Modeling Bottom Shear Stress for Transient Wave Events

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Complexity of Nonlinear Waves
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Wave induced bottom shear stress

- Motivation: Provide an accurate description of the bottom shear stress induced by long waves propagating in shallow water.

1. Accurate boundary conditions for high order model equations.
    - Energy dissipation
    - Correct wave form and wave speed
2. Calculation of bedload sediment transport fluxes.
    - Morphological changes in bathymetry due to periodic or transient waves.

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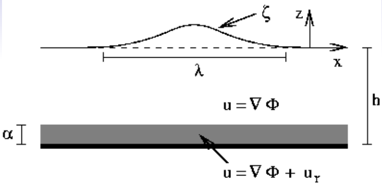
$\zeta =$ free surface displacement
$h =$ constant water depth
$\alpha =$ width of boundary layer



- Rotational motion confined to the bottom boundary layer. Flow in core region assumed to be irrotational.
- Dimensionless parameters:

$$\epsilon = \frac{a_0}{h} \qquad \text{Nonlinear effect}$$

$$\mu = \frac{h}{\lambda} \qquad \text{Wave dispersion}$$

$$\alpha^2 = \frac{\nu}{\lambda\sqrt{gh}} \qquad \text{Viscous effects } (\nu = \text{eddy viscosity})$$

- Assume weak viscous effects:

$$\mathcal{O}(\alpha) \approx \mathcal{O}(\mu^4) \approx \mathcal{O}(\epsilon^2) \ll 1$$

Boundary layer analysis

- Introduce the stretched vertical coordinate: $\eta = \frac{z+h}{\alpha/\mu}$.
- Leading-order (linearized) momentum equation for horizontal rotational velocity component $\mathbf{u}_r$:

$$\frac{\partial \mathbf{u}_r}{\partial t} = \frac{\partial^2 \mathbf{u}_r}{\partial \eta^2}$$

- Initial and boundary conditions:

$$\begin{aligned} \mathbf{u}_r(\mathbf{x}, \eta, 0) &= 0 \\ \mathbf{u}_r(\mathbf{x}, 0, t) &= -\nabla\Phi(\mathbf{x}, 0, t) \qquad \text{(no flux, no slip)} \\ \mathbf{u}_r(\mathbf{x}, \infty, t) &= 0 \end{aligned}$$

$$\Rightarrow \mathbf{u}_r(\mathbf{x}, \eta, t) = -\frac{\eta}{\sqrt{4\pi}} \int_0^t \frac{\nabla\Phi(\mathbf{x}, z=-h, T)}{(t-T)^{3/2}} \, e^{-\frac{\eta^2}{4(t-T)}} \, dT$$

Boundary layer analysis

- Irrotational horizontal velocity component at the bottom: $\mathbf{u}_b(\mathbf{x}, t) = \nabla\Phi(\mathbf{x}, z = -h, T)$.
- Bottom shear stress:

$$\tau_b = \left.\frac{\partial \mathbf{u}_r}{\partial \eta}\right|_{\eta=0} = \frac{\mathbf{u}_r(\mathbf{x}, 0)}{\sqrt{\pi t}} + \frac{1}{\sqrt{\pi}} \int_0^t \frac{\mathbf{u}_b}{(t-T)^{3/2}}\, dT$$

- Vertical rotational velocity component w_r at $\eta = 0$:

$$w_r(\mathbf{x}, 0, t) = \left.\int_\eta^\infty \nabla \cdot \mathbf{u}_r(\mathbf{x}, \xi, t)\, d\xi\right|_{\eta=0}$$

$$= -\frac{1}{\sqrt{\pi}} \int_0^t \frac{\nabla \cdot \mathbf{u}_b}{\sqrt{t-T}}\, dT$$

Equations for the core region

Traditional approach

- Bottom boundary condition:

$$\frac{\partial \Phi}{\partial z} = 0\,, \quad \text{at} \quad z = -h\,.$$

- Long wave model equations, where $\bar{\mathbf{u}}$ = depth averaged horizontal velocity.

$$\text{(A)}\ \frac{\partial \zeta}{\partial t} + \nabla \cdot ((h + \epsilon\zeta)\bar{\mathbf{u}}) = \mathcal{O}(\mu^4)$$

$$\text{(B)}\ \frac{\partial \bar{\mathbf{u}}}{\partial t} + \epsilon\bar{\mathbf{u}} \cdot \nabla\bar{\mathbf{u}} + \nabla\zeta - \frac{\mu^2}{3}\nabla\nabla \cdot \left(\frac{\partial \bar{\mathbf{u}}}{\partial t}\right) - C_f|\bar{\mathbf{u}}|\bar{\mathbf{u}} = \mathcal{O}(\mu^4)$$

Equations for the core region

Boundary layer approach

- Liu and Orfila[1]: Long wave equations including a viscous bottom boundary layer.
- Bottom boundary condition:

$$\frac{\partial \Phi}{\partial z} = \frac{1}{\sqrt{\pi}} \int_0^t \frac{\nabla \cdot \mathbf{u}_b}{\sqrt{t-T}} \, dT \,, \quad \text{at} \quad z = -h \,.$$

- Model equations with a viscous bottom boundary layer.

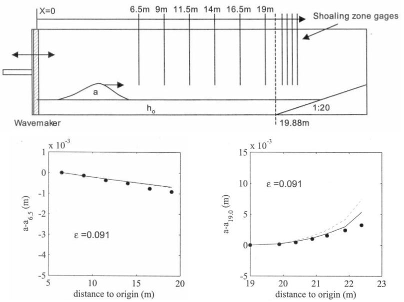
$$\text{(A)} \; \frac{\partial \zeta}{\partial t} + \nabla \cdot ((h + \epsilon\zeta)\bar{\mathbf{u}}) - \frac{\alpha}{\mu\sqrt{\pi}} \int_0^t \frac{\nabla \cdot \bar{\mathbf{u}}}{\sqrt{t-\tau}} \, d\tau = \mathcal{O}(\mu^4)$$

$$\text{(B)} \; \frac{\partial \bar{\mathbf{u}}}{\partial t} + \epsilon \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla \zeta - \frac{\mu^2}{3} \nabla\nabla \cdot \left(\frac{\partial \bar{\mathbf{u}}}{\partial t} \right) = \mathcal{O}(\mu^4)$$

[1] J. Fluid Mech.(2004), Vol. 520, pp. 83-92

Comparison with laboratory experiment

Liu et al.[2]: Combined numerical and experimental study on shoaling solitary waves.



$$h = 0.15\, m \qquad a_0 = 1.36 \cdot 10^{-2}\, m$$

$$\lambda = 2.11\, m \qquad \nu = 1.0 \cdot 10^{-6}\, m^2 s^{-1}$$

[2]Coast. Eng.(2006), Vol. 53(2-3), pp. 181-190

Numerical calculation of viscous effects

- Torsvik and Liu[3]: "An efficient method for the numerical calculation of viscous effects on transient long-waves."
- The viscous term includes a convolution integral on the form

$$A = \int_0^t \frac{\nabla \cdot \bar{\mathbf{u}}}{\sqrt{t - T}} \, dT \, .$$

- Approximate value of A is found by truncating in time and estimating the value of the residual term

$$A = A_T + A_R = \int_{t-t_N}^t \frac{\nabla \cdot \bar{\mathbf{u}}}{\sqrt{t - T}} \, dT + A_R \, .$$

[3] Coast. Eng.(2007), Vol. 54, pp. 263-269

For time step $(k+1)$ and spatial node i:

$$A_i^{(k+1)} \approx \sum_{j=1}^{N} C_j \nabla \cdot \bar{\mathbf{u}}_i^{(k+1-j)} + C_R\, R_i^{(k)}\,,$$

where

$$C_1 = \int_{t-\frac{1}{2}\Delta t}^{t} (t-T)^{-\frac{1}{2}}\, dT\,,$$

$$C_j = \int_{t-(j+\frac{3}{2}\Delta t)}^{t-(j+\frac{1}{2}\Delta t)} (t-T)^{-\frac{1}{2}}\, dT\,, \quad j \geq 2\,,$$

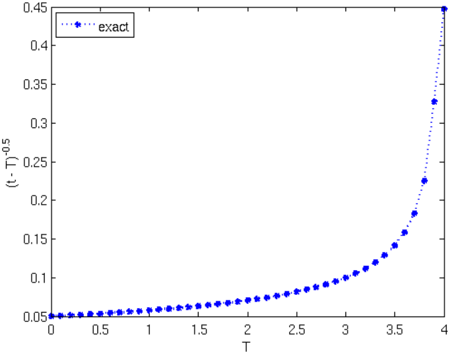
$$R_i^{(k)} = A_i^{(k)} - \sum_{j=1}^{N-1} C_j \nabla \cdot \bar{\mathbf{u}}_i^{(k-j)}\,,$$

and C_R must be found empirically.

The residual coefficient C_R

Example: $t = 4$ and $\Delta t = 0.1$

- Blue line: Weights C_j for the exact convolution integral.
- Assume four time steps are retained in memory, $N = 4$:



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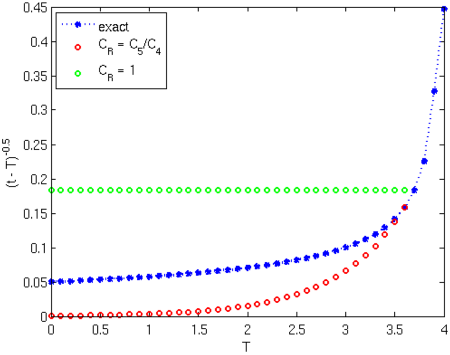

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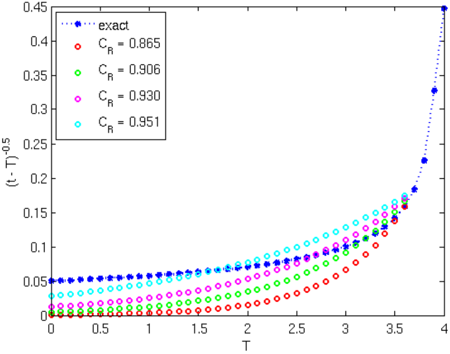
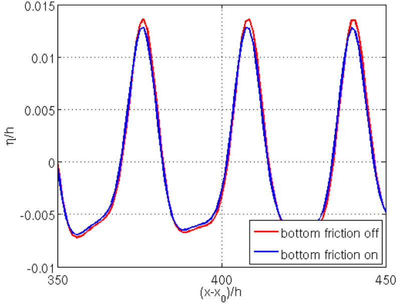
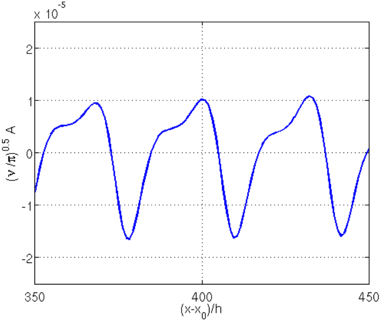


Periodic wave

Wave amplitude:



Bottom friction term:



- Error of wave amplitude (best estimate):

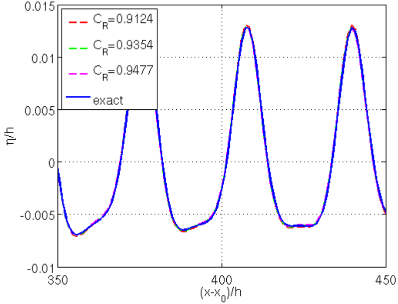
$$\frac{a_0^{(C_R=0.9354)}}{a_0^{(exact)}} = 1.1 \cdot 10^{-3}$$

- Execution time:

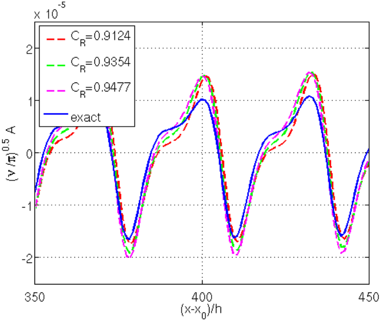
No viscous effects	26 s
Viscous effects (exact)	241 s
Viscous effects (approximate)	31 s

Periodic wave

Wave amplitude:



Bottom friction term:



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Solitary wave: one horizontal dimension

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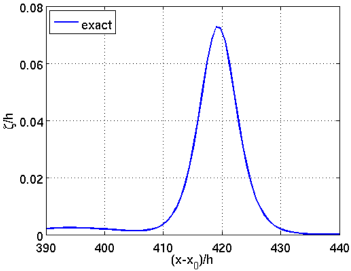

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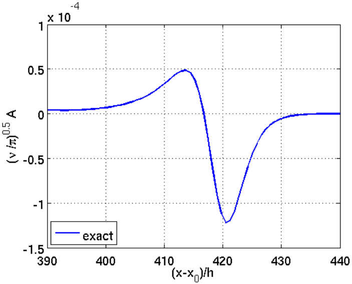

- Error of wave amplitude (best estimate):

$$\frac{a_0^{(C_R=0.9566)}}{a_0^{(exact)}} = 7.8 \cdot 10^{-3}$$

- Execution time:

No viscous effects	45 s
Viscous effects (exact)	481 s
Viscous effects (approximate)	52 s

Solitary wave: one horizontal dimension

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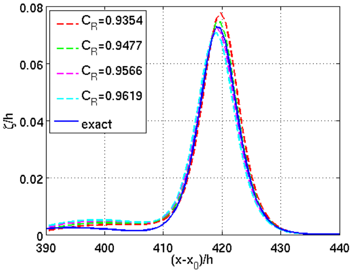
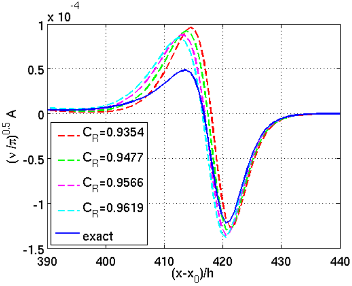


Bottom friction term:



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Example: Propagating pressure disturbance

A pressure disturbance on the free surface is propagating at the speed $U = \sqrt{gh}$, in a narrow channel.