Solitary pulse interaction theory for the generalized Kuramoto-Sivashinsky Equation

Dmitri Tseluiko

Sergey Saprykin, Serafim Kalliadasis

Imperial College London

Complexity of Nonlinear Waves
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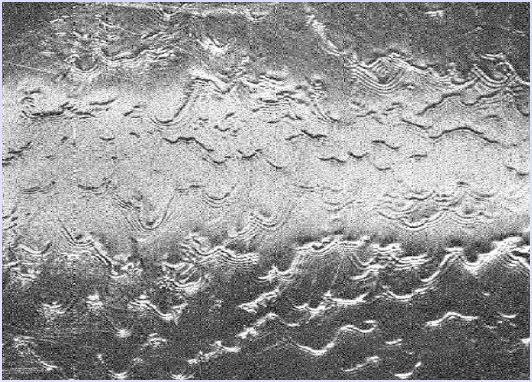
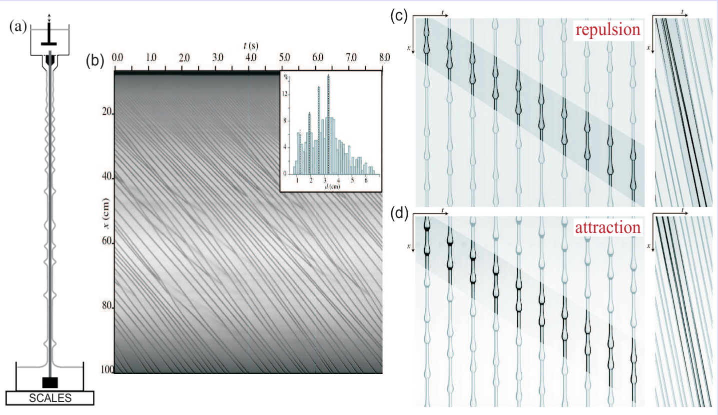
Motivation: complex dynamics on falling liquid films

- Regime of *surface turbulence* or *soliton gas*, *Tailby & Portalski, Trans. Inst. Chem. Eng. 1960*:



- Other phenomena with localized structures: *solitary vortices in plasma, Rossby waves, magmons in magma segregation in the Earth's mantle, localized rolls in nematic crystals*

Wave interactions on a film coating a fibre



(movie)

(e)

(a) experimental set-up: silicon oil v50 (density $\rho = 963\,\mathrm{kg/m^3}$*, dynamic viscosity* $\mu = 48\times10^{-3}\,\mathrm{Pa\,s}$*, surface tension* $\gamma = 20.8\times10^{-3}\,\mathrm{N/m}$*) flows down a Nylon fiber; (b) spatio–temporal diagram and histogram; (c) repulsion; (d) attraction*

Duprat, Giorgiutti-Dauphiné, Tseluiko, Kalliadasis, submitted to Phys. Rev. Lett.

Prototype: the generalized Kuramoto–Sivashinsky equation

- 1D generalized Kuramoto–Sivashinsky (gKS) equation:

$$h_t + hh_x + h_{xx} + \delta h_{xxx} + h_{xxxx} = 0 \tag{1}$$

- Previous studies: *Elphick et al., Phys. Rev. A 1991, Ei & Ohta, Phys. Rev E 1994, Chang & Demekhin 2002*

- Fluid dynamics: films flowing down inclined or vertical planes.
  Inclined plane:

$$Re = O(1), \quad Re - Re_c = O(\epsilon^2), \quad We = O(\epsilon^{-2}). \tag{2}$$

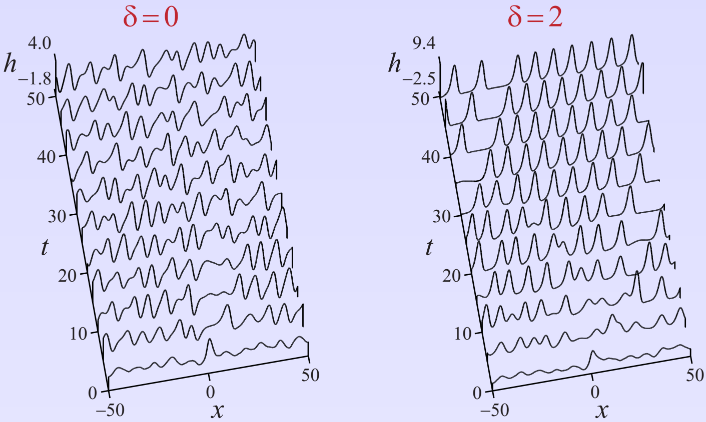
  Vertical plane:

$$Re = O(\epsilon), \quad We = O(\epsilon^{-1}). \tag{3}$$

  Dispersion parameter: $\delta = \sqrt{\frac{15}{2}\frac{1}{(Re-Re_c)We}} = O(1)$

Temporal evolution of h

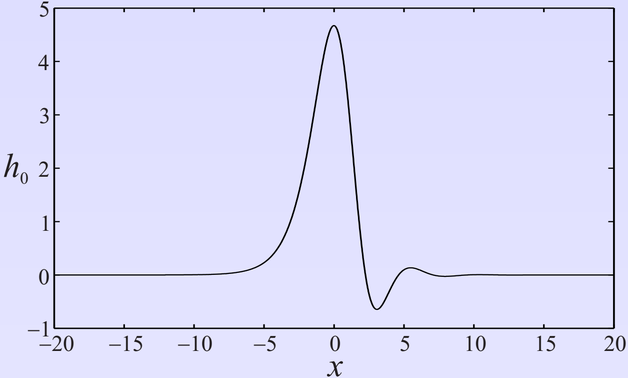
- Sufficiently large δ arrests spatio-temporal chaos (*Kawahara, Phys. Rev. Lett. 1983*):



Solitary pulses of the gKS equation

- Idea: approximate complex wave patterns by a superposition of interacting, coherent structures
- Transform the equation to the moving frame of the pulse:

$$h_t - c_\delta h_x + hh_x + h_{xx} + \delta h_{xxx} + h_{xxxx} = 0 \qquad (4)$$

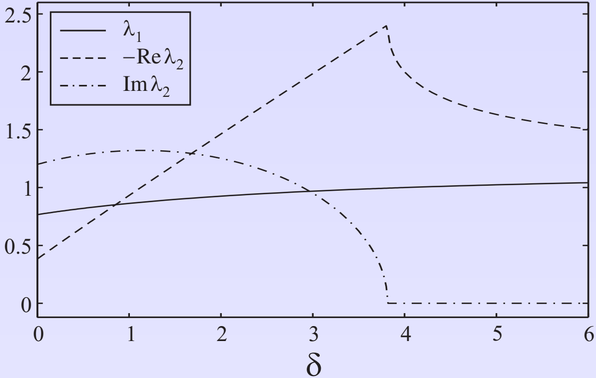


Stationary solution for $\delta = 0.5$ (velocity of he pulse is $c_\delta \approx 1.71$)

Tails of the solitary pulses

- $h_0 \sim \mathrm{Re}(C_{1,2}\mathrm{e}^{\lambda_{1,2}x})$ as $x \to \mp\infty$, where $\lambda_1 > 0$ and $\mathrm{Re}\,\lambda_2 < 0$
- $\lambda_{1,2}$ are the roots of

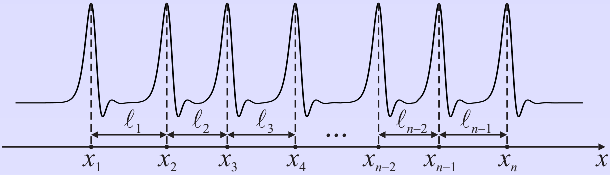
$$\lambda^3 - \delta\lambda^2 + \lambda - c_\delta = 0 \qquad (5)$$



Dependence of λ_1, $-\mathrm{Re}\,\lambda_2$ *and* $\mathrm{Im}\,\lambda_2$ *on* δ

Pulse interaction theory

- Consider weak interaction of n pulses, $h_i = h_0(x - x_i(t))$, $i = 1, \dots, n$ located at $x_1(t) < \cdots < x_n(t)$: solitons *repel* or *attract* each other by interacting through their tails



- Represent the solution as

$$h = \sum_{i=1}^{n} h_i + \hat{h}, \tag{6}$$

$\hat{h}$ – the *overlap* function

Linearized equations for the overlap function

- Evolution equation for $\hat{h}$ in the vicinity of the ith pulse:

$$\hat{h}_t - x_1'(t)h_{1x} = \mathcal{L}_1\hat{h} - (h_1h_2)_x, \qquad i = 1, \qquad (7)$$
$$\hat{h}_t - x_i'(t)h_{ix} = \mathcal{L}_i\hat{h} - (h_{i-1}h_i)_x - (h_ih_{i+1})_x, \quad 1 < i < n, \qquad (8)$$
$$\hat{h}_t - x_n'(t)h_{nx} = \mathcal{L}_n\hat{h} - (h_{n-1}h_n)_x, \qquad i = n, \qquad (9)$$

where

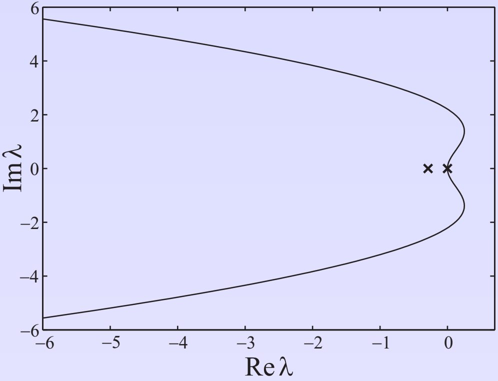
$$\mathcal{L}_i f \equiv c_\delta f_x - f_{xx} - \delta f_{xxx} - f_{xxxx} - (h_i f)_x \qquad (10)$$

- Adjoint of $\mathcal{L}_i$ with respect to L^2 inner product:

$$\mathcal{L}_i^* f \equiv -c_\delta f_x - f_{xx} + \delta f_{xxx} - f_{xxxx} + h_i f_x \qquad (11)$$

Spectra of the linear operators

- Spectrum of $\mathcal{L}_i$ for $\delta = 0.5$:



- Spectrum consists of both a point (crosses) and an essential (solid line) spectrum

Structure of the null space

- On a periodic domain $[-L, L]$, zero is an eigenvalue of algebraic multiplicity 2 and geometric 1:

$$\mathcal{L}_i \Phi_1^i = 0, \qquad \mathcal{L}_i \Phi_2^i = \Phi_1^i, \tag{12}$$

$$\mathcal{L}_i^* \Psi_1^i = 0, \qquad \mathcal{L}_i^* \Psi_2^i = \Psi_1^i, \tag{13}$$

where

$$\Phi_1^i = h_{ix}, \quad \Phi_2^i = -1, \quad \Psi_1^i = -1/2L \tag{14}$$

and

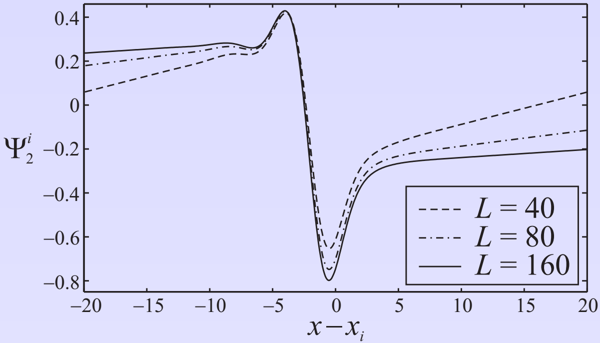
$$\langle \Phi_1^i, \Psi_1^i \rangle = 0, \qquad \langle \Phi_1^i, \Psi_2^i \rangle = 1, \tag{15}$$

$$\langle \Phi_2^i, \Psi_1^i \rangle = 1, \qquad \langle \Phi_2^i, \Psi_2^i \rangle = 0 \tag{16}$$

- $\Phi_1^i = h_{ix}$ is associated with translational invariance

Adjoint zero eigenfunctions

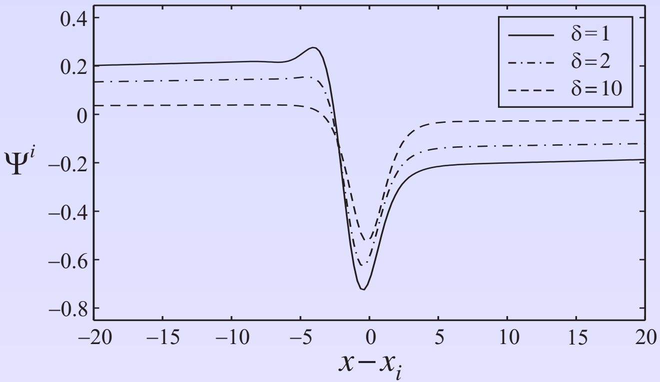
- Generalized adjoint eigenfunctions:



- As $L \to \infty$, $\|\Phi_2^i\| \to \infty$, $\Psi_1^i \to 0$, and $\mathcal{L}_i^* \Psi_2^i \to 0 \Rightarrow$ zero becomes an eigenvalue of both algebraic and geometric multiplicity 1
- Denote $\lim_{L\to\infty} \Phi_1^i$, $\lim_{L\to\infty} \Psi_2^i$ by Φ^i and Ψ^i, respectively

Generalized adjoint eigenfunctions Ψ^i

- Generalized adjoint eigenfunctions for $\delta = 1,\ 2,\ 10$:



- As $\delta \to \infty$, the jump at infinity vanishes. This is consistent with the KdV case

Projections onto translational modes

- Aim: to project dynamics onto translational modes (null spaces of $\mathcal{L}_i$'s)
- Projections can be made rigorous in a weighted space:

$$L_a^2 = \{f : \mathrm{e}^{ax} f \in L_{\mathbb{C}}^2\}, \tag{17}$$

  where $0 < a < -\mathrm{Re}\,\lambda_2$

- $\mathcal{L}_i$ in L_a^2 is equivalent to $\mathcal{L}_i^a \equiv \mathrm{e}^{ax}\mathcal{L}_i(\mathrm{e}^{-ax}(\cdot))$ in $L_{\mathbb{C}}^2$
- Zero becomes an isolated eigenvalue with an eigenfunction and an adjoint eigenfunction given by

$$\Phi_a^i = \mathrm{e}^{ax}\Phi^i, \qquad \Psi_a^i = \mathrm{e}^{-ax}(\Psi^i - \lim_{x\to-\infty})\Psi^i \tag{18}$$

- Projections are given by $P_i^a(f) = \langle f, \Psi_a^i\rangle\Phi_a^i$

Dynamics of the pulses

- For the ith pulse, project dynamics onto the null space of $\mathcal{L}_i$, assuming that $\hat{h}$ is in the null space of the projection:

$$x_1' = S_1(\ell_1), \tag{19}$$

$$x_i' = S_2(\ell_{i-1}) + S_1(\ell_i), \quad 1 < i < n, \tag{20}$$

$$x_n' = S_2(\ell_{n-1}), \tag{21}$$

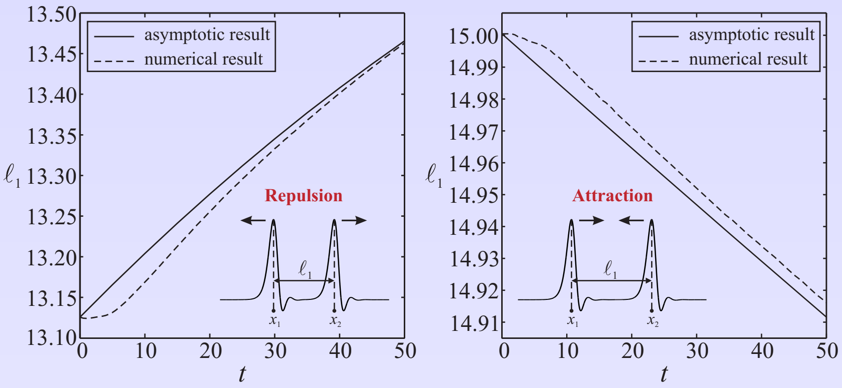
where

$$S_1(\ell) \equiv -\int_{-\infty}^{\infty} h_0(x+\ell/2)h_0(x-\ell/2)\Psi_x^0(x+\ell/2)\,\mathrm{d}x, \tag{22}$$

$$S_2(\ell) \equiv -\int_{-\infty}^{\infty} h_0(x+\ell/2)h_0(x-\ell/2)\Psi_x^0(x-\ell/2)\,\mathrm{d}x \tag{23}$$

Dynamics of two pulses

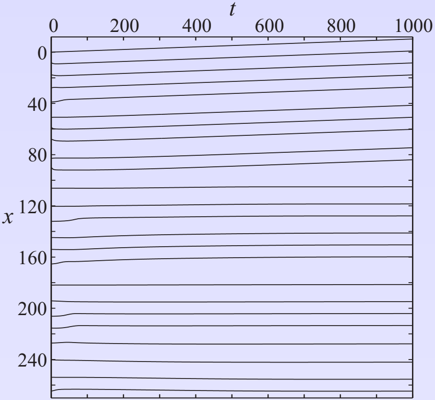
- Dynamical system for two pulses: $x_1' = S_1(\ell_1)$, $x_2' = S_2(\ell_1)$
- Dependence of the pulse separation distance on time, $\delta = 0.5$:

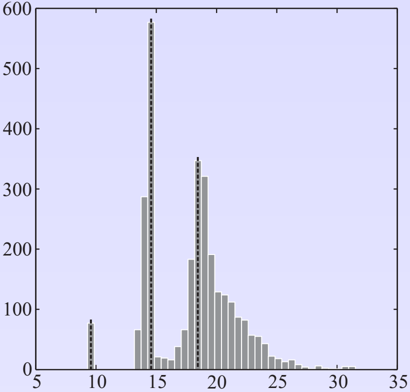


- The nearest stable bound state separation distance is 14.3

Dynamics of n pulses

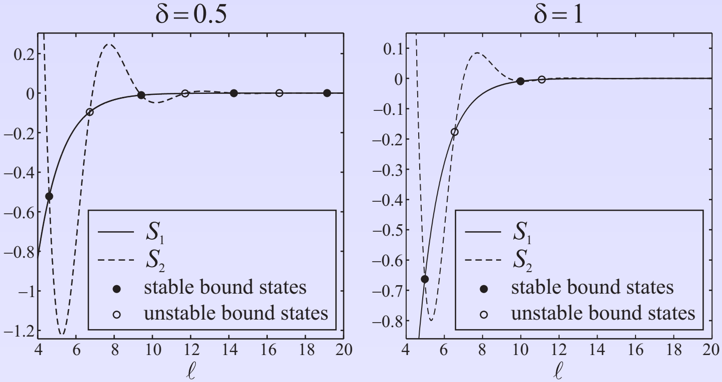
- Evolution of pulses of the gKS equation for $\delta = 0.5$ and histogram of pulse separation distances:





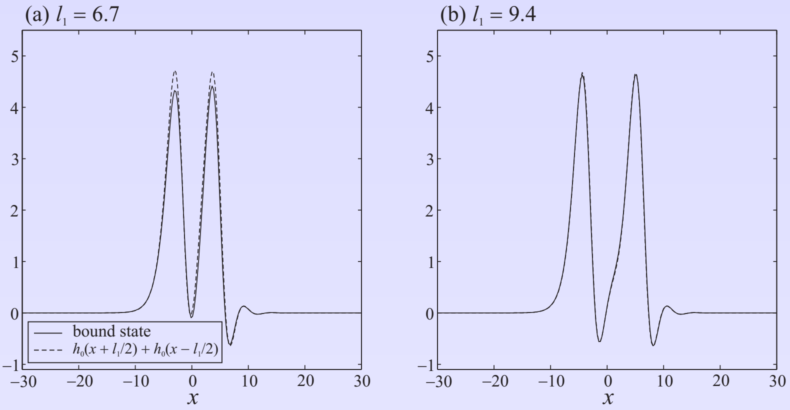
2-pulse bound states

- Formation of bound states: $x_1' = x_2' = \cdots = x_n'$
- For two pulses: $x_1' = x_2' \quad \Rightarrow \quad S_1(\ell_1) = S_2(\ell_1)$
- **Proposition:** $S_1(\ell) \sim C_1 \mathrm{e}^{-\lambda_1 \ell}$, $S_2(\ell) \sim C_2 \mathrm{e}^{\lambda_2 \ell}$ as $\ell \to \infty$
- **Corollary:** $\lambda_1 + \mathrm{Re}\,\lambda_2 > 0 \Rightarrow$ an infinite number of 2-pulse bound states. Otherwise $\Rightarrow$ a finite number of bound states



Comparison of theory with computations

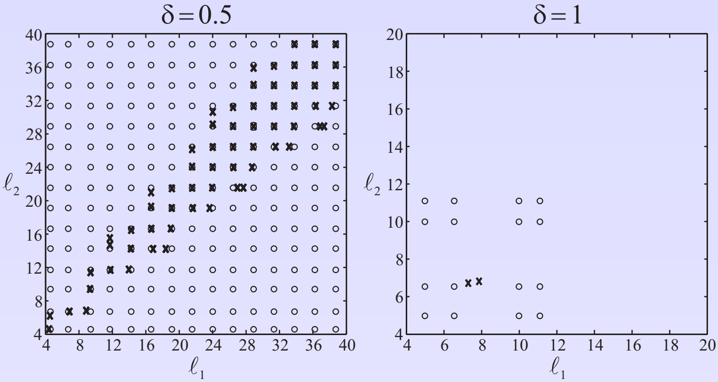
- Numerically computed bound states (solid lines) and theoretical predictions (dashed lines) for $l_1 \approx 6.7, \;\; 9.4$:



3-pulse bound states

- Formation of bound states:

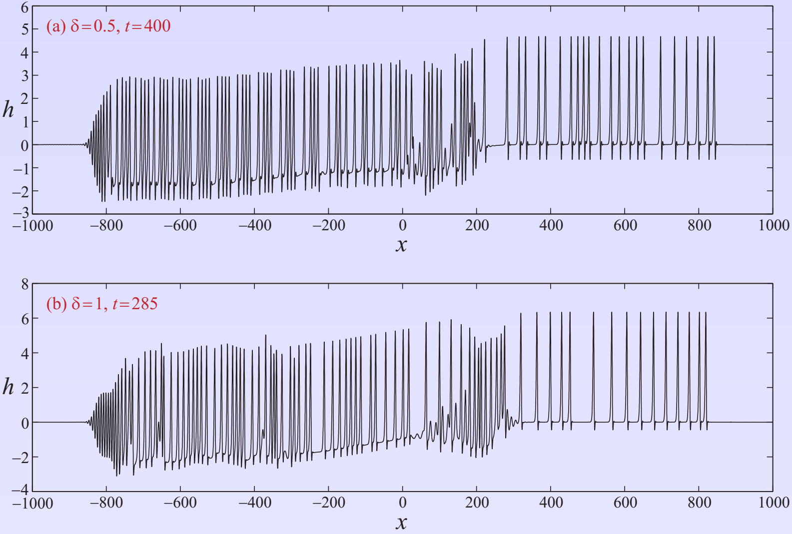
$$S_1(\ell_1) = S_2(\ell_1) + S_1(\ell_2) = S_2(\ell_2) \qquad (24)$$



- $\lambda_1 + \mathrm{Re}\,\lambda_2 > 0 \Rightarrow$ an infinite number of 3-pulse bound states. Otherwise $\Rightarrow$ a finite number of bound states

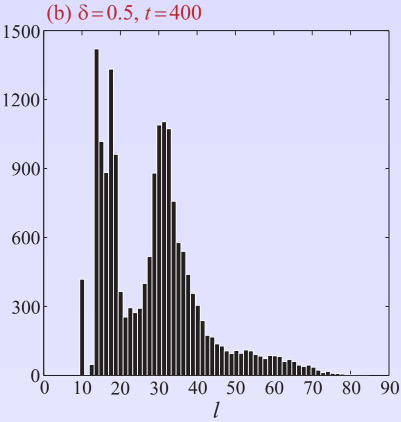
Numerical solutions of the gKS equation

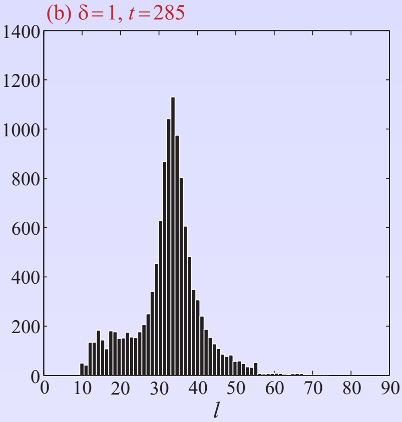
- Solutions for $\delta = 0.5$ and $\delta = 1$:



Numerical solutions of the gKS equation

- Histograms for pulse separation distances based on a series of computational experiments for $\delta = 0.5$ and $\delta = 1$:



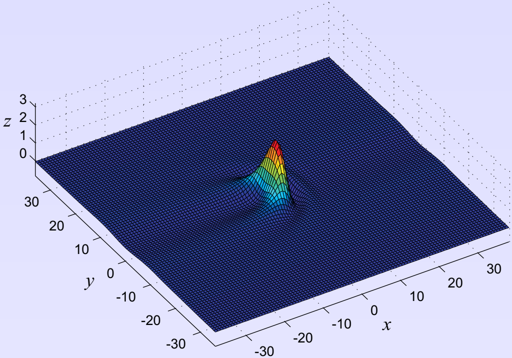


Extension to 3D

- Generalized Kuramoto–Sinvashinsky equation in 3D (describes falling liquid films):

$$h_t + hh_x + h_{xx} + \delta(\nabla^2 h)_x + \nabla^4 h = 0 \qquad (25)$$

- Solitary pulse solution for $\delta = 0.3$:



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- for the real system far from criticality, need a model that takes into account viscous dispersion (e.g. *Ruyer-Quil & Manneville, Eur. Phys. J. B 2000*)