The smash product of monoidal theories

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Higher Homotopical Structures
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- *The smash product of monoidal theories*, arXiv:2101.10361

- *The smash product of monoidal theories*, arXiv:2101.10361
- *Diagrammatic sets and rewriting in weak higher categories*, arXiv:2007.14505

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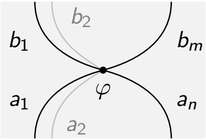

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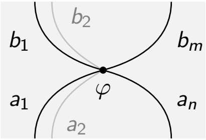

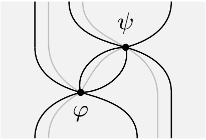

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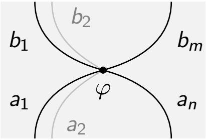

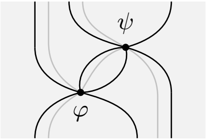

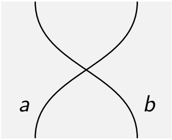

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$\rightsquigarrow$

Full subcategories of **Prop**, the category of props and symmetric monoidal functors that send sorts to sorts or the unit

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Models of $(T, \mathscr{T})$ in $\mathbf{M}$ form a category $\mathrm{Mod}_{\mathbf{M}}(T, \mathscr{T})$ with monoidal natural transformations as morphisms.

This category admits a symmetric monoidal structure.

(Idea: "run operations in parallel", use symmetry to redistribute inputs and outputs as needed)

The tensor product of props

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The tensor product $(T, \mathscr{T}) \otimes_{\mathbb{S}} (S, \mathscr{S})$ is determined universally by the requirement that

- models of $(T, \mathscr{T}) \otimes_{\mathbb{S}} (S, \mathscr{S})$ in $\mathbf{M}$

correspond naturally to

- models of $(S, \mathscr{S})$ in $\mathrm{Mod}_{\mathbf{M}}(T, \mathscr{T})$.

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The tensor product is part of a symmetric monoidal structure on **Prop** (Hackney–Robertson). The monoidal unit is the single-sorted prop $\mathbb{S}$ of permutations.

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Restricting this monoidal structure to

- *symmetric operads*, we recover the *Boardman–Vogt* product;
- *cartesian props*, we recover the "tensor product of algebraic theories".

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There is a single-sorted prop *Mon* whose models are monoids*.
Models of its dual $\mathit{Mon}^{\mathrm{co}}$ are comonoids.

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In general, $(T, \mathscr{T}) \otimes_{\mathbb{S}} \mathit{CMon}^{\mathrm{co}}$ is the free cartesian prop on $(T, \mathscr{T})$.

The smash product of pointed spaces

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$X \wedge S^1$ is the reduced suspension ΣX for each pointed space X.

Why on earth should these two be related?

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There is

- an embedding **Prop** $\hookrightarrow$ **Prob**, and
- a forgetful functor $\mathrm{U}\colon$ **Prob** $\to$ **Pro**,

with left adjoints $\mathrm{r}\colon$ **Prob** $\to$ **Prop** and $\mathrm{F}\colon$ **Pro** $\to$ **Prob**.

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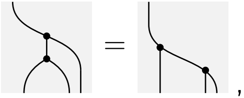
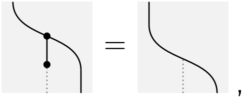
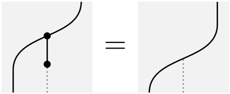

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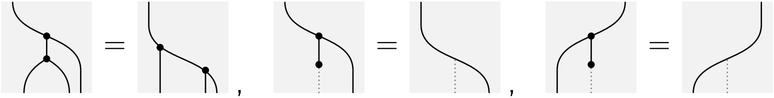
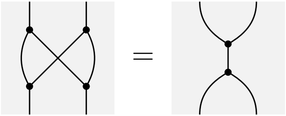


But their tensor product is not: in the theory of bialgebras,



(No "internal" tensor product on **Pro**!)

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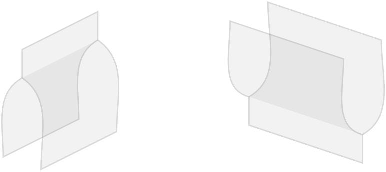

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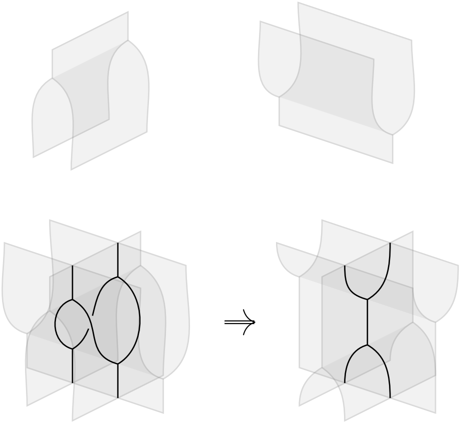

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The "sliding surfaces" picture produces

equations of diagrams in a tricategory (4d objects)

from

diagrams in a pair of bicategories (2d objects).

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There is an *external* tensor product $- \otimes - : \mathbf{Pro} \times \mathbf{Pro} \to \mathbf{Prob}$

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We recover the tensor product of props from the external product of their underlying pros, by imposing that a natural family of inclusions of the factors into their product preserve braidings.

From monoidal theories to pointed directed spaces

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If higher categories are "spaces of directed cells", we may see

- a pro as the loop space of a pointed directed 2-type,
- a prob as the 2-fold loop space of a pointed directed 3-type.

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In the directed space picture, the set of sorts should become

- a 1-dimensional cell complex structure for pros,
- a 2-dimensional cell complex structure for probs.

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The connection with smash products seems more plausible...

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5. and a “direction-forgetting” functor to pointed spaces
6. that sends smash products to smash products.

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- the "Gray smash product" of two monoidal categories (as pointed 2-categories) is not a braided monoidal category, but a highly degenerate *commutative* monoidal category;
- there is no functor from $\omega\mathbf{Cat}$ to a category of spaces that works in the intended way.

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- a notion of directed space with a combinatorial model of directed cells.

Diagrammatic sets

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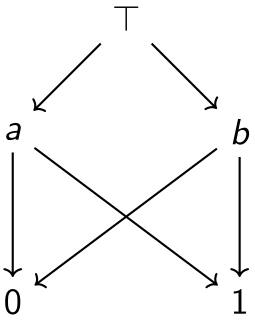

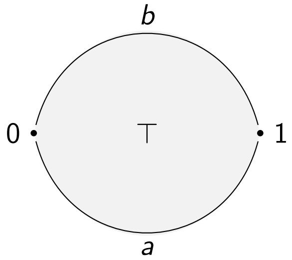

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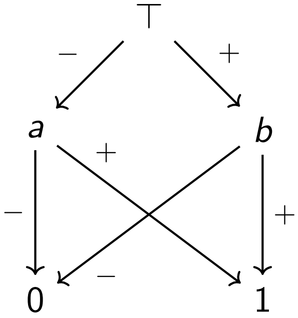

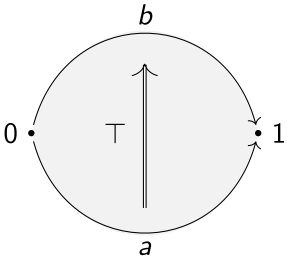

and to a higher-categorical diagram its oriented face poset.

Regular directed complexes

(After R. Steiner, *The algebra of directed complexes*, 1993)

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- An *oriented graded poset* is a finite graded poset with an orientation.
- If $U \subseteq P$ is (downward) closed, $\alpha \in \{+, -\}$, $n \in \mathbb{N}$,

$$\Delta_n^\alpha U := \{x \in U \,|\, \dim(x) = n \text{ and if } y \in U \text{ covers } x, \text{ then } o(y \to x) = \alpha\},$$
$$\partial_n^\alpha U := \mathrm{cl}(\Delta_n^\alpha U) \cup \{x \in U \,|\, \text{for all } y \in U, \text{ if } x \le y, \text{ then } \dim(y) \le n\},$$
$$\Delta_n U := \Delta_n^+ U \cup \Delta_n^- U, \qquad \partial_n U := \partial_n^+ U \cup \partial_n^- U.$$

Regular directed complexes

If U is a closed subset of P, then U is a *molecule* if either

- U has a greatest element, in which case we call it an *atom*, or
- there exist molecules U_1 and U_2, both properly contained in U, and $n \in \mathbb{N}$ such that $U_1 \cap U_2 = \partial_n^+ U_1 = \partial_n^- U_2$ and $U = U_1 \cup U_2$.

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A molecule U has *spherical boundary* if, for all $k < n$,

$$\partial_k^+ U \cap \partial_k^- U = \partial_{k-1} U.$$

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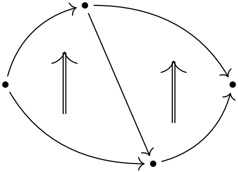

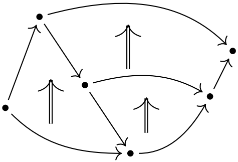

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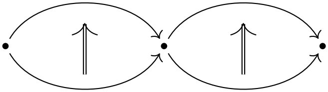

Regular directed complexes

An oriented graded poset P is a *regular directed complex* if, for all $x \in P$ and $\alpha, \beta \in \{+, -\}$,

1. $\mathrm{cl}\{x\}$ has spherical boundary,
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A map factors essentially uniquely as a *surjection* followed by an *inclusion*.

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Proposition

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In particular, the underlying poset of a regular n-dimensional atom is the face poset of a regular CW n-ball.

- Let $\odot$ be (a skeleton of) the full subcategory of $\mathbf{DCpx}^{\mathcal{R}}$ on the atoms. A diagrammatic set is a presheaf on $\odot$.

Gray product and smash product

Let P, Q be oriented graded posets.
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This is part of a monoidal structure on $\mathbf{DCpx}^{\mathcal{R}}$,
which restricts to $\odot$,
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This allows us to define a (Gray) smash product $(X, \bullet_X) \oslash (Y, \bullet_Y)$ of pointed diagrammatic sets,
part of a monoidal biclosed structure on $\odot\mathbf{Set}_\bullet$.

Gray product and smash product

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The main theorem

The bulk of the article is the definition of adjunctions relating

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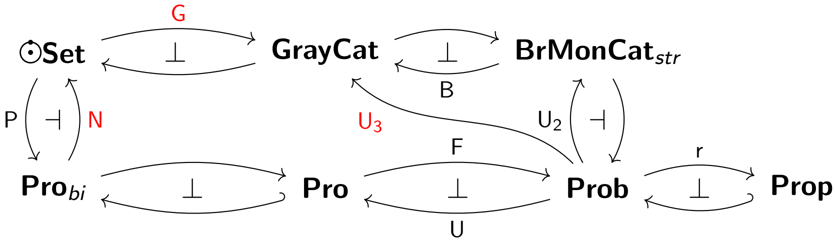

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The diagram of functors

$$
\begin{array}{ccccc}
\mathbf{Pro} \times \mathbf{Pro} & \xrightarrow{\;- \otimes -\;} & \mathbf{Prob} & & \\
 & & & \searrow^{U_3} & \\
\Big\downarrow{\scriptstyle N \times N} & & & & \mathbf{GrayCat} \\
 & & & \nearrow_{G} & \\
\odot\mathbf{Set}_\bullet \times \odot\mathbf{Set}_\bullet & \xrightarrow[\;- \owedge (-)^\circ\;]{} & \odot\mathbf{Set}_\bullet & &
\end{array}
$$

commutes up to natural isomorphism.

Picturing Gray products

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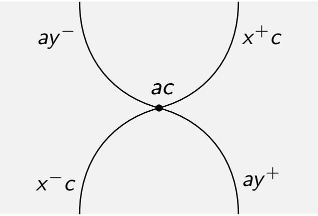
$a\colon x^- \Rightarrow x^+$ 1-cell in X, $c\colon y^- \Rightarrow y^+$ 1-cell in Y



Picturing Gray products

$\varphi\colon (a_1, \ldots, a_n) \Rightarrow (b_1, \ldots, b_m)$ 2-cell in X,

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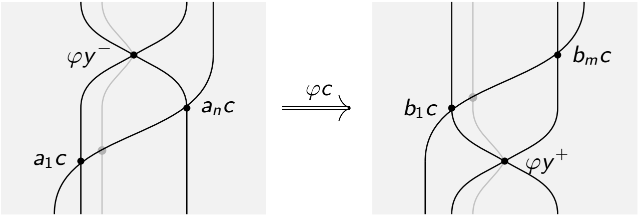

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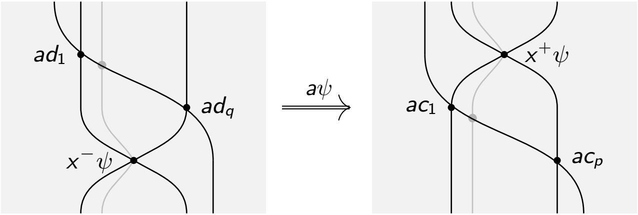

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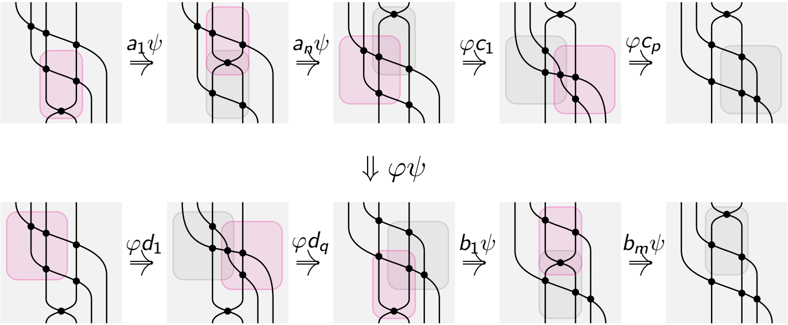

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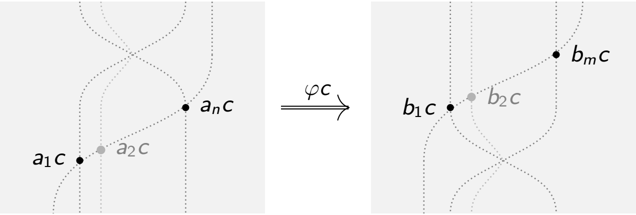

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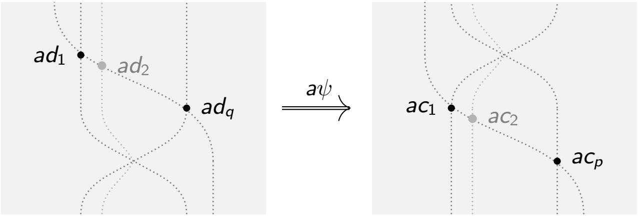

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We compute some cells in the smash product
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$1 \otimes 1$ is the only non-degenerate 2-cell in $X \owedge Y$

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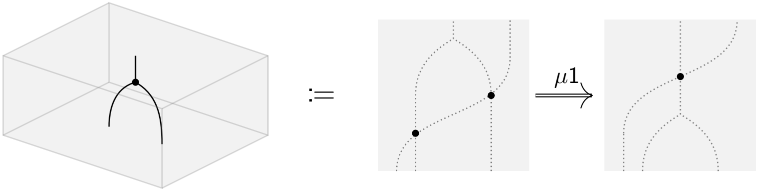

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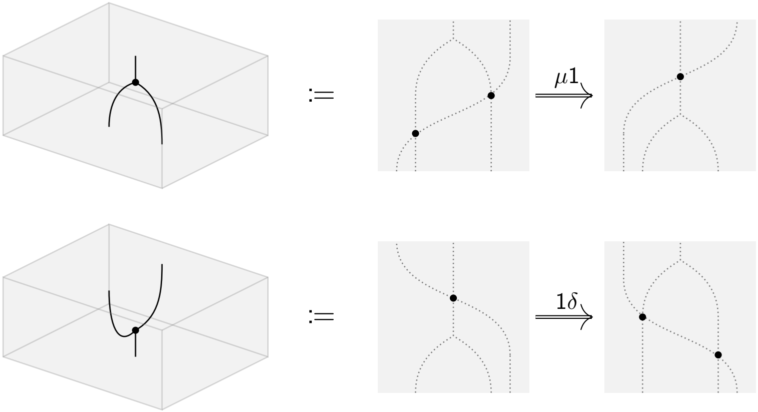

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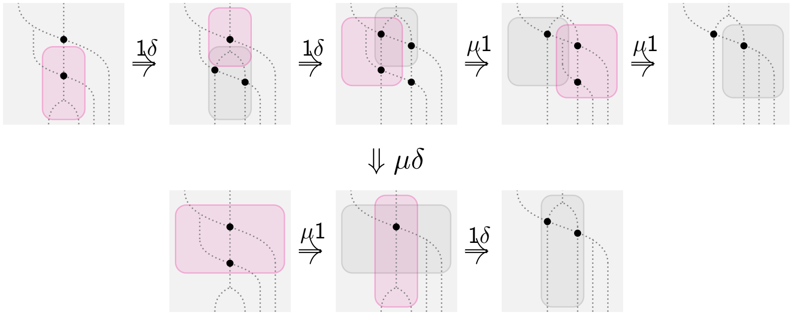

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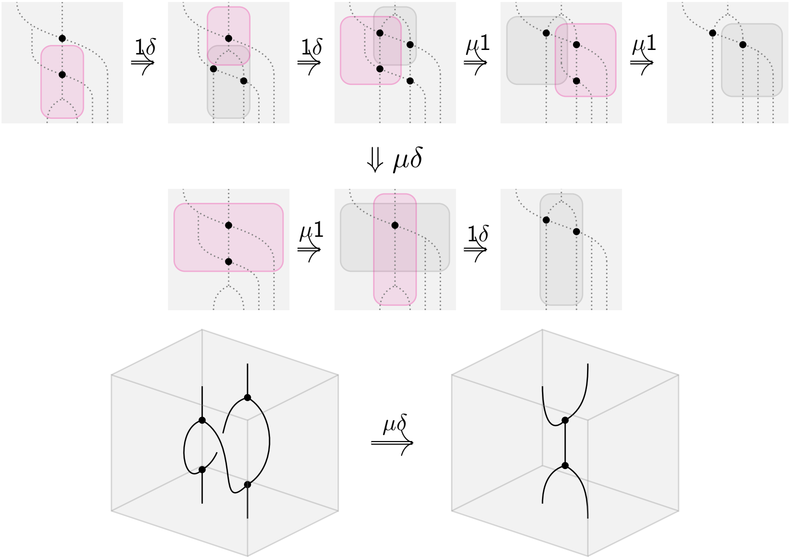

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- If X and Y have interesting oriented n-cells, then $X \oslash Y$ has interesting oriented k-cells up to $k = 2n$!

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Idea: Given presentations X of $(T, \mathscr{T})$ and Y of $(S, \mathscr{S})$, the smash product $X \owedge Y^\circ$ produces

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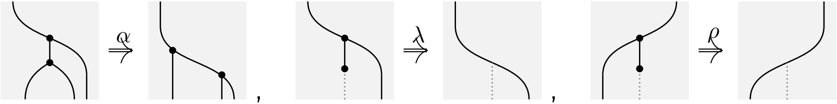
1. a presentation (with oriented equations) of $(T, \mathscr{T}) \otimes (S, \mathscr{S})$,
2. **plus** higher-dimensional coherence cells, or oriented *syzygies*, for this presentation.

Towards compositional higher rewriting

Let X be a presentation of *Mon* with the 3-cells



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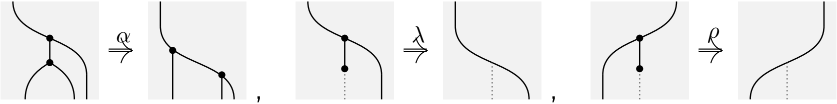

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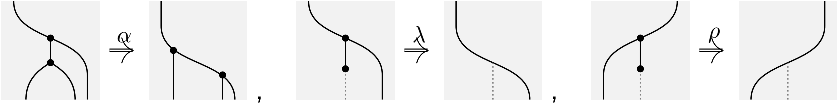

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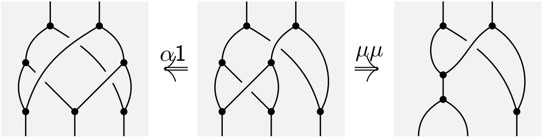
It has the following "new" critical branching:



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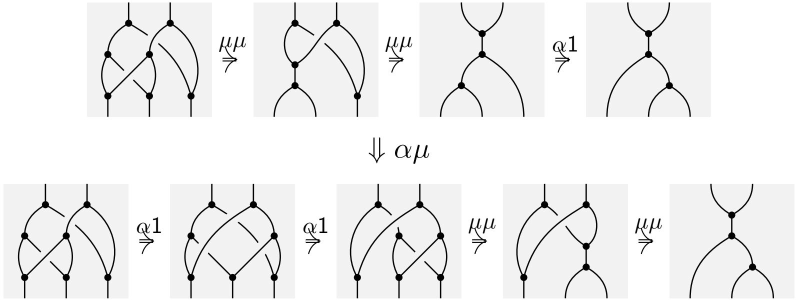

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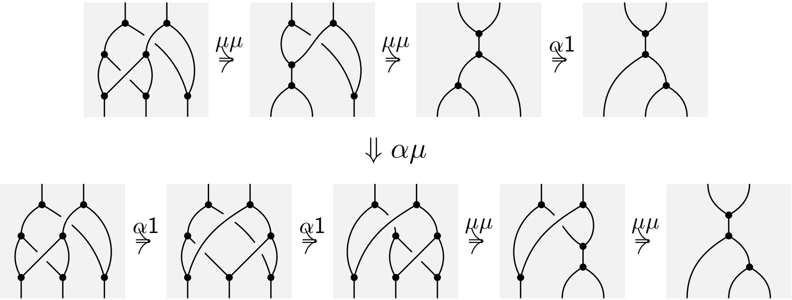

6-cells such as $\alpha \otimes \alpha$ are *higher syzygies* exhibiting confluence at critical branchings of syzygies

Towards compositional higher rewriting

Question:

If we start from presentations with nice computational properties or nice homotopical properties,

do we obtain nice presentations of their tensor product?

Outlook

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Diagrammatic sets can be a (homotopically sound) context for presentations of higher-algebraic theories with oriented generators in arbitrary dimension.

- *k-tuply monoidal* n-dimensional theories $\sim$
  k-fold directed loop spaces

  ($k = 1$: monoidal, $k = 2$: braided monoidal)

Outlook

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n-tuply monoidal $\owedge$ k-tuply monoidal $=$ $(n+k)$-tuply monoidal

Symmetric monoidal $=$ **stable** (k-tuply monoidal for each k)

...which is why props are closed under tensor products,
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Thank you for listening!