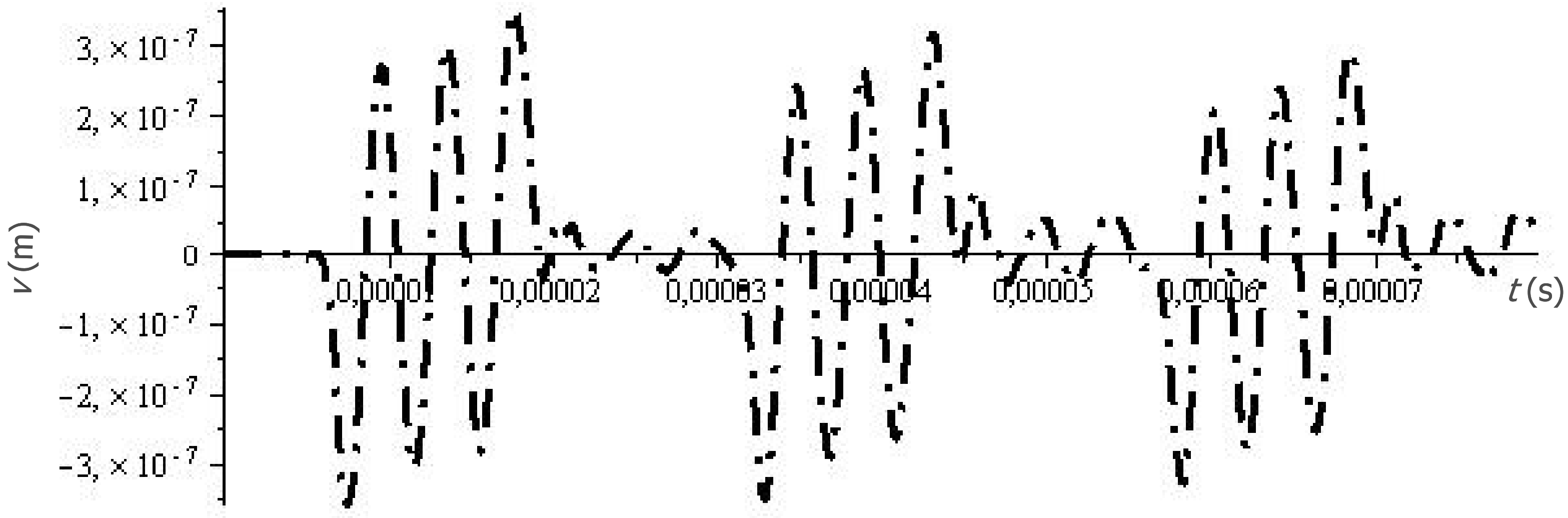
----- oscillations (v [m]) at X=h

▬▬ abs (v(0)) - abs (v(h))



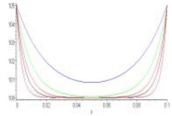
Oscillations on the axis of symmetry

Oscillations (v [m]) in asymmetrically inhomogeneous material

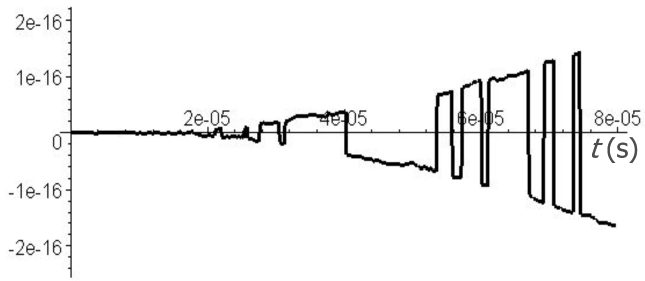


Boundary oscillations

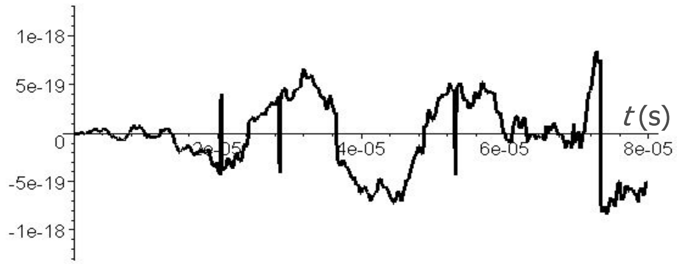
Case A: abs [$v(h)$] - abs [$v(0)$]



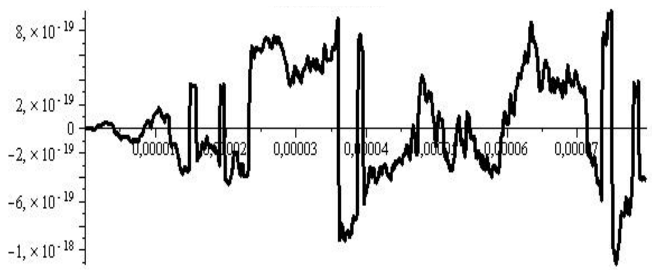
ρ



β



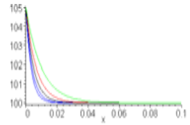
a



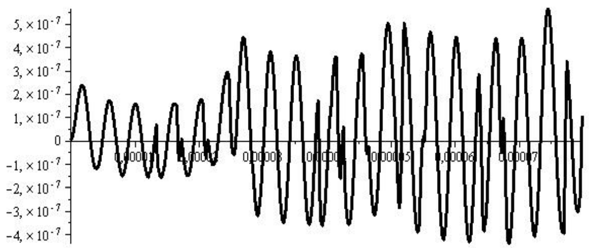
ρ, a, β

Boundary oscillations

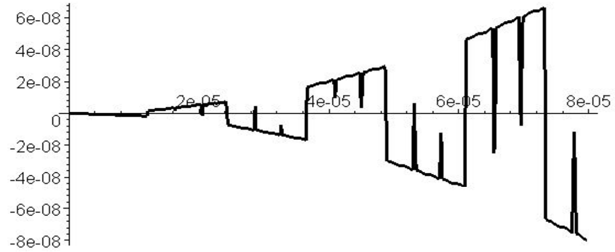
Case B: abs [$v(h)$] - abs [$v(0)$]



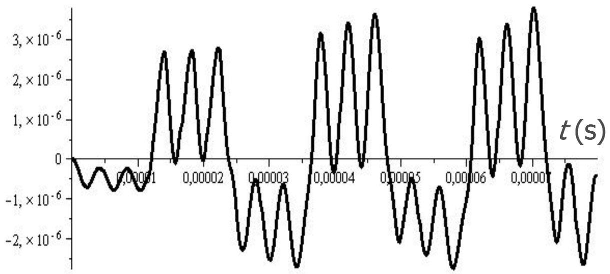
ρ



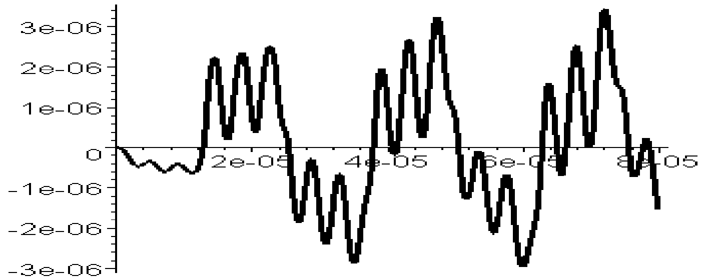
β



a

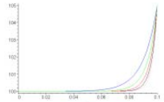


ρ, a, β

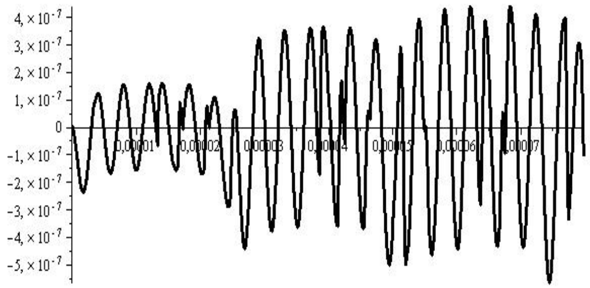


Boundary oscillations

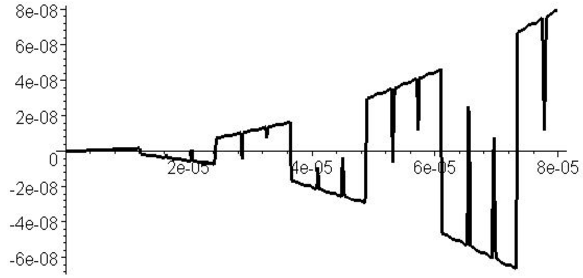
Case C: abs [$v(h)$] - abs [$v(0)$]



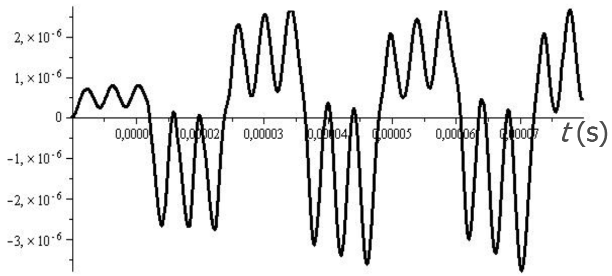
ρ



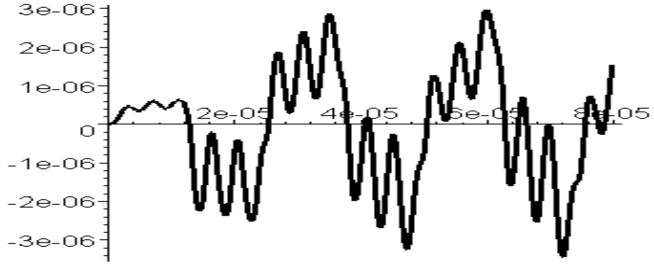
β



a

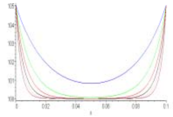


ρ, a, β

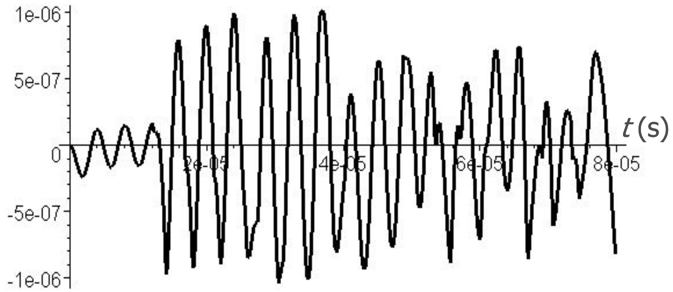


Boundary oscillations

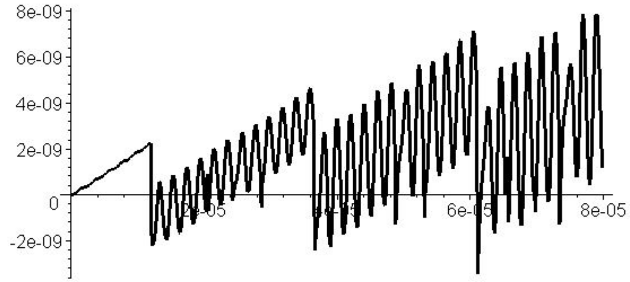
Scheme A: abs(v) - abs(u) at $X=0$ and $X=h$



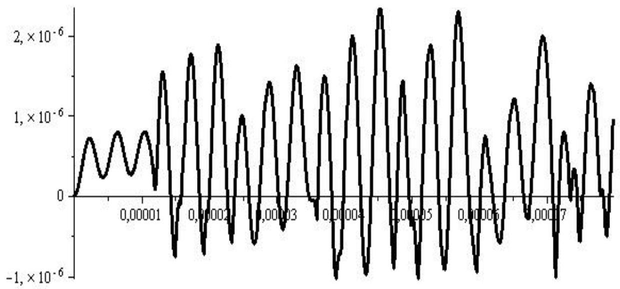
ρ



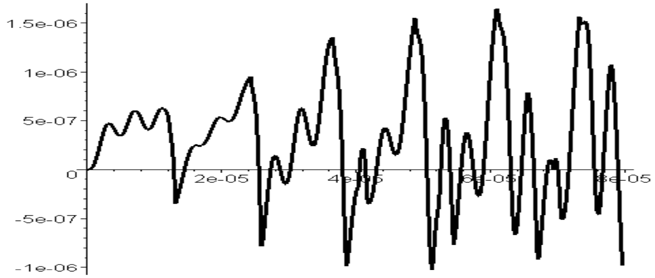
β



a

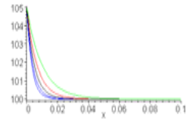


ρ, a, β

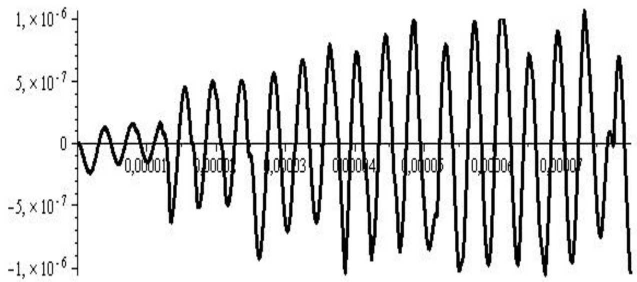


Boundary oscillations

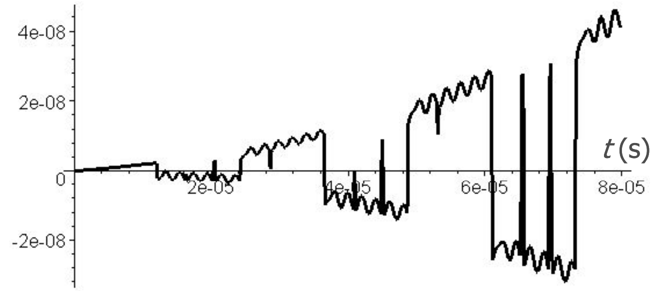
Scheme B: abs(v) - abs(u) at $X = 0$



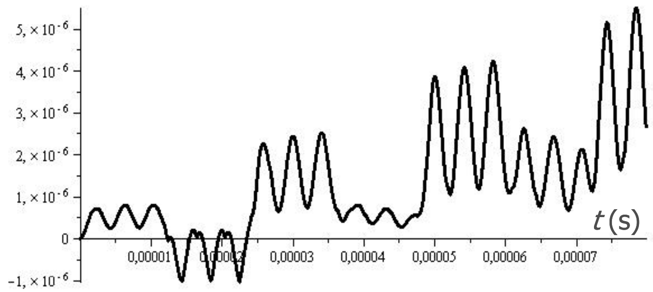
ρ



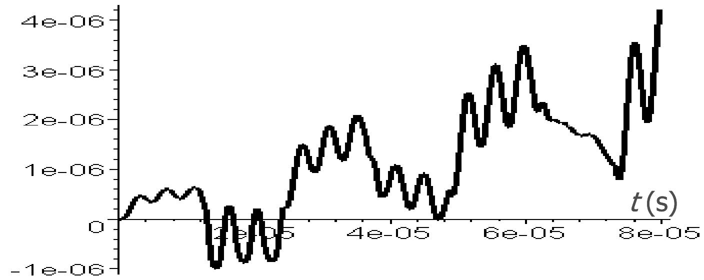
β



a

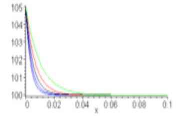


ρ, a, β

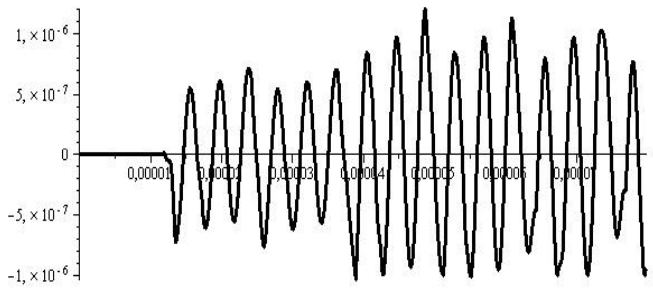


Boundary oscillations

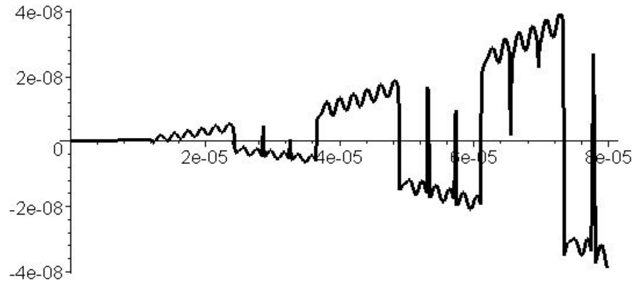
Scheme B: abs(v) - abs(u) at $X = h$



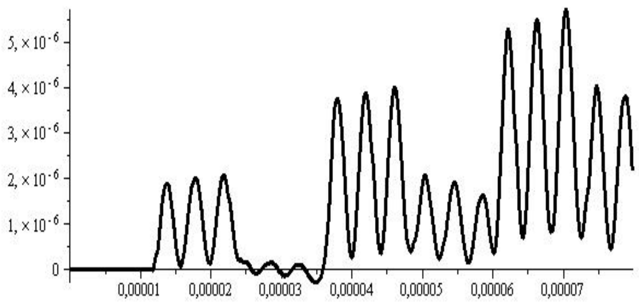
ρ



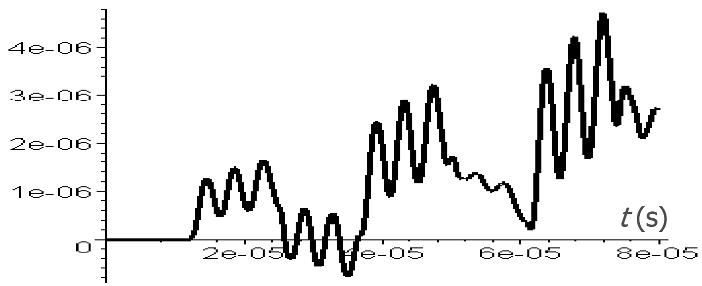
β



a

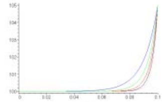


ρ, a, β

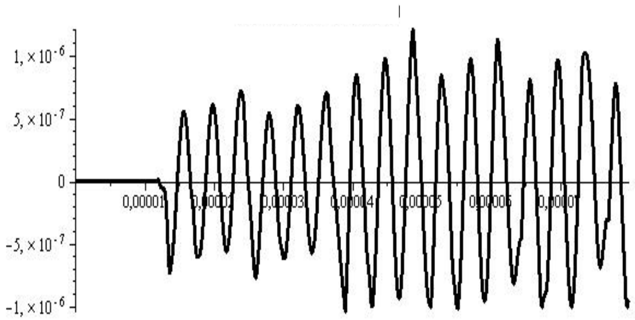


Boundary oscillations

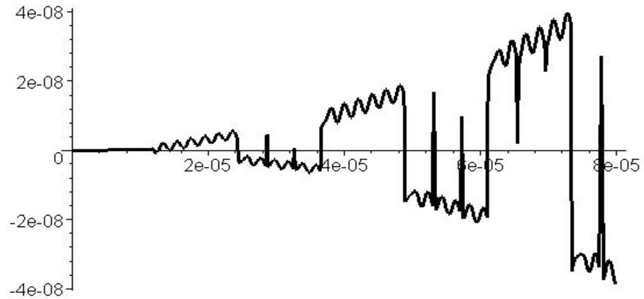
Skeem C: abs(v) - abs(u) at $X = 0$



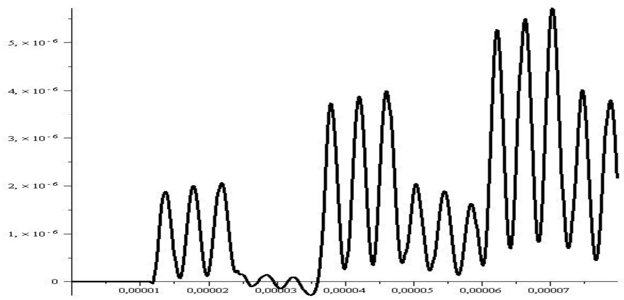
ρ



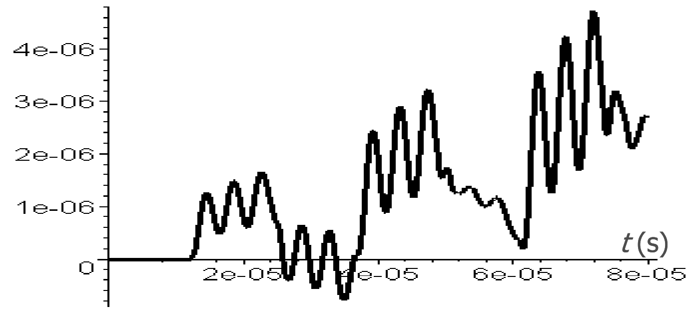
β



a

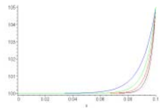


ρ, a, β

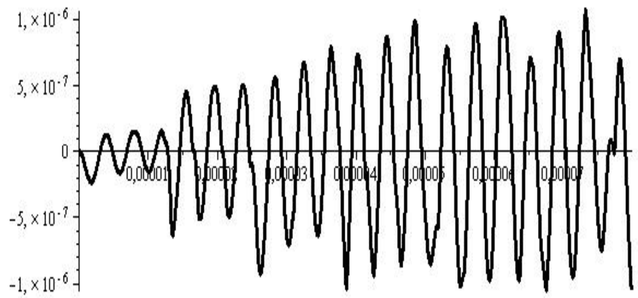


Boundary oscillations

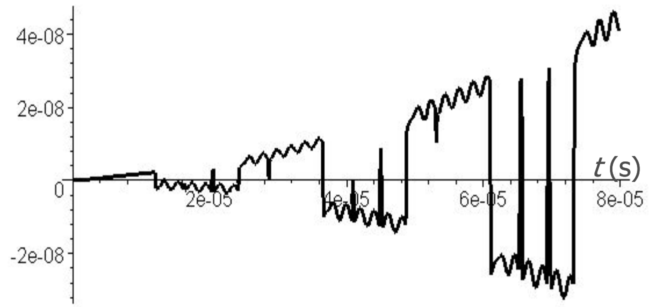
Scheme C: abs(*v*) - abs(*u*) at *X* = *h*



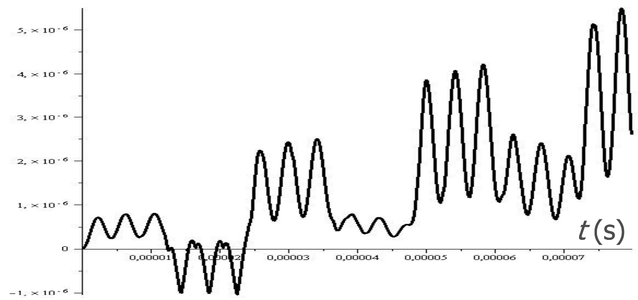
ρ



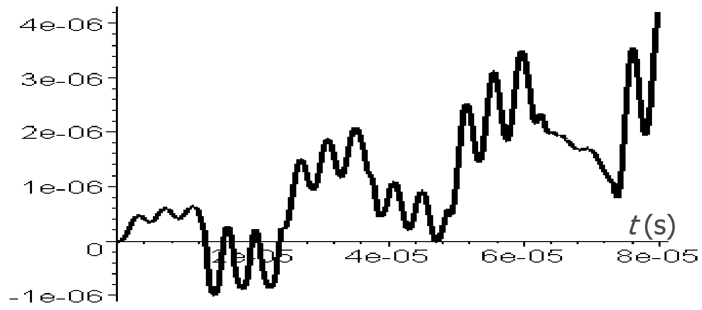
β



a



ρ, a, β



Qualitative characterization of FGM

Boundary oscillations permit to distinguish:

- homogeneous material
- inhomogeneous material
- symmetrically inhomogeneous material
- asymmetrically inhomogeneous material
- property responsible for inhomogeneity

Conclusions

- Wave interaction data are informative about the properties and states of materials
- Proposed NDT techniques are effective provided some preliminary information is available
- Extraction of information from ultrasonic wave interaction data enables to enhance the possibilities of NDT