Nonlinear
counterpropagating waves
in inhomogeneous materials

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Purpose

To use the data about complex phenomena of nonlinear wave-wave, wave-material and wave-prestress interaction for ultrasonic nondestructive characterization of inhomogeneous materials.
Problems to discuss

- Superposition or interaction
  - wave versus wave
  - wave versus prestress
  - wave versus material properties

- Wave interaction for nondestructive testing (NDT)
  - characterization of inhomogeneous prestress in nonlinear elastic material
  - characterization of inhomogeneous physical properties of functionally graded materials

Conclusions
Theory

Basic assumptions

- material is elastic with quadratic nonlinearity
- physical and geometrical nonlinearity is considered
- deformations are small but finite
Equation of motion

\[ \mathcal{T}_{KL}(X_j, t) \left( \delta_{kL} + \delta_{LM} U_{M,L}(X_j, t) \right),_K - \rho_0 \delta_{KM} U_{M,tt}(X_j, t) = 0 \]

\( T_{KL} \) - second Piola-Kirchhoff stress tensor
\( U_k \) - displacement vector
\( X_K \) - Lagrangian rectangular coordinates
\( x_K \) - Eulerian rectangular coordinates
\( t \) - time
\( \rho_0 \) - density of the material
\( \delta_{kl} \) - Euclidean shifter
\( k,K,L,M \) - 1,2,3
Superposition or interaction

\[ U^*_K = U_K + U^0_K \]

- \( U^*_K \): displacement at the present state
- \( U_K \): displacement evoked by exitation # one (wave)
- \( U^0_K \): displacement evoked by exitation # two (prestress)

\[ 2E_{KL} = U_{K,L} + U_{L,K} + U_{M,K} U_{M,L} \]

- \( E_{KL} \): Green-Lagrange strain tensor

\[ T_{KL} = T_{KL}(E_{KK} E_{KL} E_{LK} E_{ML} E_{KM}) \]

\( K, L, M = 1, 2, 3 \)
Superposition or interaction

\[ T_{KL} = (\lambda I_1 + 3 \nu_1 I_1 + \nu_2 I_2) \partial I_1 / \partial E_{KL} \]

\[ + (\mu + \nu_2 I_1) \partial I_2 / \partial E_{KL} + \nu_3 \partial I_3 / \partial E_{KL} \]

\[ I_1 = E_{kk}, I_2 = E_{kk}, I_3 = E_{kk}, I_4 = E_{kk} \]

\( I_k \) - Green-Lagrange strain tensor invariants

\( k, K, L, M = 1, 2, 3 \)

Material properties

\( \rho \) - density

\( \lambda, \mu \) - Lamé constants

\( \nu_1, \nu_2, \nu_3 \) - third order elastic constants
Superposition or interaction

🌟 Wave versus wave (prestress)
- superposition occurs in linear case
- interaction occurs by considering
  - physical nonlinearity
  - geometrical nonlinearity
  - simultaneous impact of physical and geometrical nonlinearity

🌟 Wave versus material properties
- interaction occurs in linear and nonlinear case
Wave interaction for NDT

- Counter-propagation of waves in inhomogeneously prestressed nonlinear elastic material
  - method for qualitative characterization of prestress
  - method for quantitative characterization of two-parametric prestress

- Counterpropagating waves in functionally graded materials
  - method for qualitative characterization of exponentially graded nonlinear elastic material
Prestress diagnostics

Prestressed structural element (specimen)

Transducer / Receiver
Equation of motion

\[
\left[ 1 + f_1 \right] U_{1,11} + f_2 U_{1,1} + f_3 U_{1,1} U_{1,1} - c^{-2} U_{1,tt} = 0
\]

----- linear terms
----- dispersive linear term
----- nonlinear term

\[
f_1 = k_1 U_{1,1}^0 + k_2 U_{2,2}^0
\]
\[
f_2 = k_1 U_{1,11}^0 + k_3 U_{1,22}^0 + (k_2 + k_4) U_{2,21}^0
\]
\[
f_3 = k_1 U^0 \equiv U^0(X_1,X_2)
\]
Prestress state

Equations of equilibrium

\[
(1 + k_1 U^0_{I,I} + k_2 U^0_{J,J}) U^0_{I,II} + (2 k_3 U^0_{I,J} + 2 k_4 U^0_{J,I}) U^0_{I,II} \\
+ (k_7 + k_3 U^0_{I,I} + k_3 U^0_{J,J}) U^0_{I,JJ} + (k_4 U^0_{I,J} + k_3 U^0_{J,I}) U^0_{J,II} \\
+ (k_3 U^0_{I,J} + k_4 U^0_{J,I}) U^0_{J,JJ} + (k_6 + k_5 U^0_{I,I} + k_5 U^0_{J,J}) U^0_{J,JI} \\
+ \rho_0 B_I = 0
\]

\[
k_5 = k [\lambda + \mu + 3 (2 \nu_1 + \nu_2 + \nu_3 /2)]
\]

\[
k_6 = k (\lambda + \mu), \quad k_7 = k \mu
\]

$I = 1, J = 2$ - first equation

$I = 2, J = 1$ - second equation

\[
U^0 \equiv U^0 (X_1, X_2)
\]
Prestress state

Solution

\[ U_1^0 (X_1, X_2) = \sum_{n=1}^{\infty} l^{(n)} U_1^{0(n)} (X_1, X_2) \]

First approximation

\[ U^{0(1)}_{I,II} + k_7 U^{0(1)}_{I,JI} + k_6 U^{0(1)}_{J,II} = 0 \]

\[ T^{0(1)}_{11}(0, X_2) = T^{0(1)}_{11}(h, X_2) = T^{0(1)}_{12}(0, X_2) = T^{0(1)}_{12}(h, X_2) = 0 \]

\[ T^{0(1)}_{22}(X_1, \pm L/2) = \sum_{n=0}^{5} w_n X_1^n, \quad T^{0(1)}_{21}(X_1, \pm L/2) = 0 \]
Prestress state

n\textsuperscript{th} approximation

\[ U^{0(n)}_{I,II} + k_7 U^{0(n)}_{I,JII} + k_6 U^{0(n)}_{J,II} = F(U^{0(n-1)}_{I},U^{0(n-1)}_{J}) \]

\[ T^{0(n)}_{11}(0,X_2) = T^{0(n)}_{11}(h,X_2) = T^{0(n)}_{12}(0,X_2) = T^{0(n)}_{12}(h,X_2) = 0 \]

\[ T^{0(n)}_{22}(X_1, \pm L/2) = T^{0(n)}_{21}(X_1, \pm L/2) = 0, \ n = 2, 3, \ldots \]

Final solution

\[ U_1^0 (X_1, X_2) = \xi P_1^{5.5} (X_1, X_2) + \xi P_2^{7.7} (X_1, X_2) \]

\[ U_2^0 (X_1, X_2) = \xi P_3^{5.5} (X_1, X_2) + \xi P_4^{7.7} (X_1, X_2) \]
Loading scheme

Stress

$$T_{22} = a + b X_1$$
**Equation of motion**

\[
\left[ 1 + f_1 \right] U_{1,11} + f_2 U_{1,1} + f_3 U_{1,1} U_{1,11} - c^{-2} U_{1,tt} = 0
\]

----- linear terms

----- dispersive linear term

----- nonlinear term

\[
f_1 = k_1 U_{1,1}^0 + k_2 U_{2,2}^0
\]
\[
f_2 = k_1 U_{1,11}^0 + k_3 U_{1,22}^0 + (k_2 + k_4) U_{2,21}^0
\]
\[
f_3 = k_1
\]

\[
U^0 \equiv U^0(X_1, X_2)
\]
Equation of motion

Initial and boundary conditions

\[ U_{1,t}(X_1,X_2,0) = U_1(X_1,X_2,0) = 0 \]
\[ U_{1,t}(0,X_2,t) = \varepsilon a_0 \phi(t) H(t) \]
\[ U_{1,t}(h,X_2,t) = \varepsilon a_h \psi(t) H(t) \]

| | \(\varepsilon\) | \(<\) \(<\ 1\)
| \(a_0\), \(a_h\) - constants
| \(\max |\phi(t)| = \max |\psi(t)| = 1\)

\(H(t)\) - Heaviside function
Perturbative solution

Solution

\[ U_1 (X_1, t) = \sum_{n=1}^{\infty} \epsilon^{(n)} U_1^{(n)} (X_1, t) \]

First approximation

\[ U_{1,11}^{(1)} (X_1, 0) - c^{-2} U_{1,tt}^{(1)} (X_1, 0) = 0 \]

\[ U_1^{(1)} (X_1, 0) = U_{1,t}^{(1)} (X_1, 0) = 0 \]

\[ U_{1,t}^{(1)} (0,t) = a_0 \phi(t) H(t) \]

\[ U_{1,t}^{(1)} (h,t) = a_h \psi(t) H(t) \]
Perturbative solution

\[ U^{(n)}_{1,11}(X_1,0) - \alpha^2 U^{(n)}_{1,tt}(X_1,0) = \sum_{j=1}^{m} G^{(n)}_j(X_1) F^{(n)}_j(\zeta^{(n)}_j) \]

\[ \zeta^{(n)}_j = t - g^{(n)}_j(X_1), \quad g^{(n)}_j(X_1) \geq 0 \]

\[ U^{(n)}_1(X_1,0) = U^{(n)}_{1,t}(X_1,0) = 0 \]

\[ U^{(n)}_{1,t}(0,t) = U^{(n)}_{1,t}(h,t) = 0, \quad n = 2, 3, \ldots \]
Harmonic waves

\[ U_{1,t}(X_1, t) = \sum \varepsilon^{(n)} U_{1,t}^{(n)}, \quad U_{1,t}^0(X_1, X_2) = \sum h^{(n)} U_{1,t}^{0(n)} \]

\[ U_{1,t}(X_1, X_2, t) = A_0 + A_1 \sin (\omega \zeta + \theta_1) + A_2 \sin (2 \omega \zeta + \theta_2) + A_3 \sin (3 \omega \zeta + \theta_3) \]

<table>
<thead>
<tr>
<th>( \varepsilon \rightarrow \hbar^2 )</th>
<th>( \varepsilon U_{1,t}^{(1)} )</th>
<th>( A_1^{(1)} )</th>
<th>( \cdot )</th>
<th>( \cdot )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weak wave</td>
<td>( \varepsilon^2 U_{1,t}^{(2)} )</td>
<td>( A_1^{(2)} )</td>
<td>( \theta_1^{(2)} )</td>
<td>( \cdot )</td>
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<td>( \varepsilon^3 U_{1,t}^{(3)} )</td>
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<td>( \cdot )</td>
<td>( A_2^{(2)} )</td>
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<td>( \theta_1^{(3)} )</td>
<td>( \cdot )</td>
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</table>
Numerical simulation

Input data

\[ \rho_0 = 2800 \text{ kg/m}^3 \]
\[ \lambda = 50 \text{ GPa} \]
\[ \mu = 27.6 \text{ GPa} \]
\[ \nu_1 = -136 \text{ GPa} \]
\[ \nu_2 = -197 \text{ GPa} \]
\[ \nu_3 = -38 \text{ GPa} \]
\[ h = 0.1 \text{ m} \]
\[ \varepsilon = 1 \times 10^{-4} \]
\[ \omega = 10^6, ..., 10^7 \text{ rad/s} \]
\[ a = -60, ..., 60 \text{ MPa} \]
\[ b = -1.2, ..., 1.2 \text{ GPa/m} \]
Wave interaction

Sine wave propagation
Wave interaction

Nonlinear effects

![Graphical representation](image)
Boundary oscillations

Homogeneous prestress-free material
Boundary oscillations

Homogeneously prestressed material
Boundary oscillations

Inhomogeneously prestressed material
Wave interaction technique

Qualitative prestress characterization

 Boundary oscillations permit to distinguish:

- prestress-free material
- homogeneously prestressed material
- material undergoing pure bending
- material undergoing pure bending with tension or compression
Wave interaction technique

Quantitative NDE

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Wave interaction technique

Instant $\tau_1$
Wave interaction technique

Instant $\tau_2$

$\varepsilon U_{1,1}^{(2)}$

$a = -60$ MPa
$a = -30$ MPa
$b$ (Pa/m)
$a = 0$ MPa
$a = +30$ MPa
$a = +60$ MPa
Wave interaction technique

Recoded data

$10^2 \xi U_{11}(2)$

$t/\tau$

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Quantitative NDE

Prestress evaluation, \( a = -30 \text{ MPa}, \ b = 1.2 \text{ GPa/m} \)
Elastic functionally graded materials with smoothly and arbitrarily variable nonlinear properties

Material properties by 1D

\[ \rho(X) \]
\[ \alpha(X) = \lambda(X) + 2 \mu(X) \]
\[ \beta(X) = 2[\nu_1(X) + \nu_2(X) + \nu_3(X)] \]
Functionally graded material

Inhomogeneous FGM (specimen)

Transducer / Receiver

Transducer / Receiver

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Equation of motion

\[ U_{xx} + f_1 U_x + f_2 U_x U_{xx} + f_3 U_x^2 - c^{-2} U_{tt} = 0 \]

- ---- linear terms
- ---- dispersive linear term
- ---- nonlinear terms

\[
\begin{align*}
f_1 &= f_1(a, a_1) \\
f_2 &= f_2(a, \beta) \\
f_3 &= f_3(a, a_1, \beta_1) \\
c^{-2} &= u / a
\end{align*}
\]
Equation of motion

Initial- and boundary conditions

\[ U_{,t}(X,0) = U(X,0) = 0 \]
\[ U_{,x}(0, t) = a_0 \sin(\omega t) H(t) \]
\[ U_{,x}(h, t) = a_0 \sin(\omega t) H(t) \]

\[ a_0, \omega \] - constants

\[ H(t) \] - Heaviside function
Exponentially graded material

\[ \gamma(X) = \gamma_0 \left[ 1 + \gamma_{11} \exp(-\gamma_{12}X) + \gamma_{21} \exp(\gamma_{22}(X-h)) \right] \]

\( \gamma = \rho, \alpha, \beta \)

Example

\( \gamma_{11} = \gamma_{21} = 0.05, \ \gamma_{12} = \gamma_{22} = 50, 100, 150, 200, 250, 300, i = 1, 2 \)

Case A                               Case B                           Case C

Case A

Case B

Case C

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Numerical simulation

Input data

$\rho_0 = 6000 \text{ kg/m}^3$

$\alpha_0 = 400 \text{ GPa}$

$\beta_0 = -1000 \text{ GPa}$

$\gamma_{i1} = 1$

$\gamma_{i2} = 150 \text{ m}^{-1}$

$n = 3$

$\varepsilon = 10^{-4}$

$h = 0.1 \text{ m}$
Wave interaction

Homogeneous nonlinear elastic material ($u \ [m]$)
Boundary oscillations

Homogeneous nonlinear elastic material

- oscillations \((u \text{ [m]})\) at \(X=0\)
- oscillations \((u \text{ [m]})\) at \(X=h\)

![Graph showing oscillations over time]
Boundary oscillations

Inhomogeneous nonlinear elastic material

- oscillations ($v \text{ [m]}$) at $X=0$
- oscillations ($v \text{ [m]}$) at $X=h$

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Boundary oscillations

Nonlinear constituent in boundary oscillations of homogeneous nonlinear elastic material at $X=0$

$v$ - amplitude of nonlinear oscillations (m)

$u$ - amplitude of linear oscillations (m)
Boundary oscillations

Nonlinear constituent in boundary oscillations of inhomogeneous nonlinear elastic material at X=0

$\nu$ - amplitude of nonlinear oscillations (m)

$u$ - amplitude of linear oscillations (m)
Boundary oscillations

Inhomogeneous constituent in boundary oscillations of nonlinear elastic material at X=0

v - amplitude of nonlinear oscillations in inhomogeneous material (m)
u - amplitude of nonlinear oscillations in homogeneous material (m)
Boundary oscillations

Oscillations in asymmetrically inhomogeneous material

- oscillations \((v \ [m])\) at \(X=0\)
- oscillations \((v \ [m])\) at \(X=h\)
- \(\text{abs } (v(0)) - \text{abs } (v(h))\)

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Oscillations on the axis of symmetry

Oscillations ($v \text{ [m]}$) in asymmetrically inhomogeneous material
Boundary oscillations

Case A: \( \text{abs } [\nu(h)] - \text{abs } [\nu(0)] \)

\[ \rho, \alpha, \beta \]
Boundary oscillations

Case B: \[ \text{abs } [\nu(h)] - \text{abs } [\nu(0)] \]
Boundary oscillations

Case C: abs [\(\nu(h)\)] - abs [\(\nu(0)\)]
Boundary oscillations

Scheme A: \(\text{abs}(\nu) - \text{abs}(u)\) at \(X=0\) and \(X=h\)

\[ \rho, \alpha, \beta \]
Boundary oscillations

Scheme B: $\text{abs}(\nu) - \text{abs}(u)$ at $X=0$

\begin{align*}
\rho(t) & \\
\beta(t) & \\
\alpha(t) & \\
\rho, \alpha, \beta(t) &
\end{align*}
Boundary oscillations

Scheme B: $\text{abs}(\nu) - \text{abs}(u)$ at $X = h$
Boundary oscillations

Skeem C: \( \text{abs}(\nu) - \text{abs}(u) \) at \( X=0 \)
Boundary oscillations

Scheme C: $|\nu| - |u|$ at $x = h$
Qualitative characterization of FGM

Boundary oscillations permit to distinguish:

- homogeneous material
- inhomogeneous material
- symmetrically inhomogeneous material
- asymmetrically inhomogeneous material
- property responsible for inhomogeneity
Conclusions

Wave interaction data are informative about the properties and states of materials.

Proposed NDT techniques are effective provided some preliminary information is available.

Extraction of information from ultrasonic wave interaction data enables to enhance the possibilities of NDT.