Morphisms between diagrams & Strictification of (∞, n)-categories

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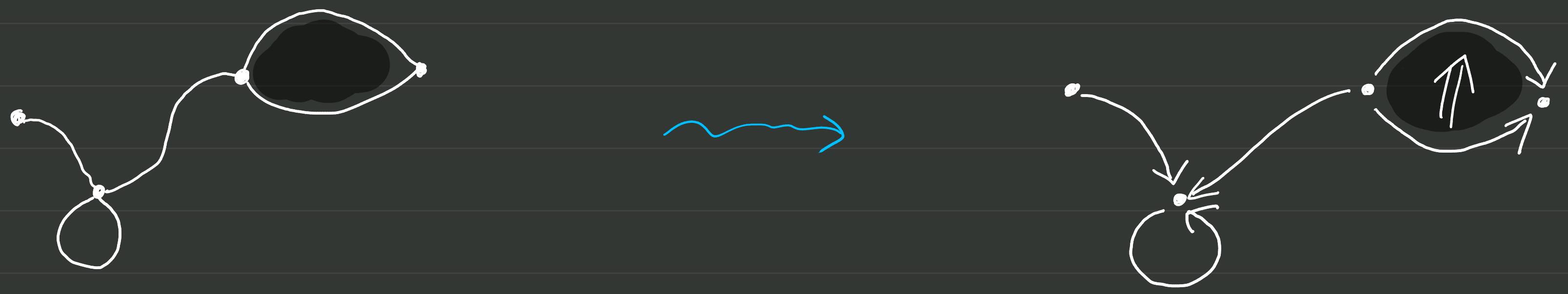
TalTech

Cambridge Logic & Semantics, 22 April 2025

The mathematical structures underpinning diagrammatic reasoning in (globular) higher categories are directed cell complexes



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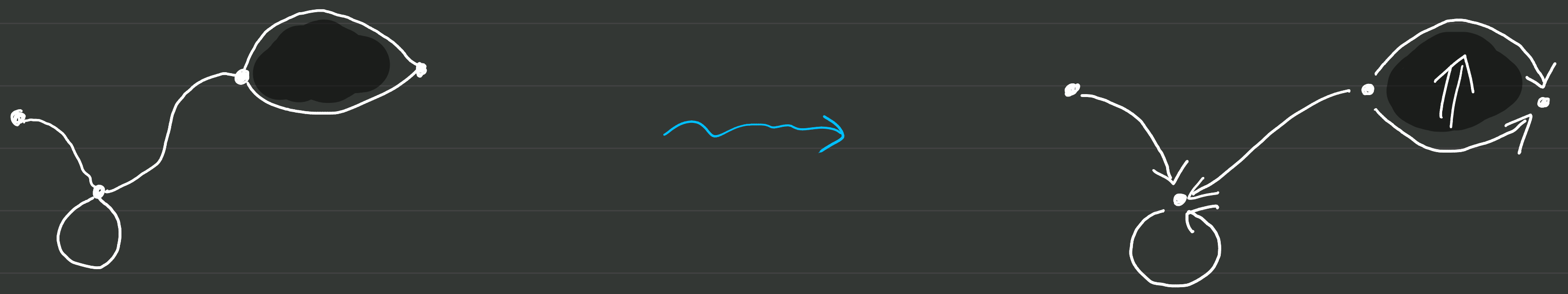


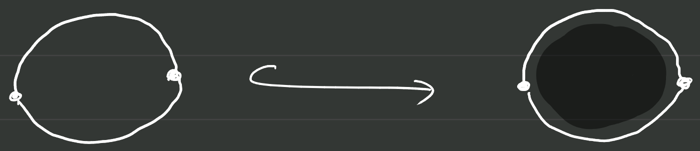
As in topology, multiple models are possible

A model of cell complexes needs

1. models of n-balls and their boundaries,

which are (n-1)-spheres

$$S^{n-1} \equiv \partial B^n \hookrightarrow B^n$$



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$$S^{n-1} \equiv \partial B^n \hookrightarrow B^n$$

2. an ambient category with a class of morphisms whose pushouts along $\partial B^n \hookrightarrow B^n$ exist, to model attaching maps

The canonical model: CW complexes

1. $B^n \equiv \{ x \in \mathbb{R}^n \mid \|x\| \leq 1 \}$,
   $\partial B^n \equiv \{ x \in \mathbb{R}^n \mid \|x\| = 1 \}$

2. attaching maps are all continuous maps

A SUBCLASS OF
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REGULAR CW COMPLEXES — CW POSETS

Björner, 1984

REGULAR CW BALLS $\cong$ CW POSETS w/ GREATEST EL.

TOPOLOGICAL EMBEDDINGS — CLOSED ORDER-EMBEDDINGS

Our approach to modelling directed cell complexes:

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1. Define an order-theoretic model of regular directed complexes, in a category with enough morphisms to model embeddings as attaching maps;
2. Find a more general, well-behaved notion of map between regular directed complexes;
3. Define more general directed cell complexes as sheaves on (r.d.cpx, maps)

1^{st} notion of morphism:

MORPHISMS of

ORIENTED GRADED POSETS

CW POSETS ARE GRADED!

- A poset P is graded if $\forall x \in P$, every maximal chain $x_0 < x_1 < \dots < x_i < \dots < x$ has the same length $\dim(x)$

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CLOSURE OF x

$\Delta x := \{ y \in P \mid x \text{ covers } y \}$

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$$cl\{x\} := \{ y \in P \mid y \leq x \} \qquad \Delta x := \{ y \in P \mid x \text{ covers } y \}$$

CLOSURE OF x FACES OF x

- An orientation on a graded poset P is, for each $x \in P$, a decomposition $\Delta x \cong \Delta^- x + \Delta^+ x$

INPUT OUTPUT

The category $\underline{\mathrm{ogPos}}$

Objects: Oriented graded posets, i.e. graded posets equipped with an orientation

Morphisms: Functions that preserve & reflect dim and Δ^α, $\alpha \in \{+, -\}$; i.e.

$$\forall x, \quad \Delta^\alpha x \xrightarrow[f_*]{\sim} \Delta^\alpha f(x)$$

Working in ogPos, we define various subclasses of oriented graded posets:

	Topologically...	Higher-categorically...
ATOMS		Shapes of cells
$\cap$		
ROUND MOLECULES	Reg. CW balls	
$\cap$		
MOLECULES	Wedges of balls	Shapes of pasting diagrams
$\cap$		
R.D. CPXs	Reg. CW complexes	

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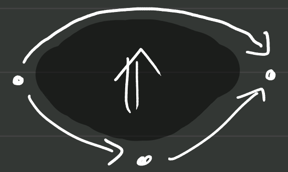
ATOMS		
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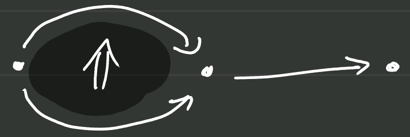
MOLECULES	Wedges of balls	
$\cap$		
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Regular directed complexes present regular CW complexes:

$$\underset{\text{rdcpx}}{P} \longmapsto \underset{\text{Top. space}}{\|P\|}$$

Theorem $\|P\|$ admits a structure of regular CW complex, whose face poset is the underlying poset of P.

Regular directed complexes present strict ω-categories

$$\underset{\text{rdcpx}}{P} \longmapsto \underset{\text{strict } \omega\text{-category}}{\mathrm{Mol}_{/P}}$$

Theorem $\mathrm{Mol}_{/P}$ admits a minimal set of composition-generators whose elements are in bijection with the elements of P.

Explicitly, the cells of $\mathcal{M}cl/P$ are

(iso-classes of) morphisms $U \xrightarrow{f} P$

where U is a molecule

in $\underline{\mathrm{ogPos}}/P$

Explicitly, the cells of Mol/P are

(iso-classes of) morphisms $U \xrightarrow{f} P$

where U is a molecule

in $\underline{\mathrm{ogPos}}/P$

This is unproblematic because molecules are rigid (no nontrivial automorphisms)

A bridge between spaces & strict ω-categories:

$$\begin{tikzcd}[column sep=small]
& \text{Regular directed complexes} \arrow[dl, "\lVert - \rVert"'] \arrow[dr, "\mathcal{M}ol/-"] & \\
\underline{\mathrm{Top}} & & \omega\underline{\mathrm{Cat}}
\end{tikzcd}$$

Desideratum: Any notion of morphism of rdcpx should make both legs functorial

Functoriality of the topological leg is easy:
it suffices for morphisms to have an
underlying order-preserving map of posets.

Q: What is the most general class of
order-preserving maps of rdcpxs that
determine functors of strict ω-categories?

2^{nd} notion of morphism:

MAPS of

REGULAR DIRECTED COMPLEXES

We want an order-preserving $P \xrightarrow{p} Q$

to determine a functor $\mathrm{Mol}/P \xrightarrow{p_*} \mathrm{Mol}/Q$.

This means we need an assignment

$$\boxed{\begin{matrix} U \\ \big\downarrow f \\ P \end{matrix}} \longmapsto \boxed{\begin{matrix} p_* U \\ \big\downarrow p_* f \\ Q \end{matrix}}$$

MOLECULES

ogPos - morphisms

compatible with boundaries & composition.

VERY NON-TRIVIAL!

Theorem Every morphism $f: P \longrightarrow Q$ in ogPos s.t. P, Q are rdcpxs is a local embedding:

$$\forall x \in P, \qquad \mathcal{cl}\{x\} \xrightarrow[f_*]{\sim} \mathcal{cl}\{f(x)\}$$

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Fact There is an OFS on $\underline{\mathbf{Pos}}$ with

$$\begin{tikzcd}
P \arrow[rr, bend left, "p"] \arrow[dr, "p_{\mathcal{F}}"'] & & Q \\
& p_* P \arrow[ur, "p_{\mathcal{L}}"'] &
\end{tikzcd}$$

FINAL MAP

LOCAL EMBEDDING

VERY NON-TRIVIAL!

Theorem Every morphism $f : P \to Q$ in ogPos s.t.

P, Q are rdcpx s is a **local embedding**:

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POSETAL VERSION OF THE COMPREHENSIVE FS ON Cat

$$\begin{tikzcd}
P \arrow[rr, bend left, "p"] \arrow[dr, "p_F"'] & & Q \\
& P_* P \arrow[ur, "p_L"'] &
\end{tikzcd}$$

FINAL MAP

LOCAL EMBEDDING

Thus it makes sense to try

$$\begin{tikzcd}
U \arrow[r, dashed, "(pf)_{\mathcal{F}}", "\text{FINAL}"'] \arrow[d, "f"'] & (pf)_* U \arrow[d, dotted, "(pf)_{\mathcal{L}}"] & \text{LOCAL EMBEDDING} \\
P \arrow[r, "p"'] & Q &
\end{tikzcd}$$

Moreover, $\exists!$ orientation on $(pf)_* U$ s.t.

$(pf)_{\mathcal{L}}$ is a morphism in <u>ogPos</u>

Proposition Given $p : P \longrightarrow Q$, TFAE:

a) This factorisation determines a functor

$$\mathrm{Mol}/P \xrightarrow{p_*} \mathrm{Mol}/Q \,;$$

b) $\forall x \in P$, $\forall \alpha \in \{+,-\}$, $\forall n \in \mathbb{N}$

- $f(\partial^\alpha_n x) = \partial^\alpha_n f(x)$,
- $\partial^\alpha_n x \xrightarrow{f_*} \partial^\alpha_n f(x)$ is final.

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$$\mathsf{Mod}/P \xrightarrow{p_*} \mathsf{Mod}/Q ;$$

b) $\forall x \in P$, $\forall \alpha \in \{+,-\}$, $\forall n \in \mathbb{N}$

- $f(\partial_n^\alpha x) = \partial_n^\alpha f(x)$,
- $\partial_n^\alpha x \xrightarrow{f_*} \partial_n^\alpha f(x)$ is final.

WHEN EITHER HOLDS, WE CALL p A <u>MAP</u> OF RDCPXS

The category RDCpx$\downarrow$

Objects: Regular directed complexes

Morphisms: Maps

- All maps are dim-non-increasing, and are dim-preserving $\Leftrightarrow$ local embeddings
- There is a ternary FS whose classes are

FINAL —— SURJECTIVE LOC. EMB. —— EMBEDDING

We would like to define more general directed cell complexes as either

a) sheaves over $\underline{\mathrm{RDCpx}}_{\downarrow}$ w/ canonical topology,

b) presheaves over $\odot_{\downarrow}$, full subcat. on the atoms

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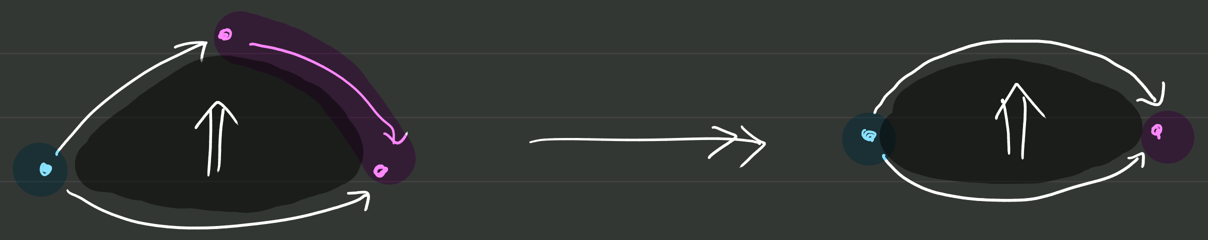
a) sheaves over $\underline{\mathrm{RDCpx}}_\downarrow$ w/ canonical topology,

b) presheaves over $\odot_\downarrow$, full subcat. on the atoms

Problem: $\odot_\downarrow$ is not Eilenberg–Zilber, so not all presheaves are "formal cell complexes"

THERE ARE "TOO MANY SURJECTIVE MAPS"!

An example of a problematic map:



3rd notion of morphism

CARTESIAN MAPS of REGULAR DIRECTED COMPLEXES

Def A map of rdcpxs is **cartesian** if it is a Grothendieck fibration of the underlying posets (seen as categories).

RDCpx := (rdcpxs, cartesian maps)

$\odot$:= full subcategory on atoms

Def A diagrammatic set is, equivalently,

a) a sheaf over $\underline{\mathrm{RDCpx}}$ w/ the canonical topology,

b) a presheaf over $\odot$

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Def A diagrammatic set is, equivalently,

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THIS IS OUR MODEL OF DIRECTED CELL COMPLEXES

- $\odot$ is Eilenberg–Zilber and a strict test category
- We recover $\underline{\mathrm{RDCpx}}$ as the full subcat. on the "cell complexes" whose attaching maps are monomorphisms

Let X be a diagrammatic set, U a rdcpx.

A morphism $U \xrightarrow{u} X$ of sheaves is ...

- a pasting diagram in X if U is a molecule,
- a round diagram in X if U is a round molecule,
- a cell in X if U is an atom.

(Pasting diagrams in X form a strict ω-category)

Cells in a diagrammatic set admit algebraic

- face operations (by pb. along embeddings of atoms),
- unit/degeneracy operations (by pb. along surjections of atoms),

but <u>no</u> composition operations ...

There is an equivalence relation $\simeq$ on (parallel) round diagrams in a diagrammatic set, with nice properties.

Def A diagrammatic set is an (∞,∞)-category if $\forall$ round diagram u,

$\exists$ cell $\langle u \rangle$ s.t. $u \simeq \langle u \rangle$

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"WEAK COMPOSITE OF u"

"EVERY ROUND DIAGRAM IS EQUIVALENT TO A SINGLE CELL"

Theorem (C. Chanavat, A. H.)

There is a (nice) model structure on diagrammatic sets s.t.

1. all dgm sets are cofibrant,
2. the fibrants are exactly the (∞,∞)-categories

This is a non-(fully-) algebraic model of higher categories

4th notion of morphism

COMAPS of

REGULAR DIRECTED COMPLEXES

What if, instead, we demand that an order-preserving map $P \xrightarrow{c} Q$ induce a functor $\mathrm{Mod}/Q \xrightarrow{c^*} \mathrm{Mod}/P$?

THIS IS ALWAYS A LOC. EMB.!

$$\begin{tikzcd}
c^*V \arrow[r, dashed] \arrow[d, dashed, "c^*f"'] \arrow[dr, phantom, very near start, "\lrcorner"] & V \arrow[d, "f"] \\
P \arrow[r, "c"] & Q
\end{tikzcd}$$

Proposition Given $P \xrightarrow{c} Q$, TFAE:

a) Pullback along c determines a functor $\mathcal{M}\mathrm{ol}/Q \xrightarrow{c^*} \mathcal{M}\mathrm{ol}/P$;

b) $\forall y \in Q$, $\forall \alpha \in \{+,-\}$, $\forall n \in \mathbb{N}$,

- $c^{-1} \mathrm{cl}\{y\}$ is a molecule,
- $c^{-1}(\partial_n^\alpha y) = \partial_n^\alpha c^{-1} \mathrm{cl}\{y\}$.

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WHEN EITHER HOLDS, WE CALL c A COMAP

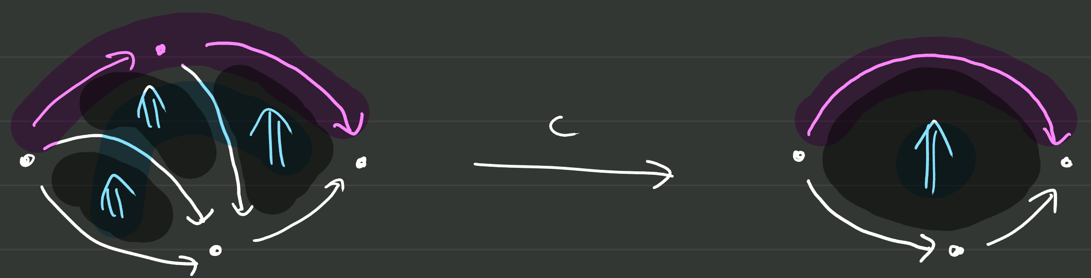
The category $\underline{\mathrm{RDCpx}}_{\uparrow}$

Objects: Regular directed complexes

Morphisms: Comaps

- a comap is always dim-non-decreasing
- a comap is also a map $\iff$ it is an isomorphism (in either category)

Comaps are dual to subdivisions of atoms into round molecules



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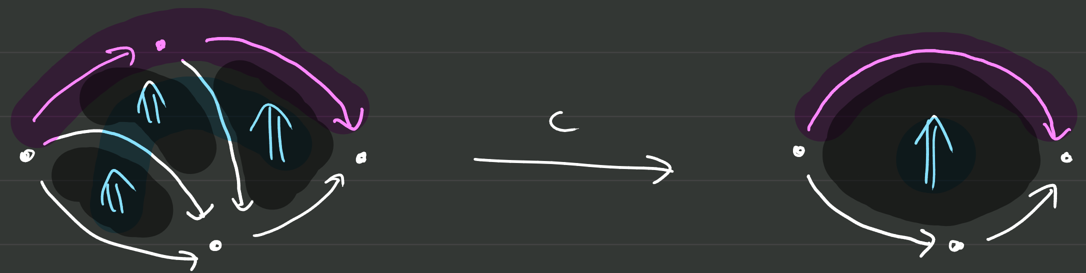

↳ Pullback of round diagrams along "dual comaps" is an algebraic notion of composition

5^{th} notion of morphism

(COMAP, CARTESIAN MAP) SPANS

$$\begin{tikzcd} & S \arrow[dl, bend right, "c"'] \arrow[dr, bend left, "p"] & \\ P & & Q \end{tikzcd}$$

For these to compose, we need, given

a pullback
$$\begin{tikzcd} c^*T \arrow[r, dashed, "d"] \arrow[d, dashed, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "p"] \\ S \arrow[r, "c"'] & P \end{tikzcd},$$

① an orientation on c^*T, s.t.

② c^*T is a regular directed complex, and

③ q is a cartesian map, d is a comap.

$$\begin{tikzcd} c^*T \arrow[r, dashed, "d"] \arrow[d, dashed, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d, "p"] \\ S \arrow[r, "c"'] & P \end{tikzcd}$$

Proposition Under the assumptions, c^*T is graded, and

$\exists!$ orientation on c^*T such that

(1) the orientation on fibres of q agrees with fibres of p up to a global sign.

BOTH NECESSARY

(2) if $\dim q(x) = \dim x$, then $q|_{cl\{x\}}$ is an isomorphism.

(3) c^*T is oriented thin ← A PROPERTY ALL RDCPXs HAVE

PROVEN IN SPECIAL CASES

Conjecture With a nice "sign gauge",

c^*T is a regular directed complex,

d is a comap, q is a cartesian map.

If the conjecture holds, we have a cospan

of wide subcategories

$$\begin{tikzcd}
 & \underline{RDC_{px}^{\updownarrow}} & \\
\underline{RDC_{px}^{\uparrow}} \arrow[ur] & & \underline{RDC_{px}} \arrow[ul]
\end{tikzcd}$$

WHAT FOLLOWS IS ONLY

WELL-DEFINED

CONDITIONALLY TO THE CONJECTURE!

By restriction, we have a functor

$$\underset{\text{COMAP-CART. MAP}}{\mathrm{PSh}(\mathrm{RDC}_{p\times}\updownarrow)} \longrightarrow \underset{\text{CART. MAP}}{\mathrm{PSh}(\mathrm{RDC}_{p\times})}$$

Def A *charted ω-category* is a presheaf on $\mathrm{RDC}_{p\times}\updownarrow$ whose restriction to $\mathrm{RDC}_{p\times}$ is a diagrammatic set.

(COMPARE: STRICT ω-CATS AS PRESHEAVES ON Θ)

We have a "free - forgetful" adjunction

$$\begin{tikzcd} \odot\underline{\mathbf{Set}} \arrow[r, bend left, "M"] \arrow[r, phantom, "\perp"] & \underline{\mathrm{ch}}\,\omega\,\underline{\mathbf{Cat}} \arrow[l, bend left, "U"] \end{tikzcd}$$

where U is monadic (charted ω-categories are algebras for a monad on diagrammatic sets)

Theorem Suppose the conjecture holds.
Then there exists a model structure on
chωCat s.t.

1. all charted ω-cats are fibrant,
2. $M \dashv U$ is a Quillen equivalence with the model structure for (∞,∞)-categories

Consequence:

Given a diagrammatic set X, the unit $X \xrightarrow{\eta_X} UMX$ is a fibrant replacement. In particular, if X is an (∞,∞)-category, η_X embeds X into an equivalent (∞,∞)-category admitting the algebraic, semistrict structure of a charted ω-category.

A HIGHER-DIMENSIONAL "MAC LANE STRICTIFICATION THEOREM"!

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THANK YOU FOR YOUR ATTENTION!

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