REWRITING IN WEAK HIGHER CATEGORIES

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CIRM, 23.2.2022

OUTLINE

1. Intro to the problem of rewriting in higher categories

2. The "diagrammatic sets" model (arXiv:2007.14505)

Central Idea of Higher-Dimensional Rewriting

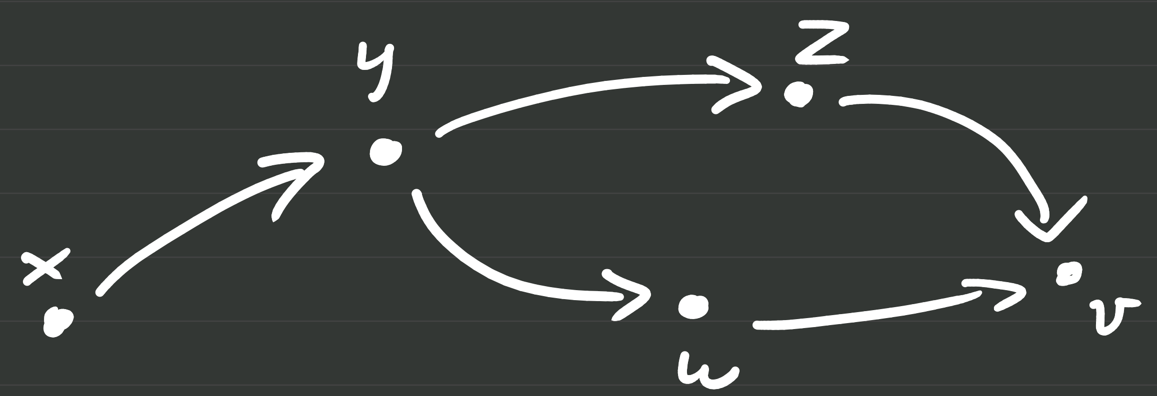
REWRITE SYSTEMS = DIRECTED* CELL COMPLEXES

* Oriented, in such a way that the boundary of a cell is split into an "input" and an "output" half.

$\boxed{1D}$ Abstract Rewrite Systems

~ Directed Graphs

~ Directed 1D cell complexes

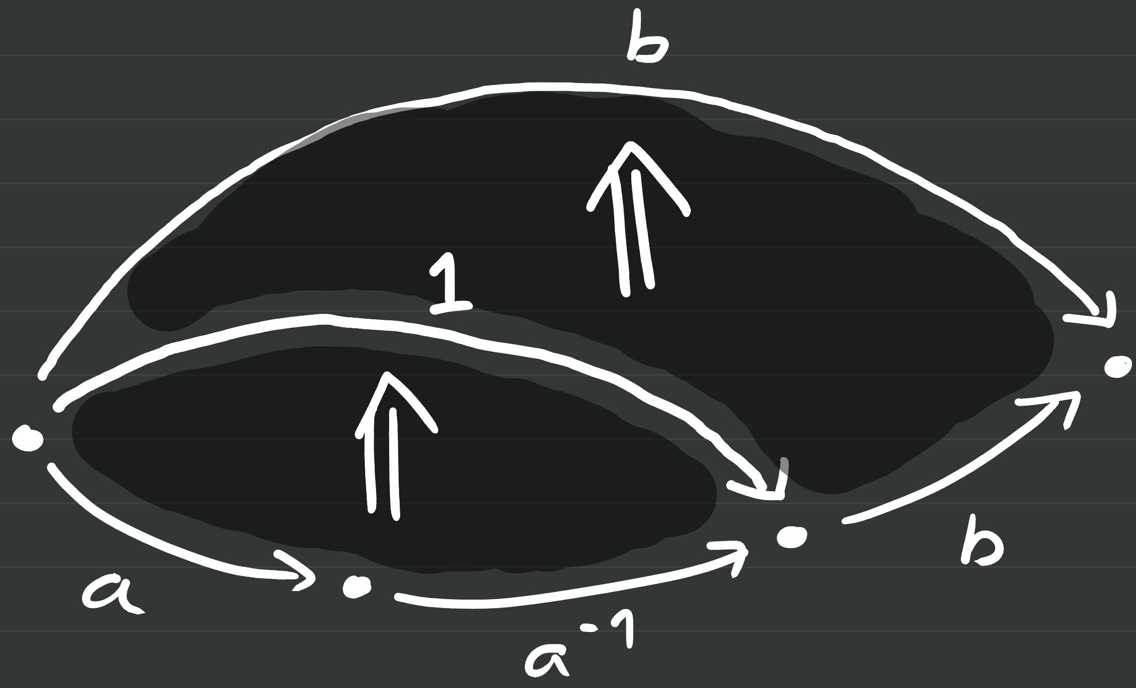


Rewrite sequence ~ Directed homotopy

$\boxed{2D}$ String Rewrite Systems

~ Rewriting paths in graphs

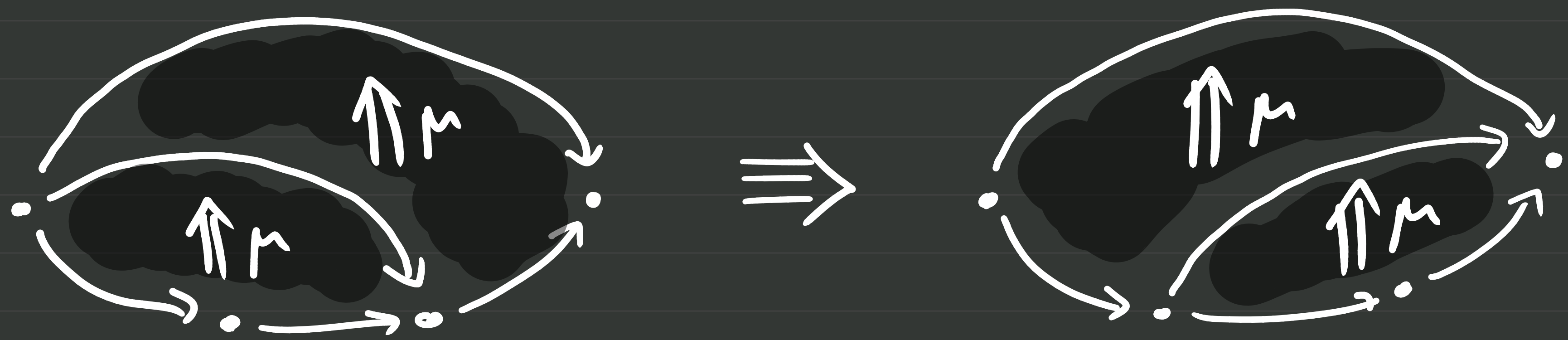
~ Directed 2D cell complexes



$\boxed{3D}$ Term Rewriting Systems

~ Diagram rewriting in PRO(P)s

~ Directed 3D cell complexes



$$\mu(\mu(x,y),z) \Rrightarrow \mu(x,\mu(y,z))$$

To model a (directed) cell complex, we need:

1. Models of n-cells & their boundaries

2. Models of "gluing maps", specifying how cells can be put together

Models can be...

- POINT-SET
- COMBINATORIAL
- ALGEBRAIC
- LOGICAL/SYNTACTIC

Example: POLYGRAPHS / COMPUTADS

- An n-cell is modelled by the n-globe

- It can be glued along any functor of strict ω-categories

3 PROBLEMS FOR A HIGHER-DIMENSIONAL REWRITING THEORY

$\boxed{\#0}$ EXPRESSIVENESS

$\boxed{\#1}$ TOPOLOGICAL SOUNDNESS

$\boxed{\#2}$ HIGHER-CATEGORICAL SEMANTICS

$\boxed{\#0}$ EXPRESSIVENESS

A HDRT has to adequately address the existing practice of rewriting theory.

$\boxed{\#1}$ TOPOLOGICAL SOUNDNESS

A directed cell complex also presents a (topological) cell complex.

A well-formed rewrite sequence induces a (cellular) homotopy in the presented space.

$\boxed{\#2}$ HIGHER-CATEGORICAL SEMANTICS

Higher-dimensional rewrite systems should admit a suitably wide class of "semantic universes" in which they can be interpreted

(e.g. when used to present higher algebraic theories)

Higher Rewrite Systems $\longrightarrow$ Higher Categories

SYNTAX $\longrightarrow$ SEMANTICS

HDRSs can be interpreted in higher categories, but they themselves aren't necessarily higher categories

HDRSs need pasting, but not composition of diagrams:

- PASTING changes the type
  (e.g. concatenation of paths:
  $$X^{[0,1]} \times X^{[0,1]} \to X^{[0,2]}$$)

- COMPOSITION preserves the type
  (e.g. composition of paths in an ∞-groupoid:
  $$X^{[0,1]} \times X^{[0,1]} \to X^{[0,1]}$$)

Relative to a model of higher categories, solving problem #2 amounts to proving a **PASTING THEOREM**:

Given a suitable interpretation of its constituents, each diagram in a HDRS has a unique interpretation in a higher category.

The "FUNCTORIAL SEMANTICS" approach:

Both SYNTAX and SEMANTICS living in the same category;

an INTERPRETATION is simply a morphism from a "syntactic" to a "semantic" object.

A typical (and relevant) setting:

- A cofibrantly generated model category;
- Generating cofibrations model "cells" and their boundary inclusions;
- Cofibrants are "CELL COMPLEX"-like;
- Fibrants are "HOMOTOPY TYPE"-like.

Example: POLYGRAPHS are cofibrant in the folk model structure on ωCat (Lafont-Métayer-Worytkiewicz).

ALL ω-CATEGORIES are fibrant

- the lines between syntax & semantics, and pasting & composition, are blurred...

HOW DO POLYGRAPHS DO?

$\boxed{\#0}$ GREAT!

$\boxed{\#1}$ NO SOUND INTERPRETATION of all gluing maps (due to "strict Eckmann-Hilton")

$\boxed{\#2}$ SO-SO...

Semantics only in STRICT higher categories

Notice that #1 fails because polygraphs are TOO EXPRESSIVE

(too many "cell complexes")

while #2 fails because strict ω-categories are NOT EXPRESSIVE ENOUGH

(too few "semantic universes")

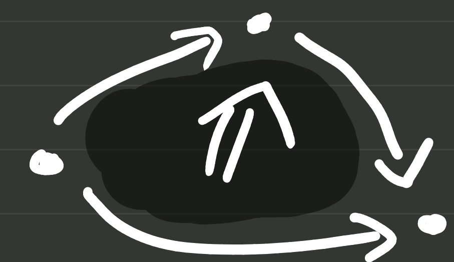
Example: COMPLICIAL SETS are fibrant in a model structure on marked simplicial sets.

These seem to capture general, weak higher categories:

$\boxed{\#2}$ GREAT!

ALL MARKED SIMPLICIAL SETS are cofibrant. They have a geometric realisation as CW complexes:

$\boxed{\#1}$ GREAT!

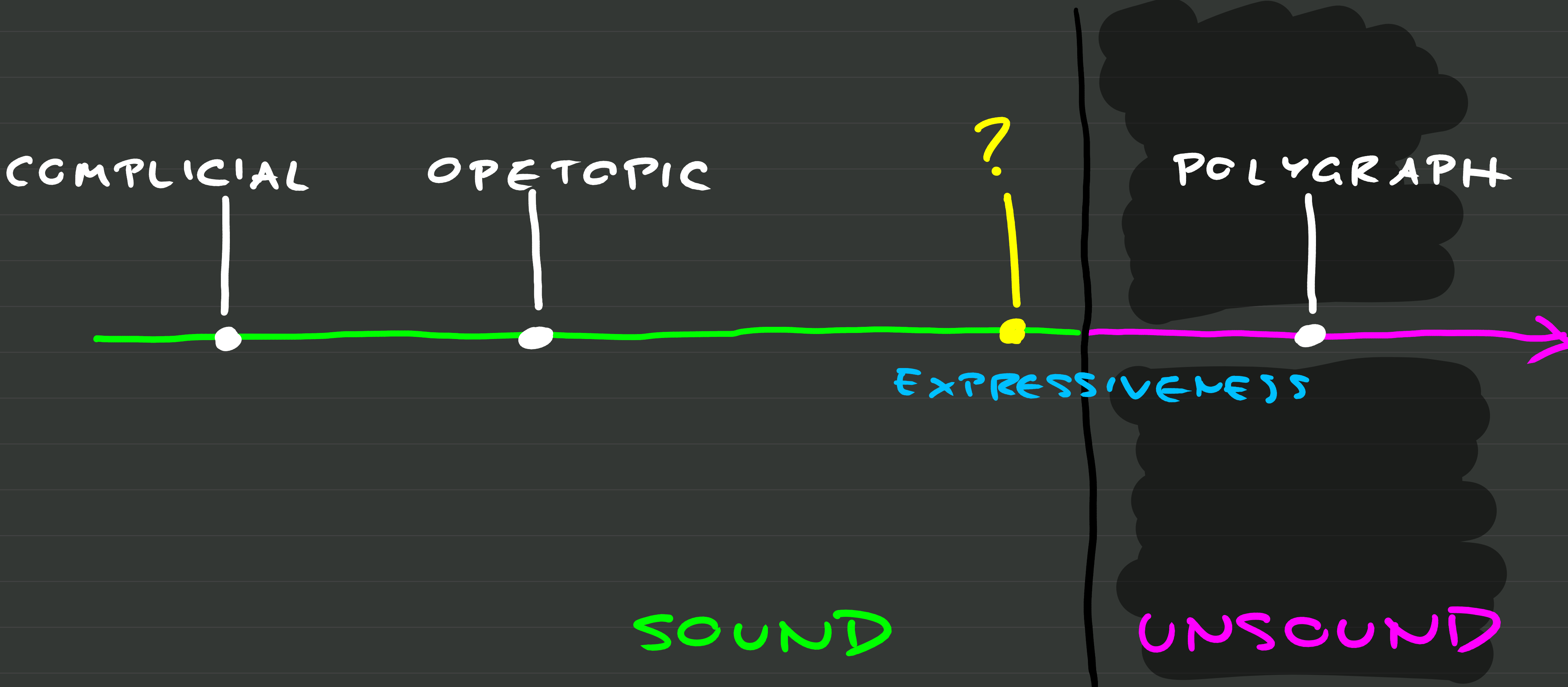
On the other hand, 

only simplex-shaped cells ...

E.g. in 2D, only $1 \mapsto 2$ or $2 \mapsto 1$ rewrites (depending on orientation)

$\boxed{\#0}$ NOT GREAT!

A TRADE-OFF:


COMPLICIAL
OPETOPIC
?
POLYGRAPH
EXPRESSIVENESS
SOUND
UNSOUND

GOAL: Find a LARGE, but WELL-BEHAVED class of shapes, closed under useful rewriting operations

One forerunner in considering "big" shape categories: the unfortunate Kapranov-Voevodsky '91.

I borrowed their name: DIAGRAMMATIC SETS

A string of works in the late '80s, early '90s, on combinatorial descriptions of pasting diagrams (Power, Johnson, Street, Steiner ...)

GOALS OF THESE WORKS:

- CLASSIFYING ω-CATEGORICAL (GLOBULAR) COMPOSITIONS;
- PRESENTING POLYGRAPHS COMBINATORIALLY

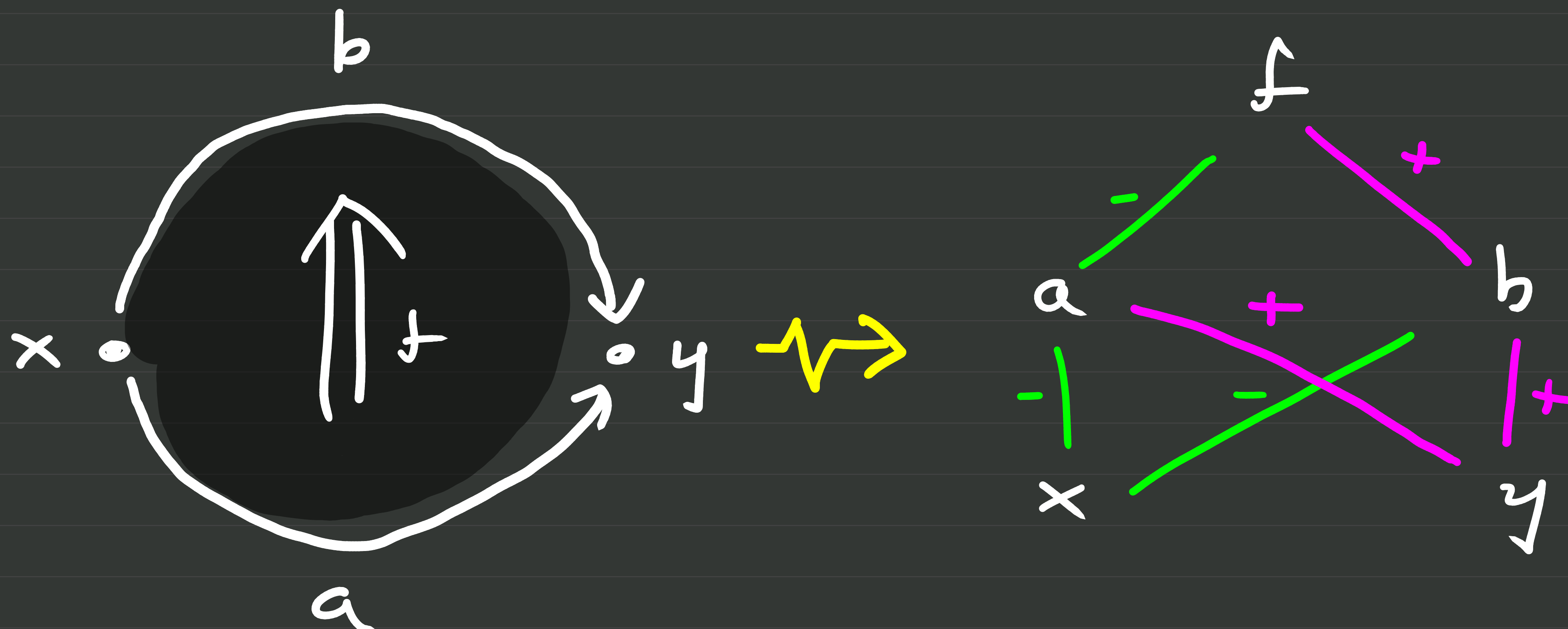
We return to them with different aims

A subtle idea hidden in Steiner '93,
"THE ALGEBRA OF DIRECTED COMPLEXES":

Use the globular $\#_k$ operations
NOT as an "ALGEBRA OF COMPOSITION"
in higher categories;

but as an "ALGEBRA OF PASTING"
of combinatorial "diagram shapes",
for which the equations of ω-cats
are simply TRUE "in the model"!

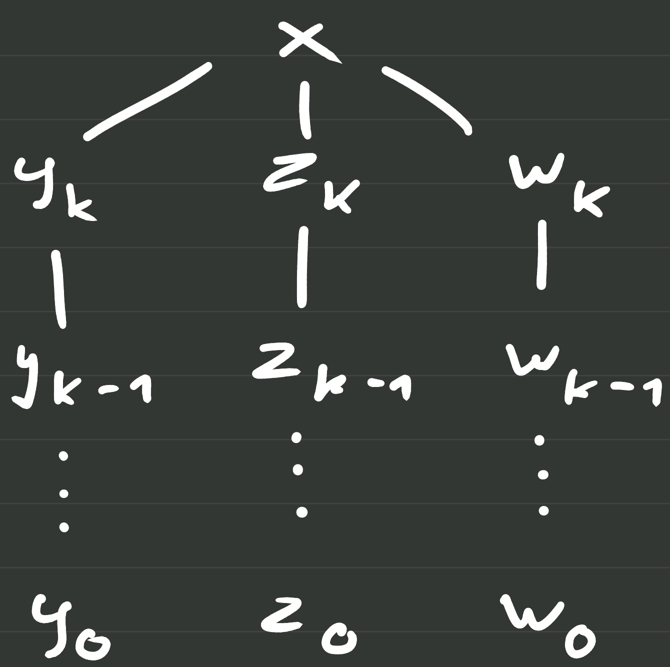
ORIENTED FACE POSET of a shape



− Source/Input

+ Target/Output

Def A finite poset P is **graded** if $\forall x \in P$, all maximal descending chains under x have the same length.



Rank / Dimension of x : $k+1$

Def An oriented graded poset is a graded poset together with an edge-labelling of its Hasse diagram in $\{-, +\}$.

We work mainly with (downwards) closed subsets of an o.g. poset (which inherit an o.g. poset structure)

Def Boundaries of a closed subset U

$k \in \mathbb{N}$ $\qquad\qquad$ $\alpha \in \{-, +\}$

$$\Delta_k^\alpha U := \left\{ x \in U \;\middle|\; \dim(x) = k, \text{ and } \forall y \in U \quad \begin{array}{c} y \\ | \\ x \end{array} \Rightarrow \begin{array}{c} y \\ |\alpha \\ x \end{array} \right\}$$

$$\mathrm{Max}_j U := \left\{ x \in U \;\middle|\; \dim(x) = j, \text{ and } x \text{ is maximal} \right\}$$

$$\partial_k^\alpha U := \mathrm{cl}(\Delta_k^\alpha U) \cup \bigcup_{j<k} \mathrm{cl}(\mathrm{Max}_j U)$$

Notation: for $x \in P$, $\partial_k^\alpha x := \partial_k^\alpha \mathrm{cl}\{x\}$

Def A map $f : P \to Q$ of o.g. posets is a function satisfying

$$\boxed{f(\partial_k^\alpha x) = \partial_k^\alpha f(x)}$$

for all $x \in P$, $k \in \mathbb{N}$, $\alpha \in \{-,+\}$.

Prop A map of o.g. posets is
- order-preserving, and
- dimension-non-increasing.

The class $\mathcal{R}$ of REGULAR MOLECULES:

1. (POINT) The terminal o.g. poset $\bullet \in \mathcal{R}$.

2. (ATOM) If $U, V \in \mathcal{R}$,
   a) $\dim(U) = \dim(V) = n$,
   b) $\partial^{\alpha}_{n-1} U \simeq \partial^{\alpha}_{n-1} V$ for all $\alpha \in \{-,+\}$,
   c) U, V have spherical boundary,

   then $U \Rightarrow V \in \mathcal{R}$.

3. (PASTE) If $U, V \in \mathcal{R}$, $\partial^{+}_{k} U \simeq \partial^{-}_{k} V$, then $U \#_k V \in \mathcal{R}$.

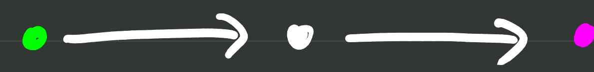
2. (ATOM):

- Glue U, V along the (unique!) isomorphism of their boundaries

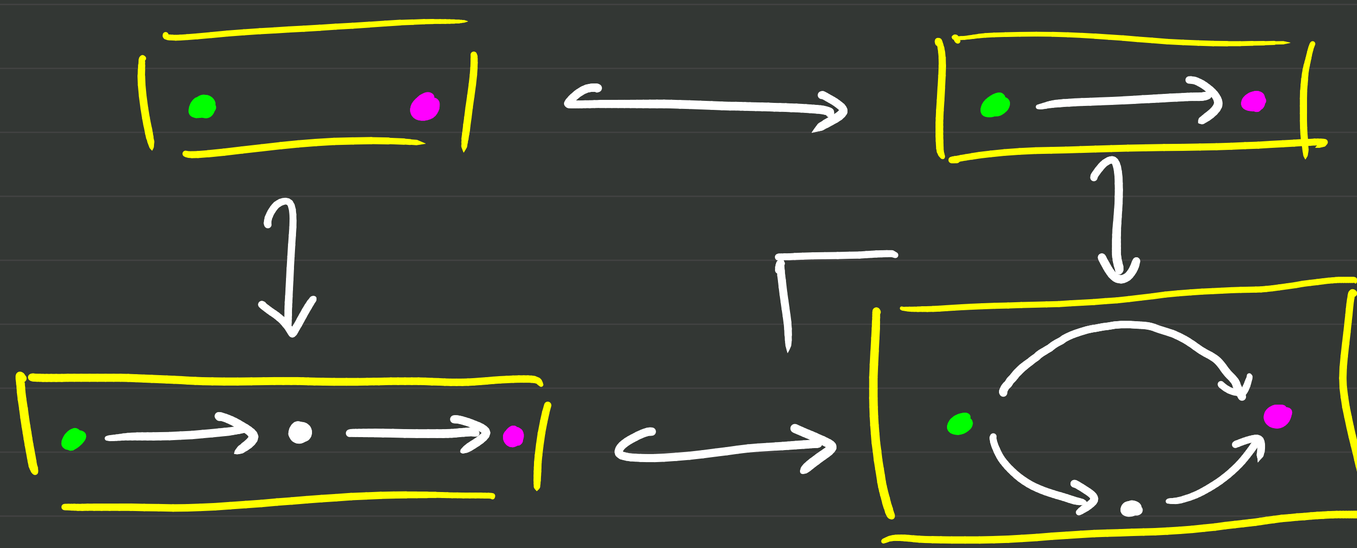
$$\begin{tikzcd}
\partial_{n-1} U \arrow[r, "\sim"] \arrow[d, hook] & \partial_{n-1} V \arrow[r, hook] & V \arrow[d] \\
U \arrow[rr] & & \partial(U \Rightarrow V) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

- Add a greatest element $\top$ with $\partial_n^- \top = U, \quad \partial_n^+ \top = V$

Example :

$U =$ 

$V =$ 



$U \Rightarrow V =$ 

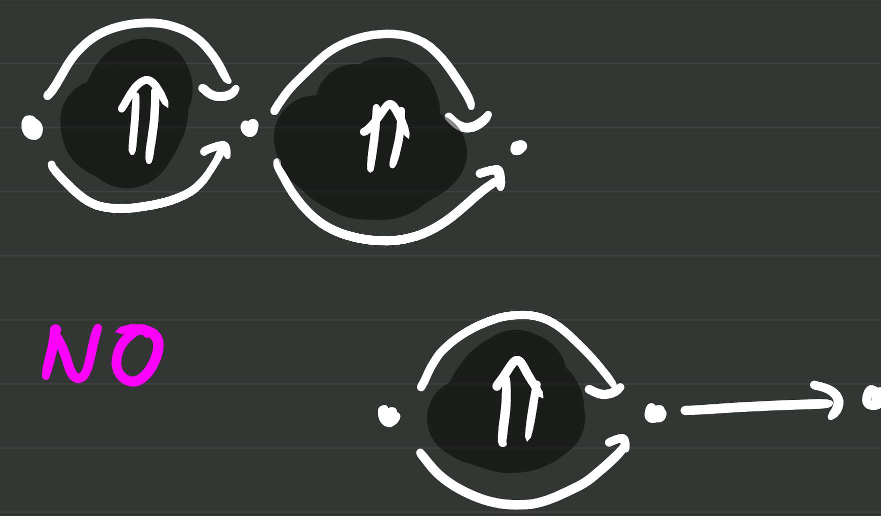
For all $U \in \mathcal{R}$, $k < \dim(U)$,

$$\partial_{k-1} U \subseteq \partial_k^- U \cap \partial_k^+ U .$$

Def U has **spherical boundary** if these are equalities.

(cfr S. Henry, "REGULAR POLYGRAPHS")

YES



NO

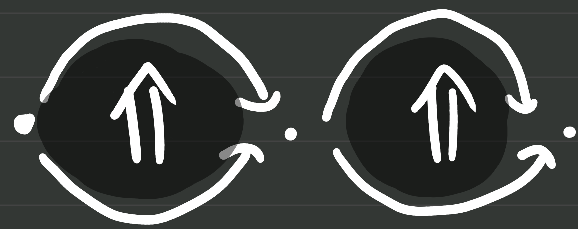
3. (PASTE):

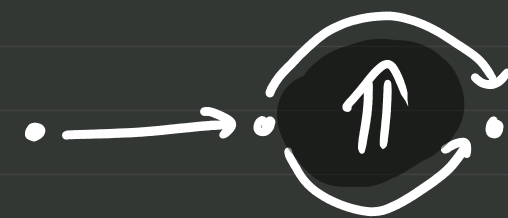
Glue U, V along the (unique!) isomorphism of $\partial_k^+ U$, $\partial_k^- V$.

$$\begin{tikzcd}
\partial_k^+ U \arrow[r, hook, "\sim"] \arrow[d, hook] & \partial_k^- V \arrow[r, hook] & V \arrow[d, hook] \\
U \arrow[rr, hook] & & U \#_k V \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

The equations of globular composition hold up to unique isomorphism.

Example:

 as

1.  $\#_1$ 

2.  $\#_1$ 

3.  $\#_0$ 

THE SHAPE CATEGORY $\odot$ (ATOM)

- **Objects:** Regular atoms
- **Morphisms:** Maps of o.g. posets

Factor as surjective (co-degeneracies)
followed by injective (co-faces)

$\odot\underline{\mathrm{Set}} := [\odot^{op}, \underline{\mathrm{Set}}]$

$\odot$ contains the following as full subcategories:

- The category of SIMPLICES;
- The REFLEXIVE GLOBE category;
- The category of CUBES WITH CONNECTIONS;
- The category of POSITIVE OPETOPES WITH CONTRACTIONS.

$\odot$ is closed under

- All DIRECTION-REVERSING dualities (like CUBES, GLOBES);
- SUSPENSIONS (like GLOBES);
- GRAY PRODUCTS (like CUBES);
- JOINS (like SIMPLICES)

Regular molecules & their maps can be Yoneda-embedded in $\odot\underline{\mathbf{Set}}$.

Terminology:

U molecule, X diagrammatic set

- A DIAGRAM in X of SHAPE U is a morphism $U \longrightarrow X$.
- It is COMPOSABLE if U has spherical boundary.
- It is a CELL if U is an atom.

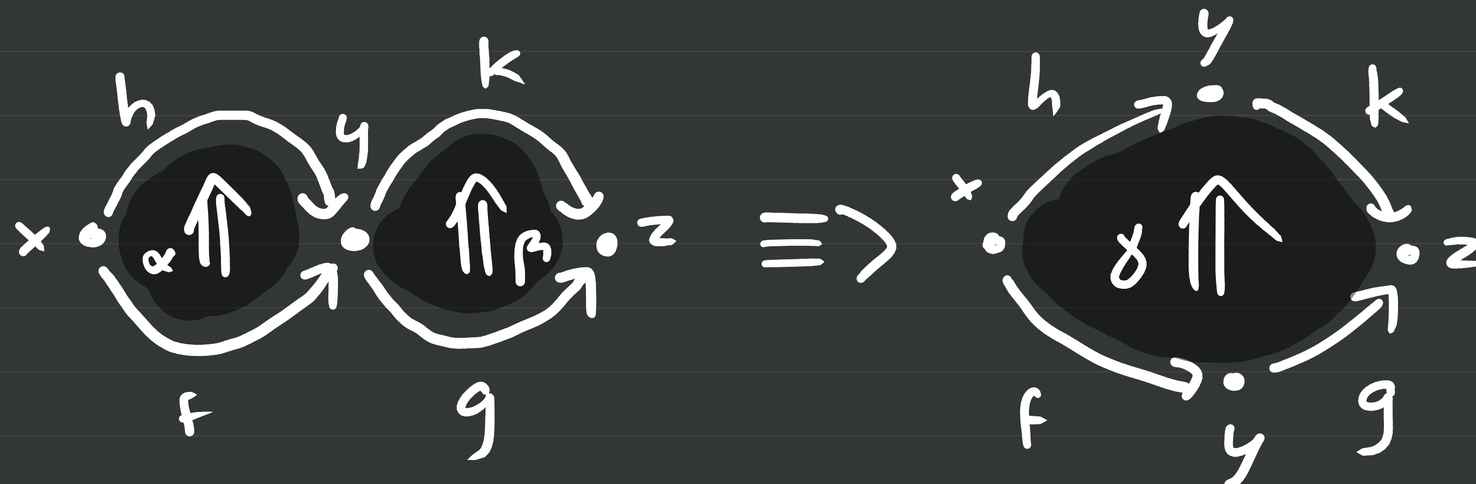
$\boxed{\#0}$ EXPRESSIVENESS

Very similar to polygraphs; the main restriction is the "spherical boundary" constraint on cell shapes

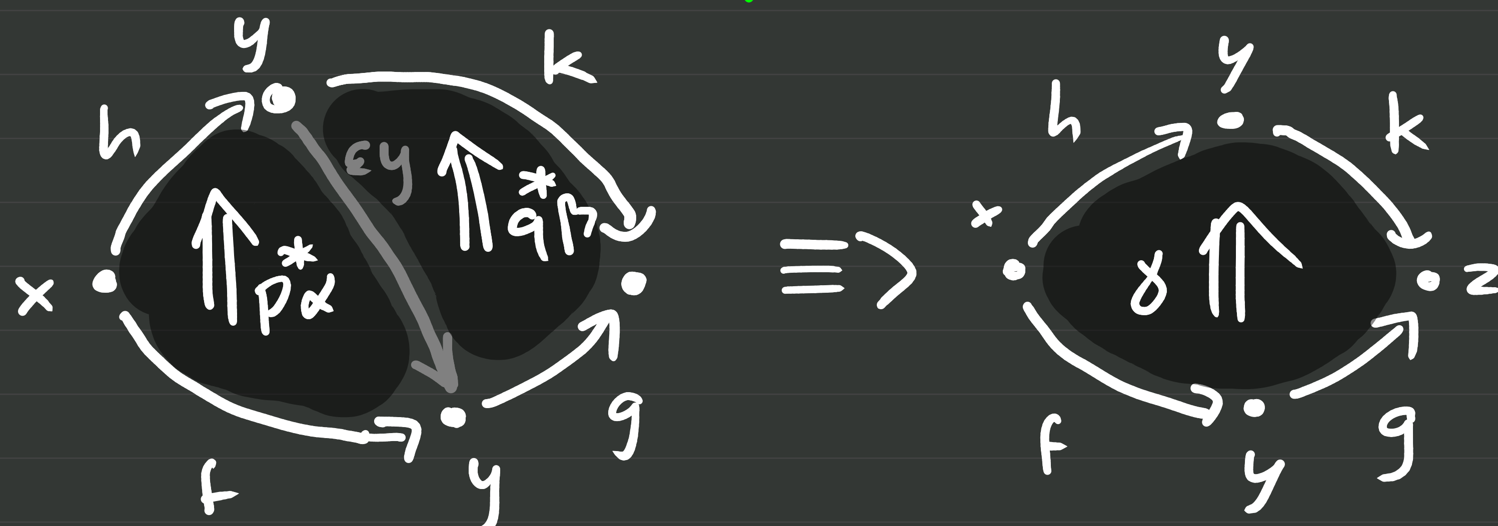
$\mapsto$ No "strictly degenerate" boundaries!

However, DEGENERACIES give access to "WEAK UNITS & UNITORS" that can be used to "regularise" shapes.

Example:



can be replaced with



$\boxed{\#1}$ TOPOLOGICAL SOUNDNESS

$$
\begin{tikzcd}
\odot \arrow[r, "\text{Forget}"] \arrow[d, "\text{Yoneda}"] & \underline{\mathrm{Pos}} \arrow[r, "\text{Nerve}\ N"] & s\underline{\mathrm{Set}} \arrow[r, "\text{Realise}\ |-|"] & \underline{\mathrm{Spaces}} \arrow[lll, bend left, "S_{\odot}"] \\
\odot\underline{\mathrm{Set}} \arrow[rrru, bend left=10, "\text{Lan}_y\ |-|"] \arrow[rrru, phantom, "\perp"] & & &
\end{tikzcd}
$$

Prop If U is an n-dim. atom,

- $|U| \cong D^n$
- $|\partial U| \cong S^{n-1}$.

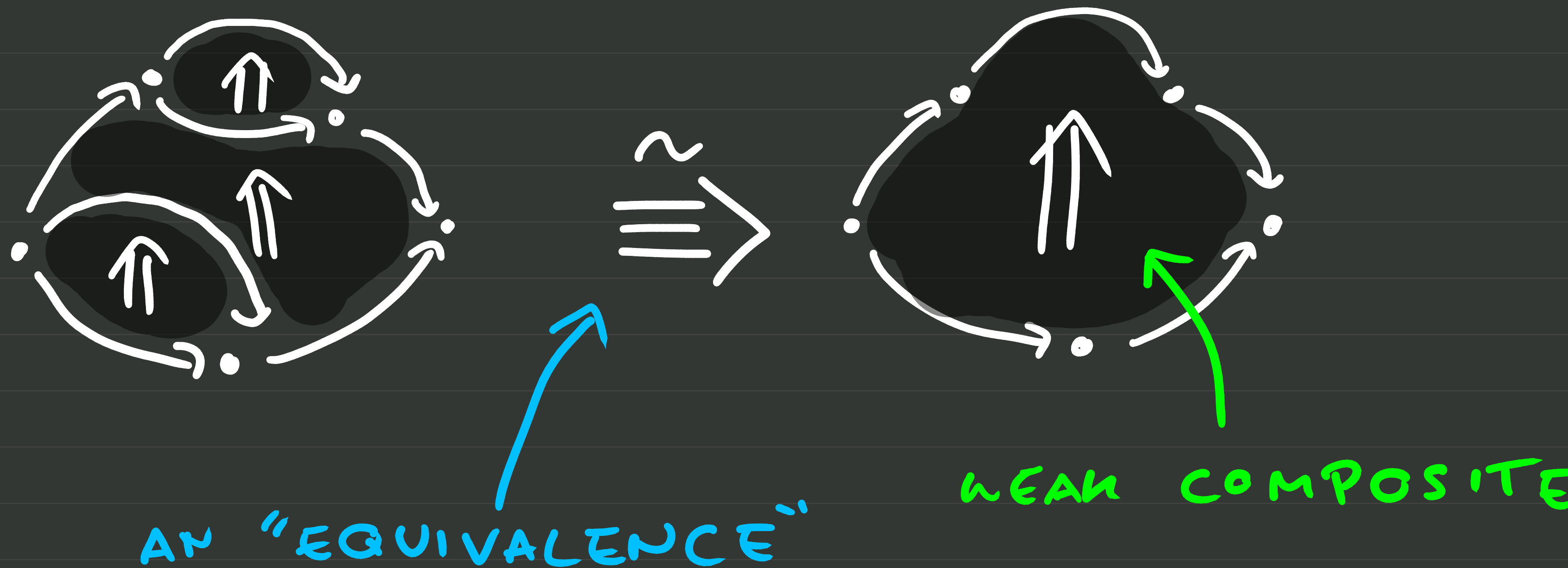
(Almost) **Corollary** If X is a "cell complex" with generating cells $\{x_i : U_i \longrightarrow X\}_{i\in I}$, then $|X|$ is a CW complex with generating cells $\{|x_i| : |U_i| \longrightarrow |X|\}_{i \in I}$.

$\boxed{\#2}$ HIGHER-CATEGORICAL SEMANTICS

Idea (shared with complicial & opetopic):

> A diagrammatic set is a higher category if every composable diagram is equivalent to a single cell (its weak composite).

The equivalence is exhibited by a higher diagram (compositor).



A BIG COMPUTATIONAL ADVANTAGE:

The combinatorics of shapes is rich enough to support an algebraic definition of equivalences as pseudoinvertible diagrams (after E. Cheng).

- ALL MORPHISMS preserve equivalences.

- We can localise a diagrammatic set at a set of cells simply by attaching "pseudoinverses" & higher witnesses of weak inversion.

- Let $U \simeq V$ be the localisation of the atom $U \Rightarrow V$ at the "tautological" cell.
Then morphisms $U \simeq V \longrightarrow X$ classify equivalences of shape $U \Rightarrow V$ in X.

If U is a molecule with spherical boundary, let $\langle U \rangle$ be the unique atom with $\partial \langle U \rangle$ isomorphic to ∂U.

Prop. The following are equivalent:

- X has weak composites;
- for all composable $x : U \to X$, there exists an extension

acyclic cofibrations

$$\begin{tikzcd}
U \arrow[rr, "x"] \arrow[d, hook] & & X \\
U \simeq \langle U \rangle \arrow[urr, "c(x)"'] & &
\end{tikzcd}.$$

Conjecture Diagrammatic sets with weak composites are equivalent to other models of (∞,∞)-cats (in the "coinductive" sense)

"Strategy":

- Prove equivalence with complicial sets (I have "candidate" Quillen adjunctions);
- Wait for proof that complicial sets are equivalent to Segal models...

FURTHER WORK

- "THE SMASH PRODUCT OF MONOIDAL THEORIES": A topological construction applied to presentations of higher algebraic theories

  (arXiv: 2101.10361)

- Joint WIP with Diana Kessler: a proof assistant for HDR based on diagrammatic sets (similar to homotopy.io)
- Developing more RT in this framework – effect of "weak unit" handling?
- Stronger semi-strictification results?

MERCI POUR VOTRE ATTENTION !

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