Merge-bicategories: towards semi-strictification of higher categories

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The goal:

A combinatorial understanding of semi-strictification for higher categories, in the style of Mac Lane's coherence theorem, or the coherence proofs based on string diagrams

Preliminary observations

1

No **full** strictification (of associativity, interchange, and units) is possible.

We are pursuing strictification of associativity and interchange in the style of "C. Simpson's conjecture for regular compositions" (S. Henry, soon to appear)

Preliminary observations

2

Notions of "fully weak" higher categories and functors are usually non-algebraic (e.g. presheaves with horn filling properties), but **strictness** is always relative to an algebraic theory.

However: certain algebraic higher categories may have an underlying space with a "horn filling" property, which makes it a non-algebraic higher category. Then one can speak of the property of the underlying space of **admitting** the algebraic structure of a semi-strict higher category

Preliminary observations

3

The combinatorial coherence theorems often rely on the following step: they define a combinatorial notion of composable diagram of "elementary cells" of X, and then promote it to an elementary cell of the semi-strictification of X.

Thus it is expected that we should work with a combinatorial presentation of higher-dimensional pasting diagrams.

Preliminary observations

4

There is no satisfactory "shape category" for arbitrary ω-categorical pasting diagrams (because the category of polygraphs is not a presheaf category).

We work with more restrictive composable diagrams, which nevertheless are "enough" in the presence of weak units: they have the property that composable diagrams of n-cells have k-boundaries "shaped like topological k-balls", for all $k \leq n$.

Combinatorics of regular polygraphs in arbitrary dimension:

A combinatorial-topological shape category for polygraphs, arXiv 1806.10353

This talk: "2-truncated" regular polygraphs
(merge-bicategories)

Weak units, divisible cells, and coherence via universality for bicategories, arXiv 1803.06086

The category **MrgBiCat**

Merge-bicategories

Regular 2-polygraphs with an associative composition of 2-generators (in the regular sense), and maps of polygraphs that preserve it

2-cells ($n, m > 0$):

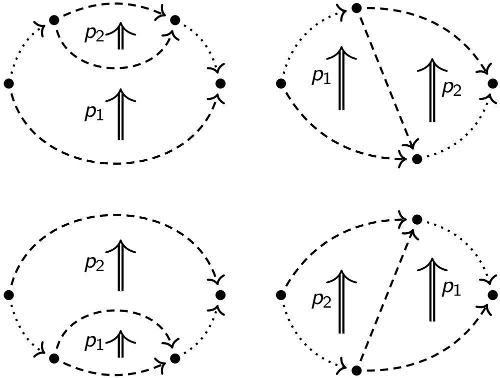
$$
\begin{tikzcd}
 & x_2^+ \arrow[r, dashed] & x_m^+ \arrow[dr, "b_m"] & \\
x^- \arrow[ur, "b_1"] \arrow[dr, "a_1"'] \arrow[rrr, phantom, "\Uparrow p"] & & & x^+ \\
 & x_2^- \arrow[r, dashed] & x_n^- \arrow[ur, "a_n"'] &
\end{tikzcd}
$$

Also write: $p : (a_1, \ldots, a_n) \to (b_1, \ldots, b_m)$,

$$\partial_i^- p = a_i, \qquad \partial_j^+ p = b_j$$

Mergers of 2-cells

Composition:



Representable regular polygraphs

A regular polygraph X is *representable* if, for all composable n-dimensional diagrams x of X, there exist an n-generator $\underline{x}$ and an $(n+1)$-**equivalence** y with $\partial^\alpha y = x$ and $\partial^{-\alpha} y = \underline{x}$.

A *strong map* of regular polygraphs is a map which preserves **equivalences**.

Representable merge-bicategories

Representable merge-bicategory

The 2-truncation of a representable regular polygraph.

$\rightsquigarrow$ Instead of 3-equivalences, we will have “equalities” via the algebraic composition

Equivalences from divisibility

We define a notion of **divisibility** of a cell x of shape U at a sub-diagram V of its boundary $\partial^\alpha U$.
An equivalence is a cell x of shape U which is divisible at $\partial^- U$ and at $\partial^+ U$.

Divisibility of 1-cells

$e : x \to x'$ divisible at ∂^+

For all $a : x \to y$, $a' : x' \to y$, there exist

$$
\begin{tikzcd}
x \arrow[rr, "a"] \arrow[dr, "e"'] & & y \\
& x' \arrow[ur] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
x \arrow[rr] \arrow[dr, "e"'] & & y \\
& x' \arrow[ur, "a'"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
$$

which are **equivalences**, and are divisible at the marked location (+ same thing with 2-cells pointing in the other direction)

Divisibility of 1-cells

$e : x \to x'$ divisible at ∂^-

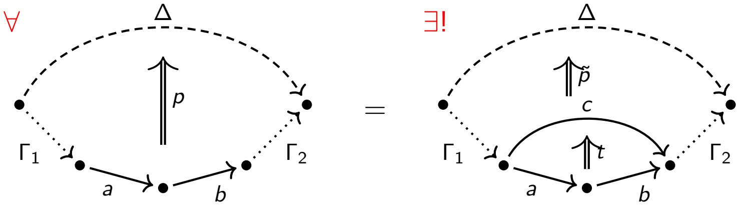
For all $b : z \to x$, $b' : z \to x'$, there exist

$$\begin{tikzcd} z \arrow[rr, "b'"] \arrow[dr] & & x' \\ & x \arrow[ur, "e"'] \arrow[u, phantom, "\Uparrow"] & \end{tikzcd} \quad , \quad \begin{tikzcd} z \arrow[rr] \arrow[dr, "b"'] & & x' \\ & x \arrow[ur, "e"'] \arrow[u, phantom, "\Uparrow"] & \end{tikzcd}$$

which are **equivalences**, and are divisible at the marked location
(+ same thing with 2-cells pointing in the other direction)

Divisibility of 2-cells

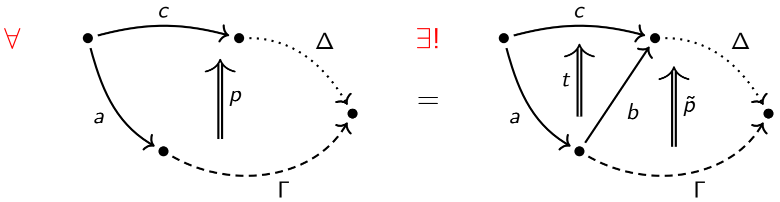
$t : (a, b) \to (c)$ divisible at ∂_1^+



From 2-truncation: **equivalence** $\rightsquigarrow$ equality,
divisibility at location of $\tilde{p}$ $\rightsquigarrow$ uniqueness

Divisibility of 2-cells

$t : (a, b) \to (c)$ divisible at ∂_2^-



From 2-truncation: **equivalence** $\rightsquigarrow$ equality,
divisibility at location of $\tilde{p}$ $\rightsquigarrow$ uniqueness

Representable merge-bicategories and bicategories

There is a functor $G : \mathbf{MrgBiCat} \to \mathbf{2Gph}$, restricting to single-input, single-output cells.

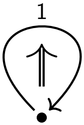
Theorem

If X is a representable merge-bicategory, GX admits the structure of a bicategory.
If $f : X \to Y$ is a strong morphism of representable merge-bicategories, $Gf : GX \to GY$ admits the structure of a pseudofunctor of bicategories.

This can be made into one side of an equivalence of large categories.

Weak units

The definition of a weakly associative composition of 1-cells is the same as in the opetopic approach (Hermida '00 for the 2-truncated case).
However, the opetopic approach builds units out of universal cells with "degenerate" boundaries:



2-out-of-3 for 1-equivalences

Theorem

Suppose

$$
\begin{tikzcd}
x \arrow[rr, "e''"] \arrow[dr, "e"'] & & x'' \\
& x' \arrow[ur, "e'"'] \arrow[u, phantom, "p \Uparrow"] &
\end{tikzcd}
$$

is divisible at (the location of) two 1-cells in its boundary. If two of e, e', e'' are 1-equivalences, then the third is also a 1-equivalence, and p is divisible at each of its 1-cells.

Units from equivalences

Dividing a 1-equivalence by itself,

$$\begin{tikzcd} x \arrow[rr, "e"] \arrow[dr, "e"'] & {} \arrow[d, phantom, "\Uparrow p"] & x' \\ & x' \arrow[ur, "1_{x'}"'] & \end{tikzcd}$$

we obtain a 1-equivalence $1_{x'}$. Using the divisibility properties of p, we see that $1_{x'}$ is also idempotent.

The 1-equivalence $1_{x'}$ is a cancellable (pseudo-)idempotent on GX (a *Saavedra unit*, proved by J. Kock to be equivalent to a "triangle equation" unit)

Higher morphisms

MrgBiCat has a natural monoidal biclosed structure with the lax Gray tensor product; for each X, Y, there is

- a merge-bicategory $[X, Y]_l$ whose 1-generators are *oplax* transformations, and
- a merge-bicategory $[X, Y]_r$ whose 1-generators are *lax* transformations.

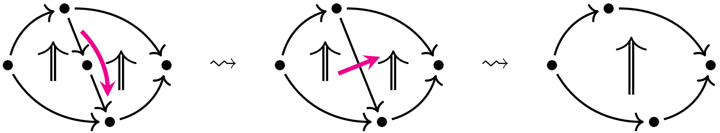
Theorem

If Y is representable, then $[X, Y]_l$ and $[X, Y]_r$ are representable.

From here we can recover higher morphisms of bicategories, with some *caveats*, and a notion of weak equivalence

Mergers

A notion of composition



Merge

Let X be a regular polygraph. Then $\mathcal{M}X$ is a regular polygraph with an n-generator $m_* x$ of shape U' for all n-cells x of X of shape U, and mergers $m : U \overset{*}{\rightsquigarrow} U'$.

Collapses

A notion of degeneracy/unit

$$\begin{tikzcd} & \bullet \arrow[dr, bend left] & \\ \bullet \arrow[ur, bend left] \arrow[dr, bend right] & \Uparrow & \bullet \\ & \bullet \arrow[ur, bend right] & \end{tikzcd} \quad \rightsquigarrow_c \quad \bullet \longrightarrow \bullet \longrightarrow \bullet$$

Inflate

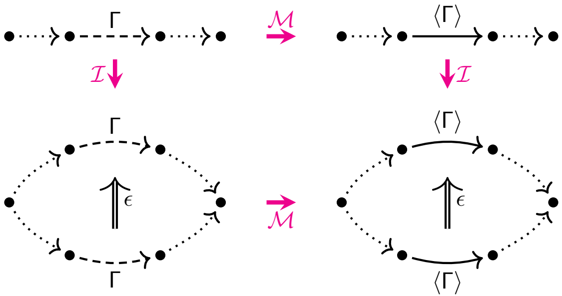
Let X be a regular polygraph. Then $\mathcal{I}X$ is a regular polygraph with a generator c^*x of shape U' for all diagrams x of X of shape P, and collapses $c : U' \overset{*}{\rightsquigarrow}_c P$.

Distributive law

Both $\mathcal{I}$ and $\mathcal{M}$ define monads on **MrgBiCat**.

Proposition

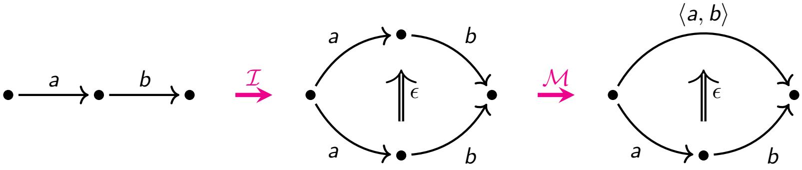
There is a distributive law from $\mathcal{I}\mathcal{M}$ to $\mathcal{M}\mathcal{I}$. Consequently, $\mathcal{T} := \mathcal{M}\mathcal{I}$ has the structure of a monad on **MrgBiCat**.



$\mathcal{T}$-algebras as semi-strict bicategories

Proposition

Let $\alpha : \mathcal{T}X \to X$ be a $\mathcal{T}$-algebra such that α is a strong morphism. Then X is representable, and GX admits canonically the structure of a strictly associative bicategory.



The semi-strictification argument

Let X be a representable merge-bicategory. Then:

1. X admits an $\mathcal{I}$-algebra structure $\beta : \mathcal{I}X \to X$ such that β is a strong morphism;
2. $\mathcal{M}X$ is representable, and the unit $X \to \mathcal{M}X$ is a weak equivalence (*a fortiori*, a strong morphism);
3. $\mathcal{M}X$ admits a $\mathcal{T}$-algebra structure $\alpha : \mathcal{T}(\mathcal{M}X) \to \mathcal{M}X$ such that α is a strong morphism.

The unit of $\mathcal{M}$ on X realises the semi-strictification of X.

Higher dimensions

Many things still to be checked, but there is no obvious reason (that I see) why this should not generalise to higher dimensions.

Work in progress: *Representable regular polygraphs*