Emergence of Solitary Waves in Mindlin-type Microstructured Solids.

Kert Tamm, Andrus Salupere

CENTRE FOR NONLINEAR STUDIES

Institute of Cybernetics at TUT
TTÜ Küberneetika Instituut

1918
TALLINNA TEHNIKAÜLIKOOL
TALLINN UNIVERSITY OF TECHNOLOGY

Outline

- Model equations [a] [b]
- Statement of the problem and numerical method
- Results
  1. Case 1 - difference between dispersion curves is less than 5%
  2. Case 2 - difference between dispersion curves is 10% or more
- Summary and discussion

[a] J.Engelbrecht, F. Pastrone, Waves in microstructured solids with nonlinearities in microscale, Proc. Estonian Acad. Sci. Phys. Math., 2003, 52, 1, 12 – 20.

[b] J. Janno, J. Engelbrecht, An inverse solitary wave problem related to microstructured materials. Inverse Problems, 2005, 21, 2019 – 2034.

Euler – Lagrange equations

$$\left(\frac{\partial L}{\partial u_t}\right)_t + \left(\frac{\partial L}{\partial u_x}\right)_x - \frac{\partial L}{\partial u} = 0,$$

$$\left(\frac{\partial L}{\partial \varphi_t}\right)_t + \left(\frac{\partial L}{\partial \varphi_x}\right)_x - \frac{\partial L}{\partial \varphi} = 0.$$

Lagrangian, kinetic and potential energy

$L = K - W, \quad K = 0.5\rho u_t^2 + 0.5 I \varphi_t^2$ and $W = W(u_x, \varphi, \varphi_x)$.

ρ – macrodensity, I – microinertia
u – macrodisplacement, φ – microdeformation.

Taking corresponding partial derivatives and inserting them into Euler – Lagrange equations; defining macrostress, microstress and interaction force from potential enery W

$$\sigma = \frac{\partial W}{\partial u_x}, \quad \eta = \frac{\partial W}{\partial \varphi_x}, \quad \tau = \frac{\partial W}{\partial \varphi},$$

one arrives at equations of motion

$$\rho u_{tt} = \sigma_x,$$

$$I\varphi_{tt} = \eta_x - \tau.$$

In nonlinear case the simplest potential is

$$W = \underbrace{\frac{A}{2}u_x^2 + \frac{B}{2}\varphi^2 + \frac{C}{2}\varphi_x^2 + D\varphi u_x}_{\text{linear part}} + \underbrace{\frac{N}{6}u_x^3 + \frac{M}{6}\varphi_x^3}_{\text{nonlinear part}}.$$

Equations of motion will take form

$$\rho u_{tt} = Au_{xx} + Nu_x u_{xx} + D\varphi_x,$$

$$I\varphi_{tt} = C\varphi_{xx} + M\varphi_x\varphi_{xx} - Du_x - B\varphi.$$

Model equations

For dimensionless form following change of variables is defined

$$X = \frac{x}{L_o}, \quad T = \frac{tc_0}{L_o}, \quad U = \frac{u}{A_0}, \delta = \frac{l^2}{L_o^2}, \quad \varepsilon = \frac{A_0}{L_o},$$

A_0 – amplitude of initial exitation, L_o – wavelenght of initial exitation, l – characteristic scale of microstructure.
Application of the slaving principle results in a single hierarchical equation for displacement U

$$\underbrace{U_{TT} - bU_{XX} + 0.5\mu(U_X^2)_X}_{\text{marcoscale}} = \underbrace{\delta(\beta U_{TT} - \gamma U_{XX} - 0.5\sqrt{\delta}\lambda U_{XX}^2)_{XX}}_{\text{microscale}},$$

$$b = 1 - \frac{D^2}{AB}; \mu = \frac{NA_0}{AL_o}; \beta = \frac{ID^2}{\rho l^2 B^2}; \gamma = \frac{CD^2}{AB^2 l^2}; \lambda = \frac{D^3 M A_0}{AB^3 l^3 L_o}.$$

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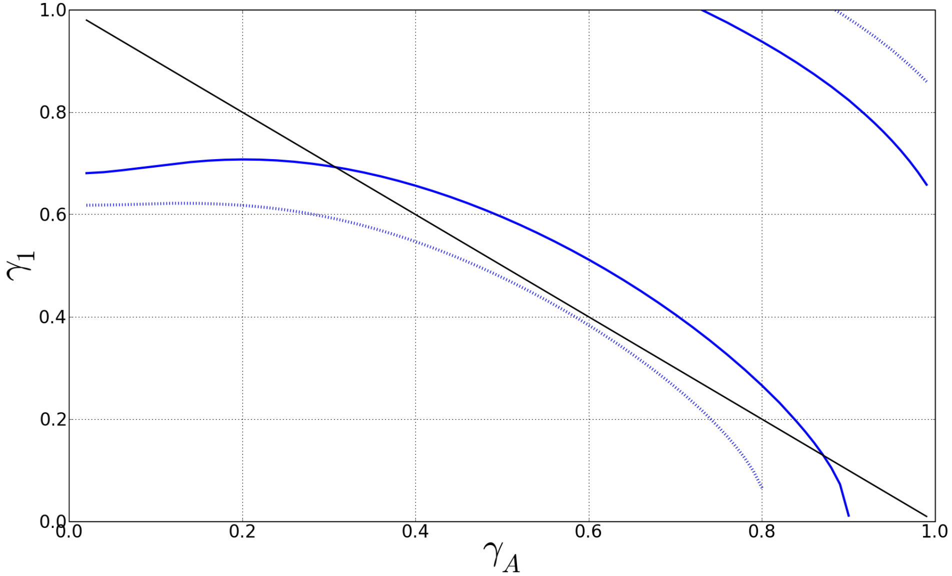
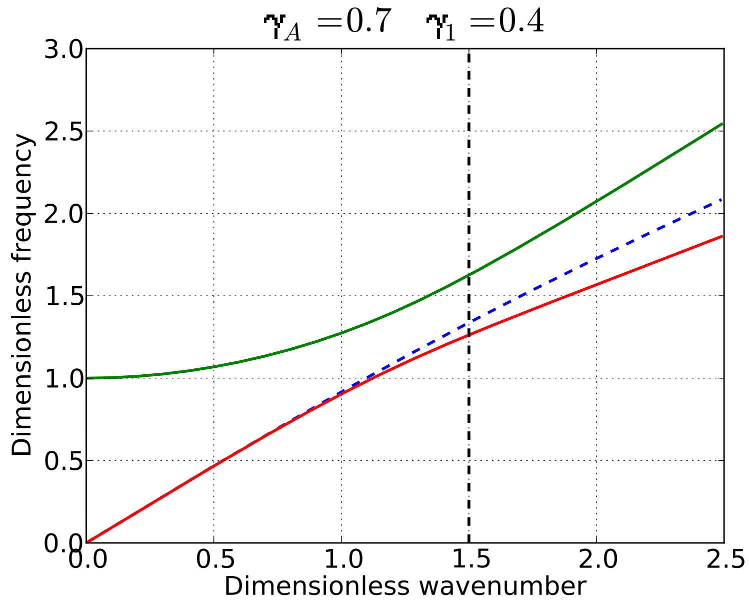
Statement of the problem





$$\kappa = 1.5, \quad \kappa = pc_o k, \quad \eta = p\omega, \quad c_o^2 = \frac{A}{\rho}, \quad p^2 = \frac{I}{B}$$

T.Peets,M. Randrüüt,J. Engelbrecht; On modelling dispersion in microstructured solids. Wave Motion (2008) vol. 45 (4) pp. 471-480

Statement of the problem

Full system of equations (FSE) in dimensionless form

$$U_{TT} = \frac{D}{A\varepsilon}\varphi_X + \frac{N\varepsilon}{A}U_X U_{XX} + U_{XX}$$
$$\varphi_{TT} = \frac{C\rho}{AI}\varphi_{XX} - \frac{B\rho L_o^2}{AI}\varphi - \frac{D\rho A_o L_o}{AI}U_X + \frac{M\rho}{AIL_o}\varphi_X\varphi_{XX}.$$

Hierarchical approximation (HE) in dimensionless form

$$U_{TT} - bU_{XX} + 0.5\mu(U_X^2)_X = \delta(\beta U_{TT} - \gamma U_{XX} - 0.5\sqrt{\delta}\lambda U_{XX}^2)_{XX},$$

ρ – macrodensity, I – microinertia, U – macrodisplacement, φ – microdeformation, c_0 - characteristic speed, ε and δ - geometrical parameters, l - characteristic scale of microstructure ($l = 1$)

Main goals

- To solve the hierarchical model equation (HE) (approximation) and the full system of equations (FSE) under localized initial conditions (linear and nonlinear cases).
- To compare solutions of HE and FSE for case 1 (linear dispersion curves apart less than 5%) and case 2 (10% or more).
- To put special emphasis on effects caused by nonlinearity.

$$W = \underbrace{\frac{A}{2}u_x^2 + \frac{B}{2}\varphi^2 + \frac{C}{2}\varphi_x^2 + D\varphi u_x}_{\text{linear}} + \underbrace{\frac{N}{6}u_x^3 + \frac{M}{6}\varphi_x^3}_{\text{nonlinear}}.$$

DFT based pseudospectral method

- Space – DFT
- Time – Standard ODE (lsoda)
- Boundary conditions (periodic):
  $U(X + 2n\pi, T) = U(X, T); \quad n = 0, \pm 1, \pm 2, \ldots$
- Initial conditions
  - Localized:
    $U(X, 0) = U_0 \cdot \mathrm{sech}^2(\kappa \cdot X/2); \quad 0 \leq X \leq 2k\pi$
  - $U_T = -c \cdot U_X$ assuming $U(X, T) = U(\xi); \quad \xi = X - cT$
  - Full system $\varphi(X, 0) = 0; \quad \varphi(X, 0)_T = 0$
  - Phase speed initial estimate: $c = 0$ (peak of interaction)

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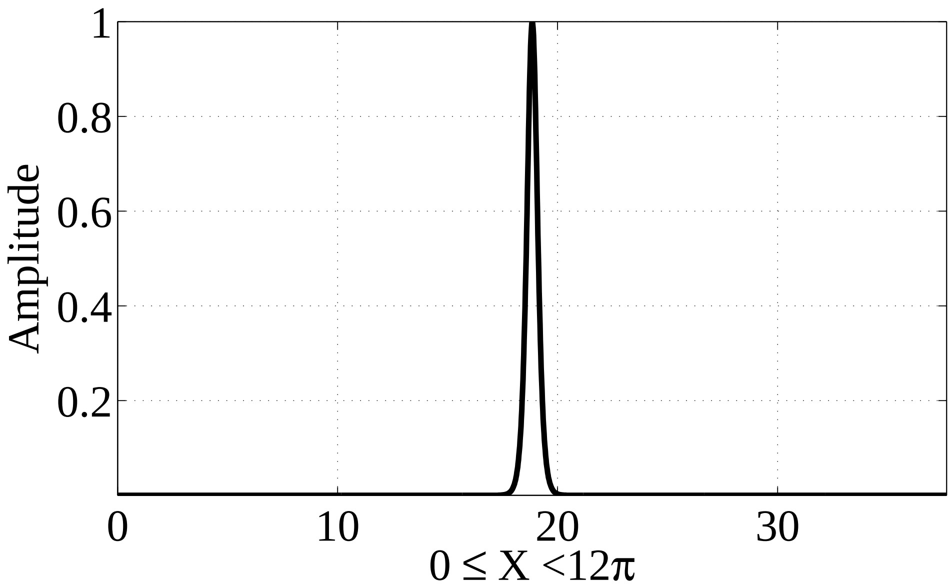
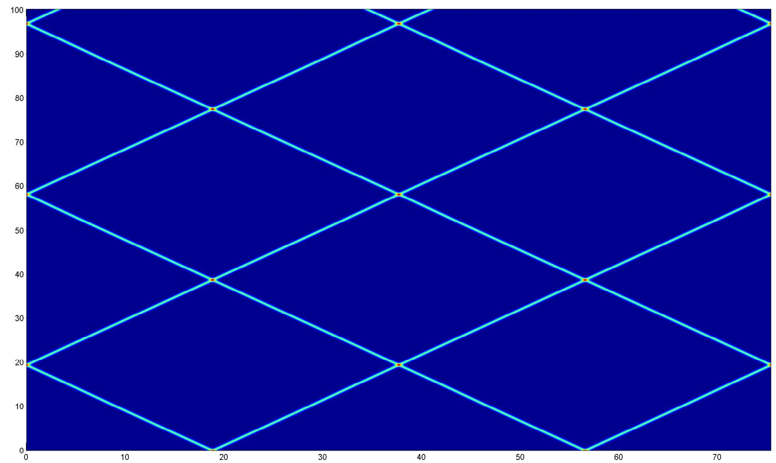
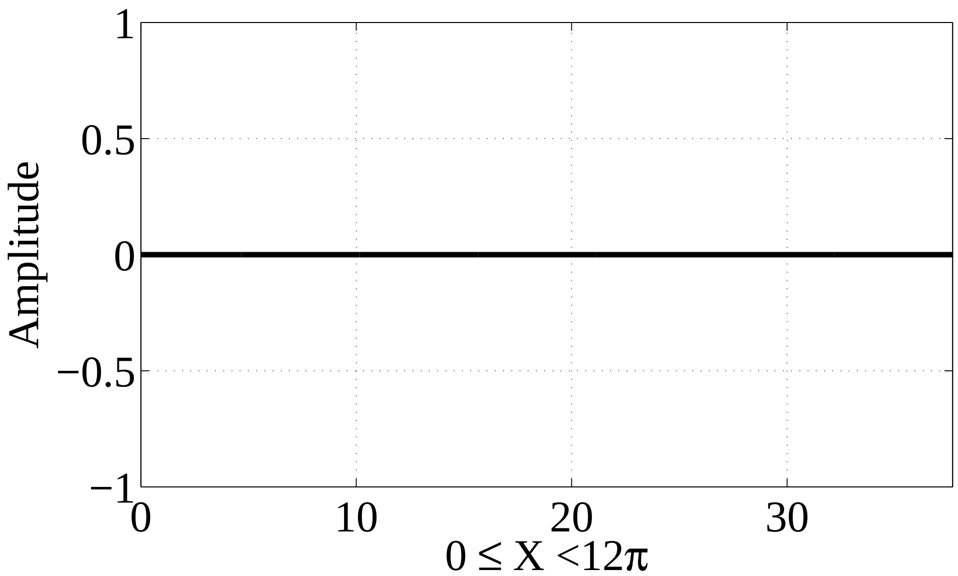
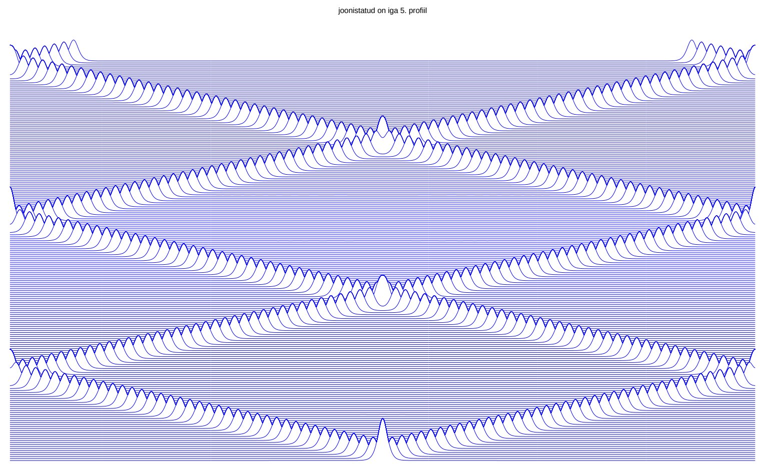

Amplitude
1
0.8
0.6
0.4
0.2
0
10
20
30
$0 \leq X < 12\pi$




Amplitude
1
0.5
0
−0.5
−1
0
10
20
30
$0 \leq X < 12\pi$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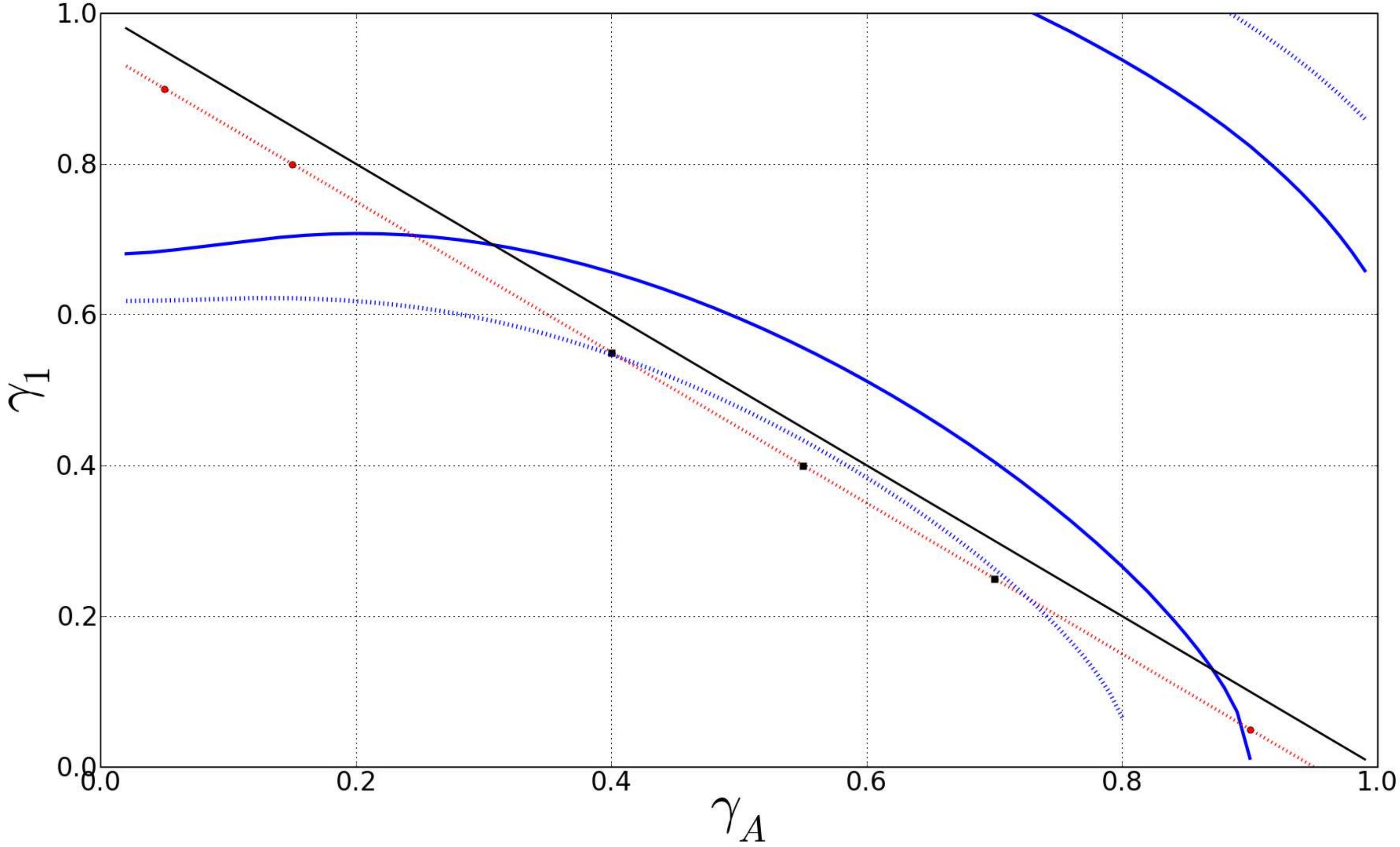
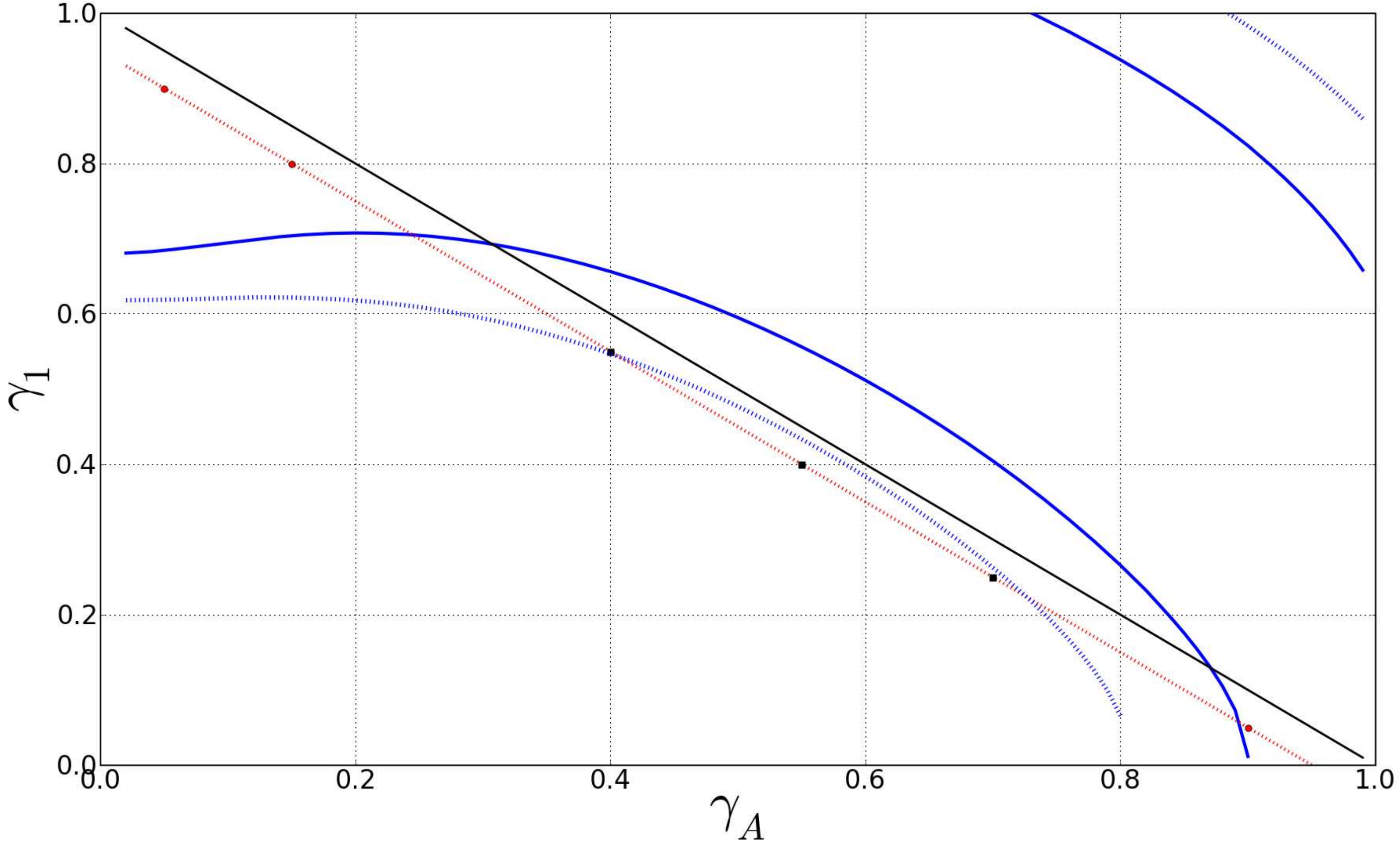
$\Gamma = 1 - \gamma_1 - \gamma_A$, where $\gamma_A = \frac{D^2}{AB}$, $\gamma_1 = \frac{\rho C}{AI}$



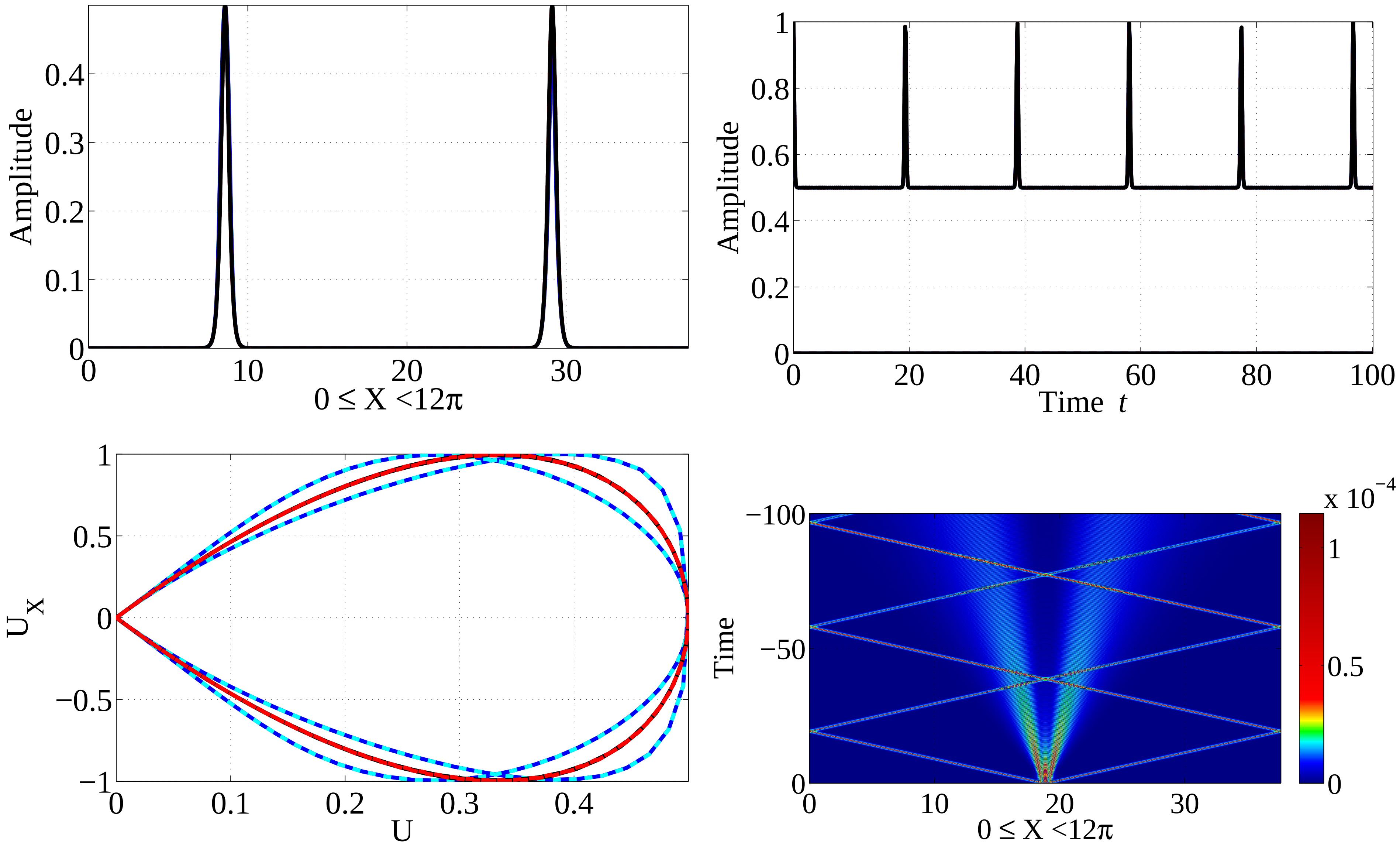
- Snaspshots (waveprofiles, phasespace) are taken at $T = 88$
- HE is represented by dashed lines (Red for linear, Cyan for nonlinear)
- FSE is represented by solid lines (Black for linear, Blue for nonlinear)
- Linear case $N = 0$ (macroscale nonlinearity) and $M = 0$ (microscale nonlinearity)
- Nonlinear case $N = 1$ and M varies so that $\gamma_N = 0.5$
- FSE vs HE plot $|FSE(X,T) - HE(X,T)|$

$$\gamma_N = \frac{\lambda}{\mu} = \frac{D^3 M}{l_o^3 B^3 N}$$

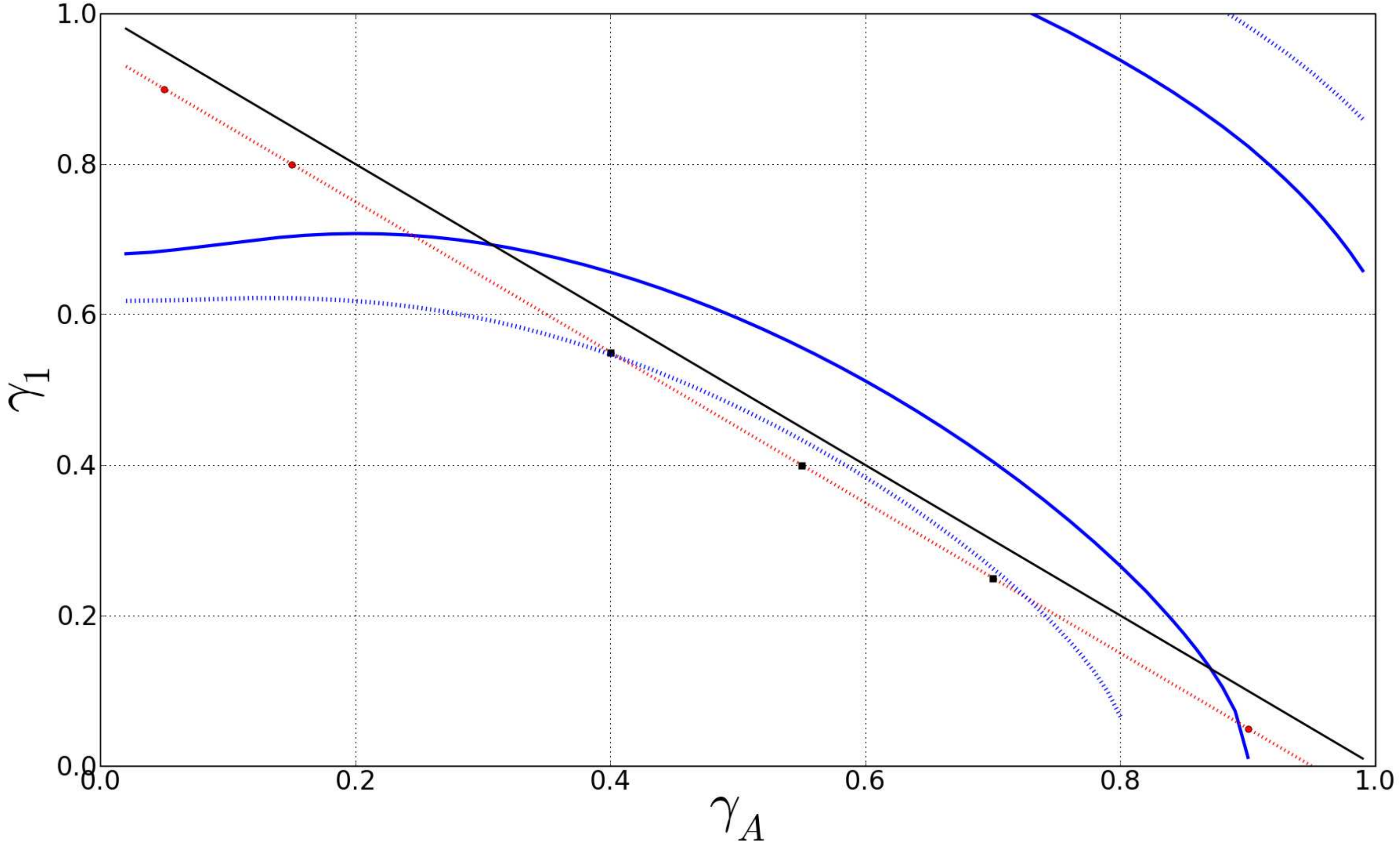
Case 1 - point 1



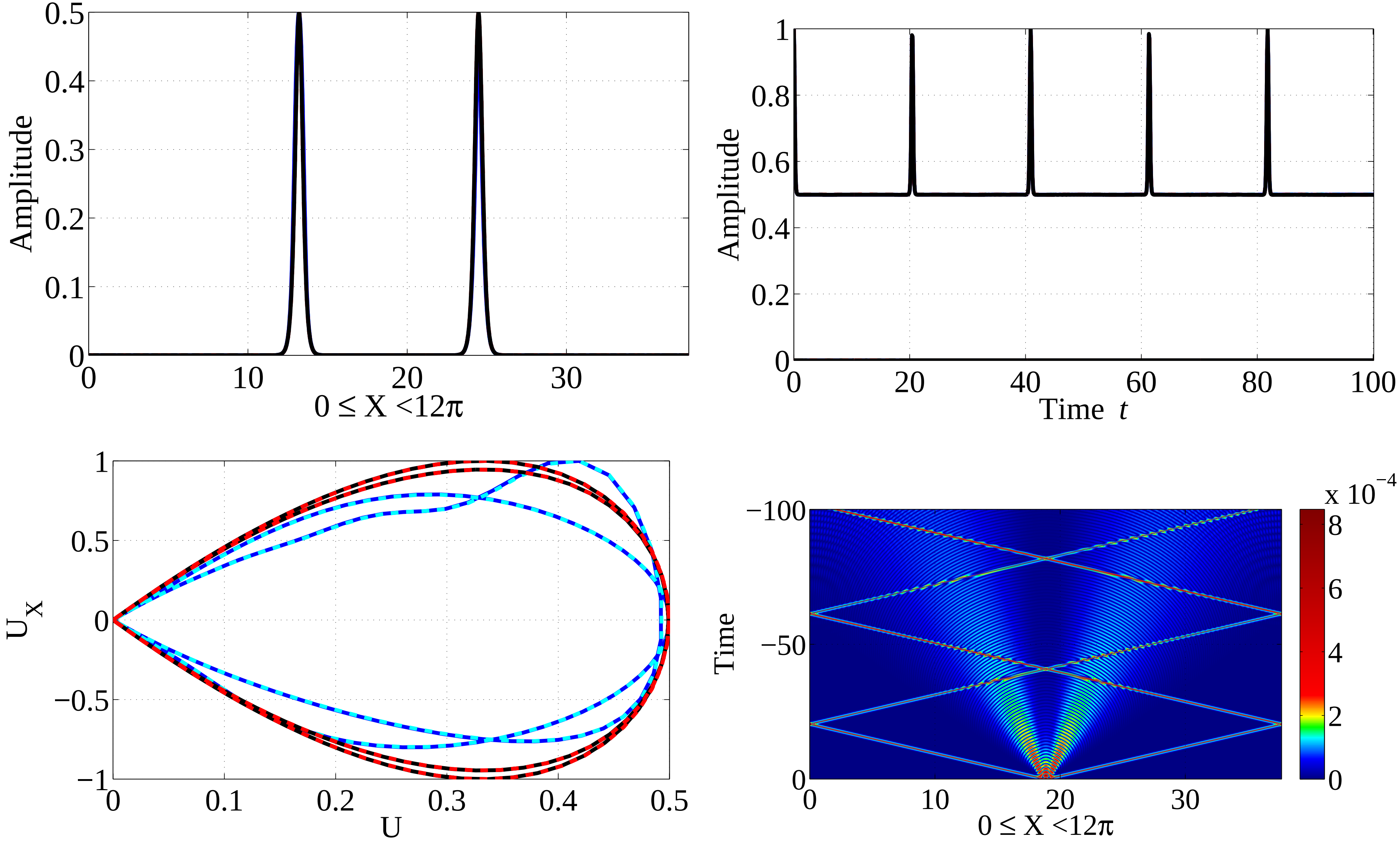
[0.0115%] Results - Case 1 $\gamma_A = 0.05$, $\gamma_1 = 0.90$



Case 1 - point 2



[0.085%] Results - Case 1 $\gamma_A = 0.15$, $\gamma_1 = 0.80$



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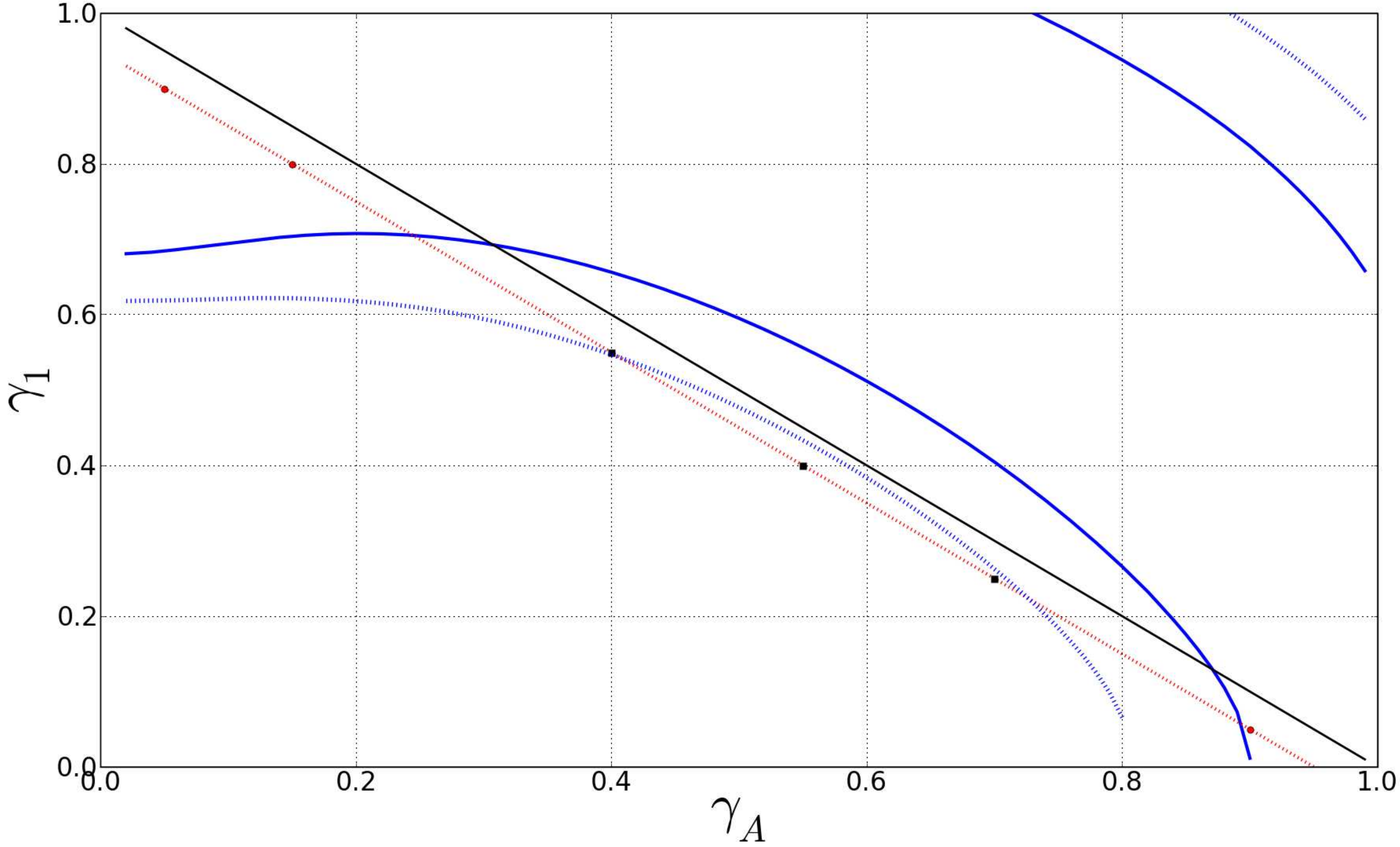
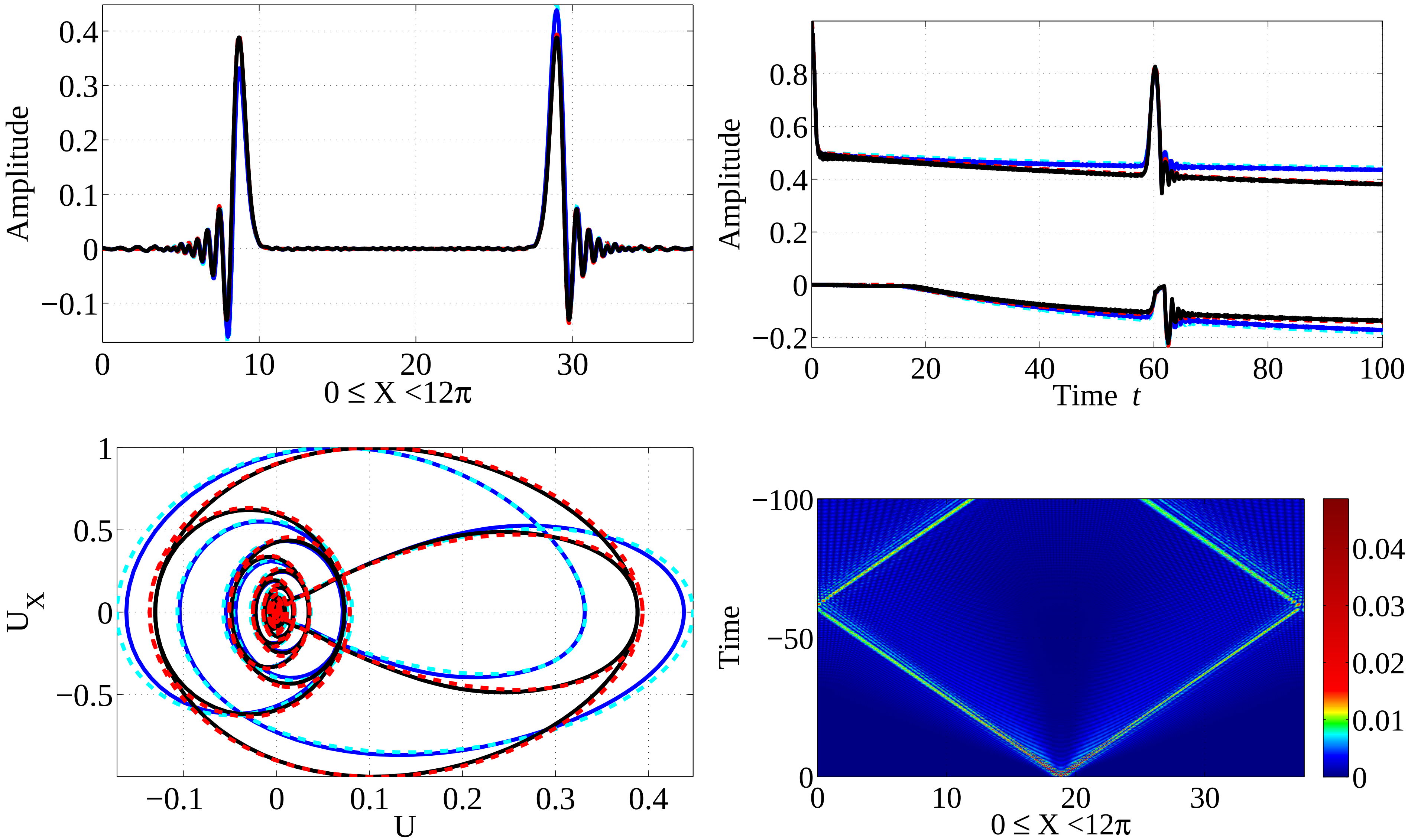
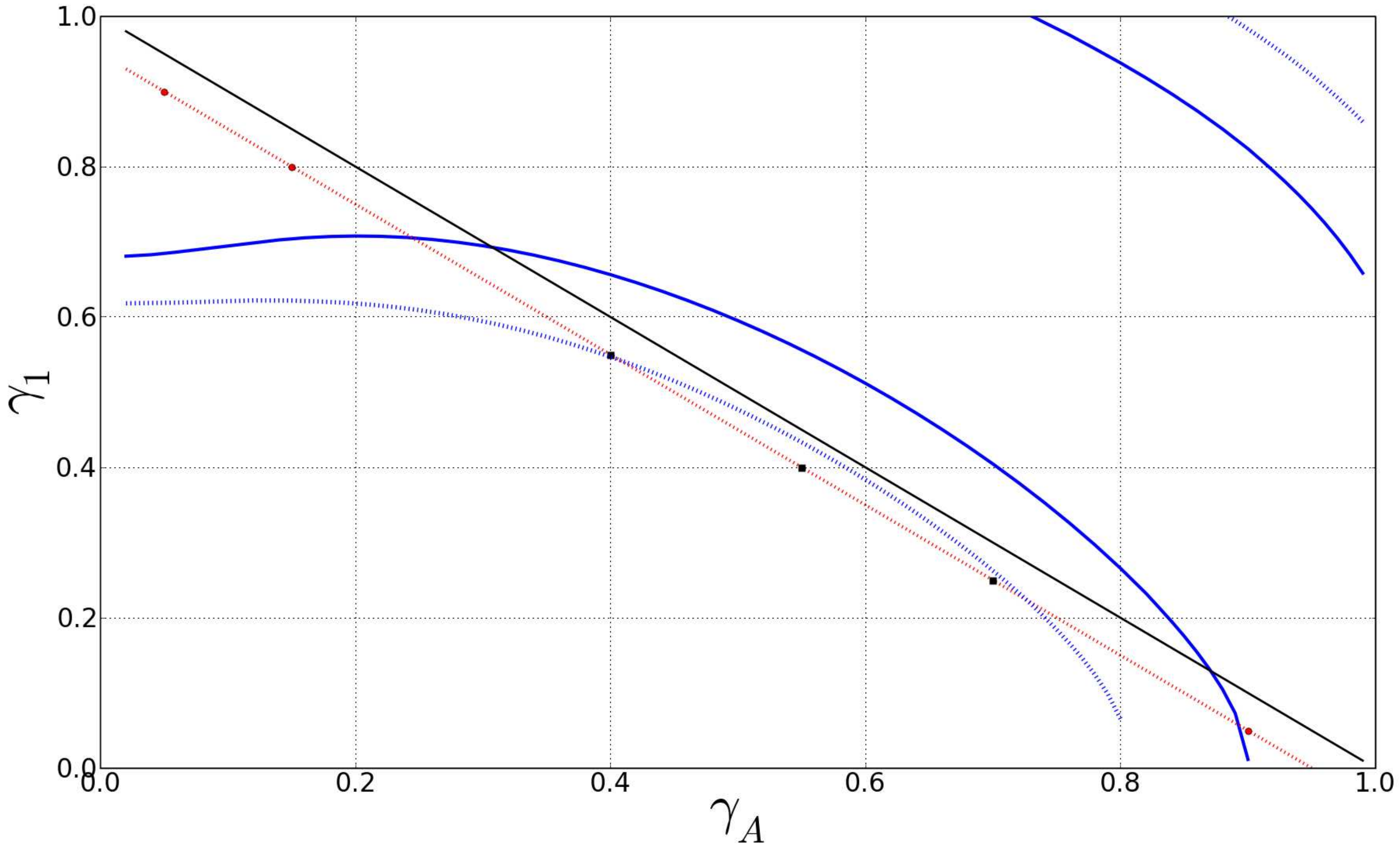
Case 1 - point 3



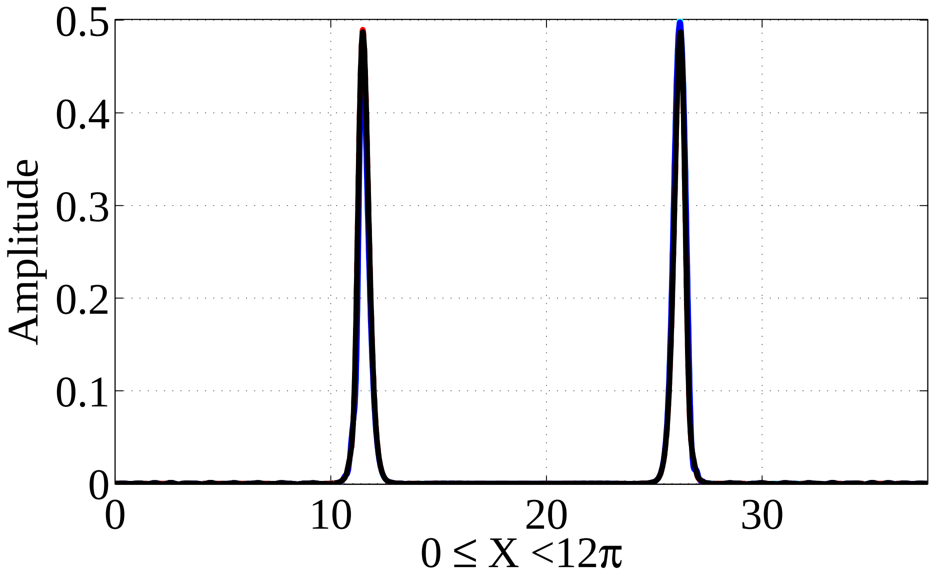
[4.86%] Results - Case 1 $\gamma_A = 0.90$, $\gamma_1 = 0.05$

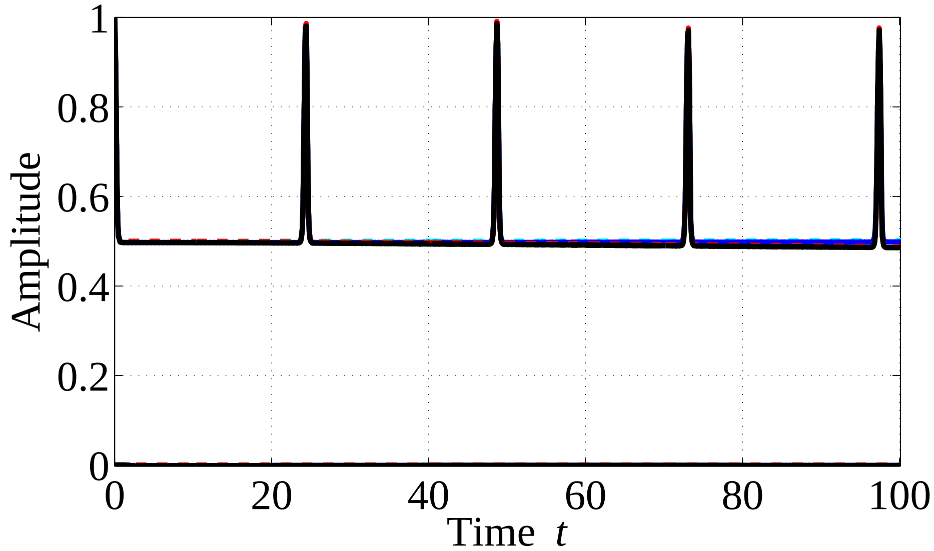


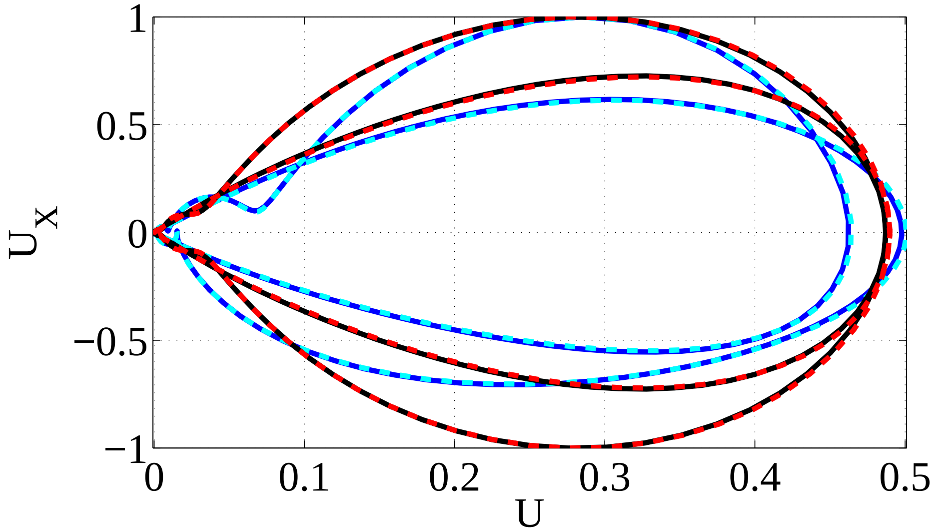
Case 2 - point 1

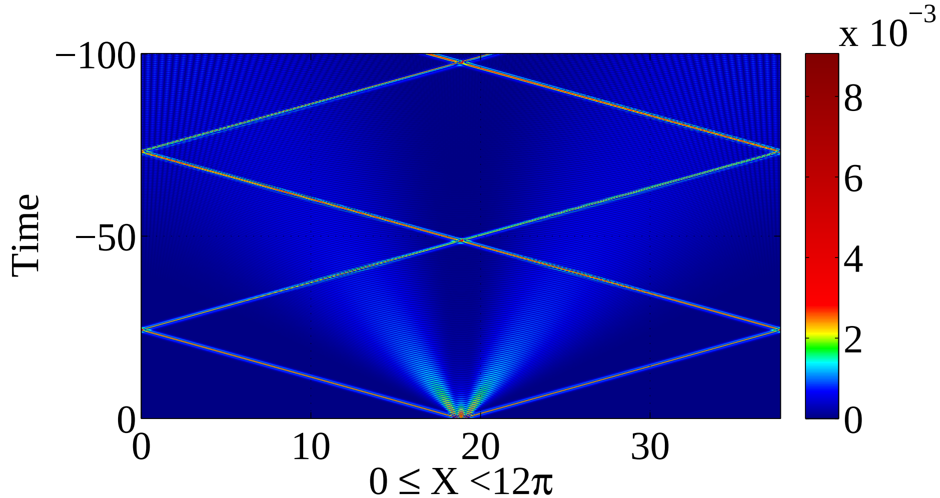


$[0.91\%]$ Results - Case 2 $\gamma_A = 0.40$, $\gamma_1 = 0.55$

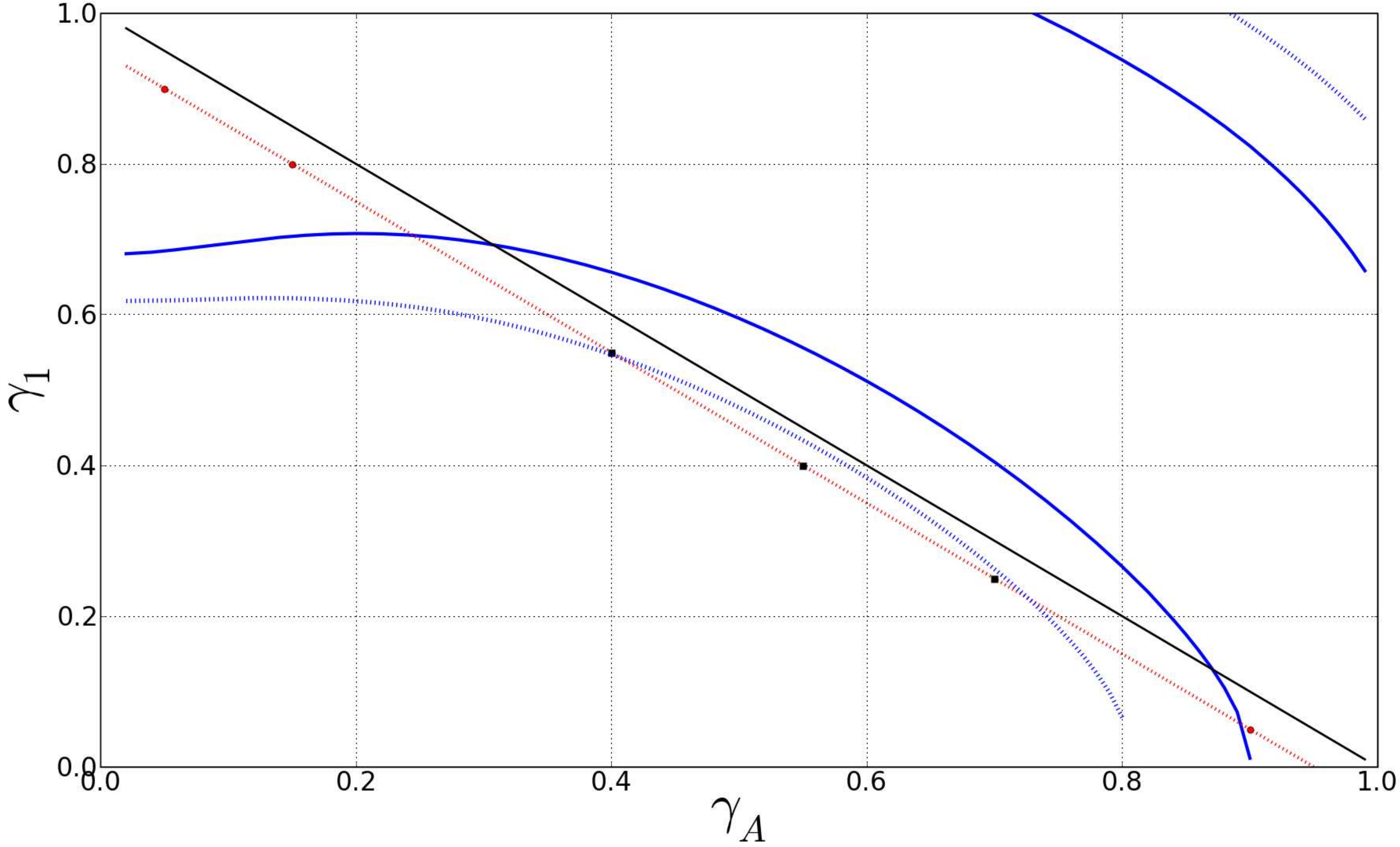




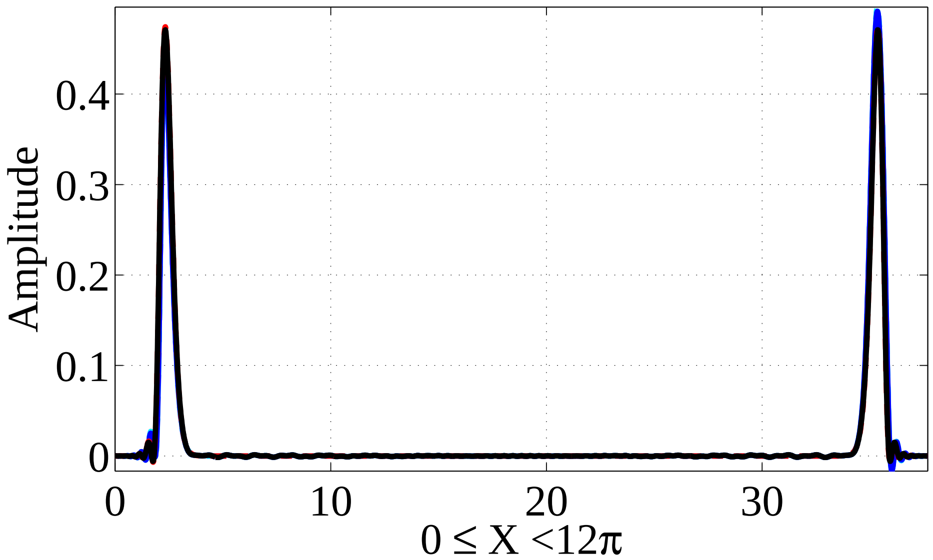


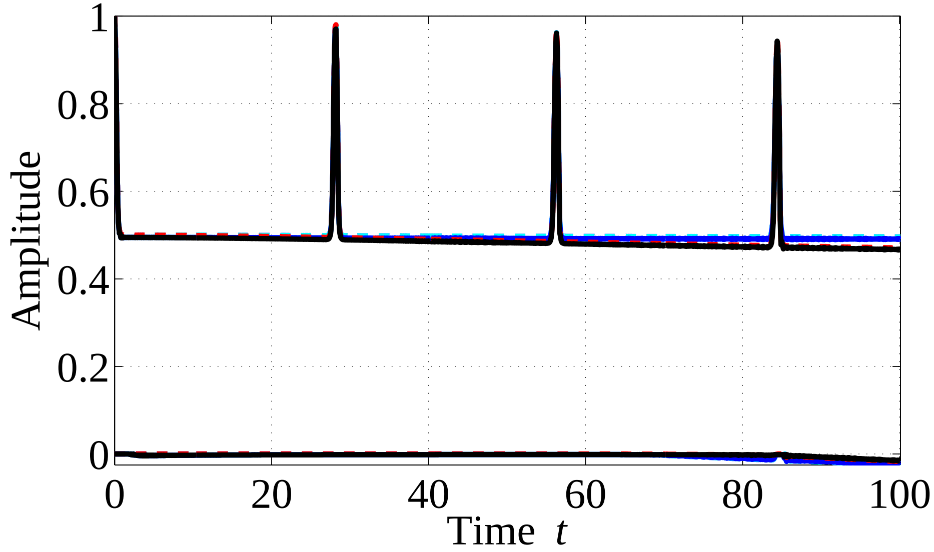


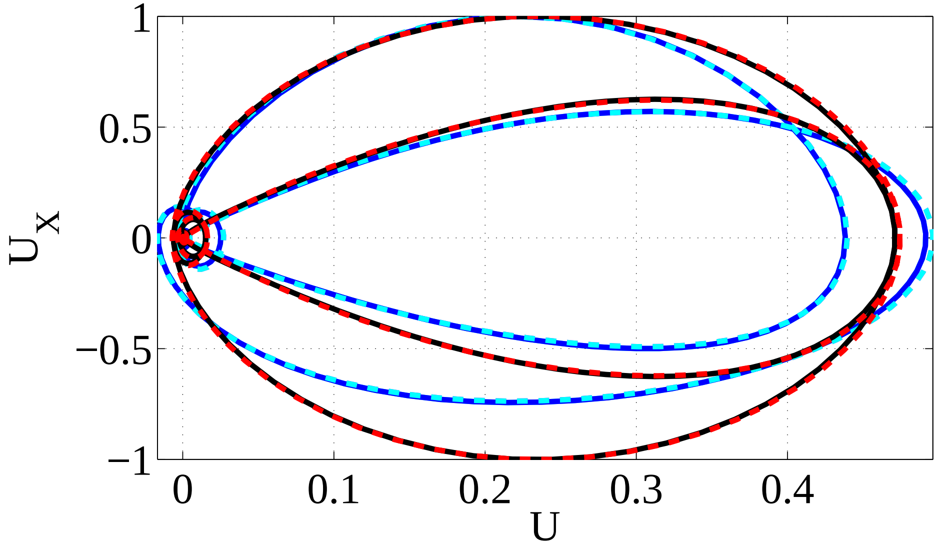
Case 2 - point 2

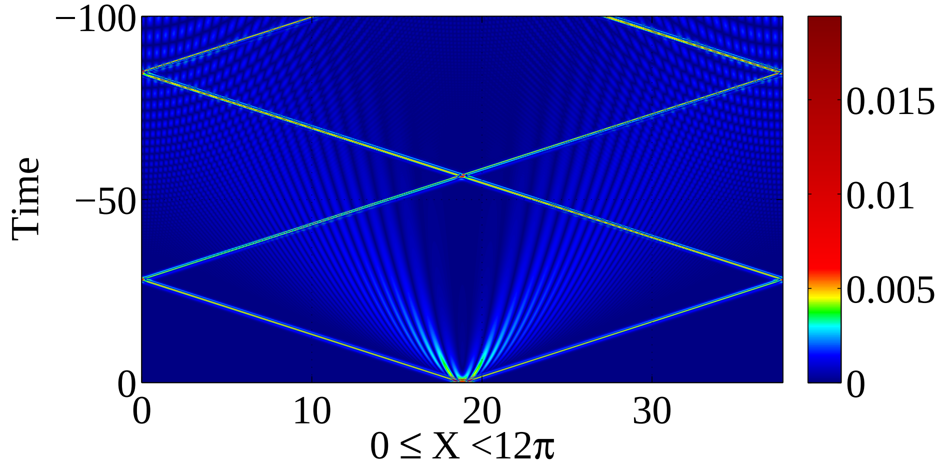


[1.94%] Results - Case 2 $\gamma_A = 0.55$, $\gamma_1 = 0.40$

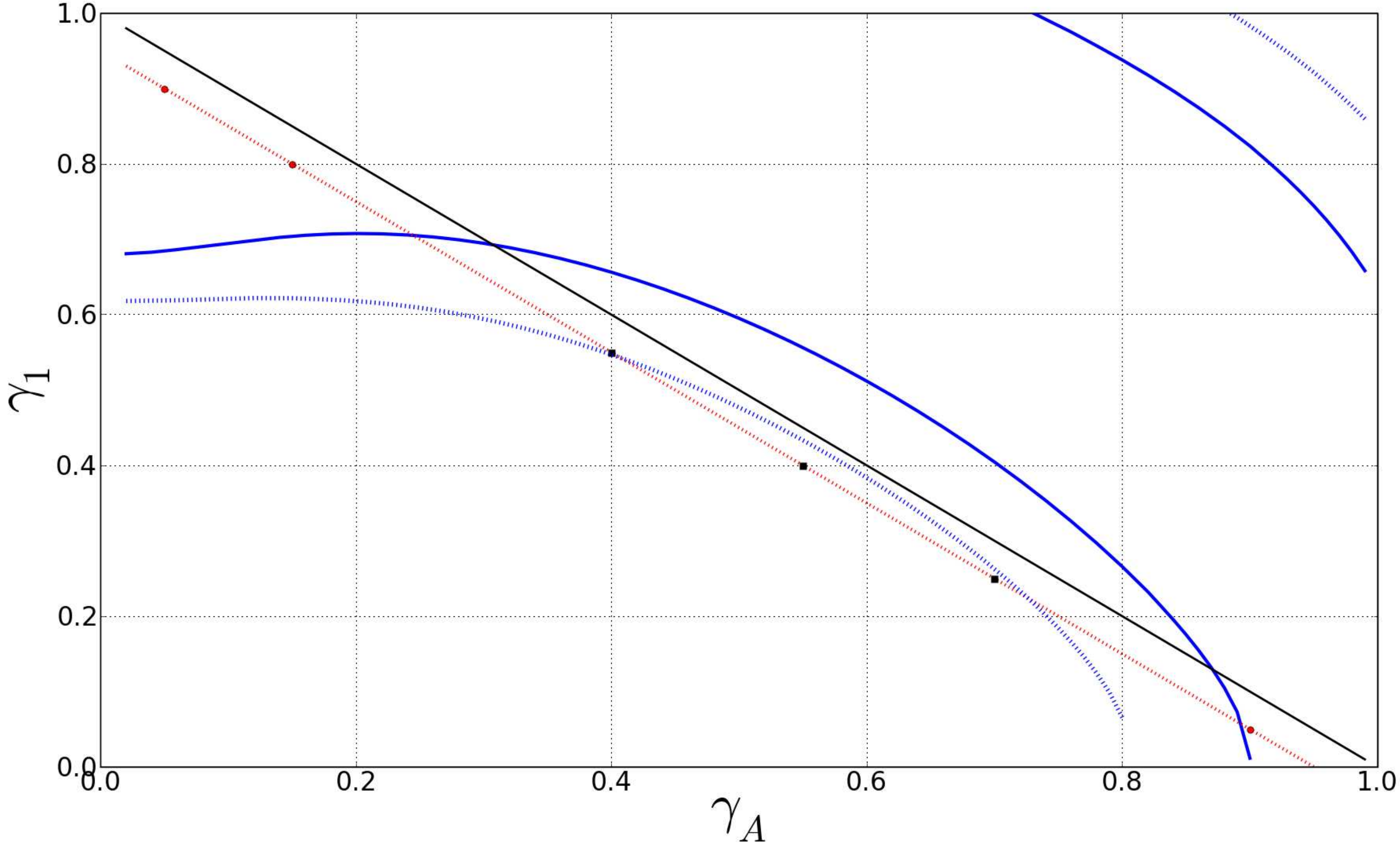




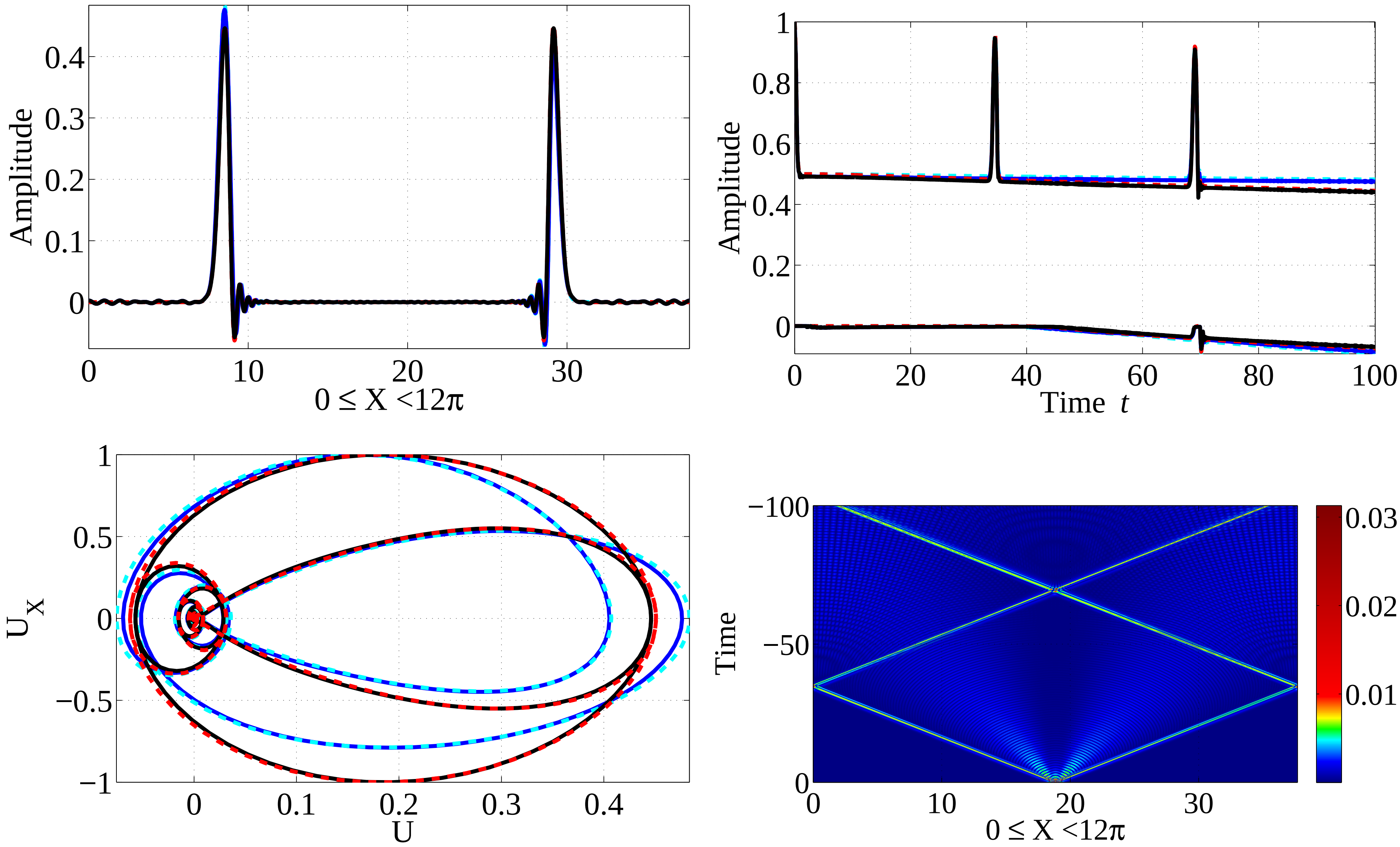




Case 2 - point 3



[3.13%] Results - Case 2 $\gamma_A = 0.70$, $\gamma_1 = 0.25$



- Case 1 - difference between dispersion curves is less than 5%
  1. Both linear and nonlinear cases have good agreement between HE and FSE solutions
  2. Following the weak normal dispersion line quality of agreement between solutions of HE and FSE weakens if γ_A increases.
- Case 2 - difference between dispersion curves is 10% or more
  1. Solutions of HE and FSE have good agreement for 'main parts' of waveprofiles,
  2. Qualitative shift from having wider deviation in waveprofile propagating in negative coordinate direction to waveprofile propagating in positive direction having bigger difference between solutions.

Summary

- Nonlinearity accelerates the altering of waveprofile shape
- Nonlinearity amplifies assymmetry between waveprofiles propagating in opposite directions.
- Predictions from dispersion analysis hold also for nonlinear cases, however nonlinearity introduces additional effects not taken into account by linear dispersion analysis.

THANK YOU!