Bringing compositionality to rewriting theory via polygraphs

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RIMS — Kyoto University

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My focus: **rewriting theory** v **homotopy theory**

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Homotopy ideas appear in “non-geometric” areas of mathematics when there is a need to store some “intensional” information, while keeping access to its “extensional” projection

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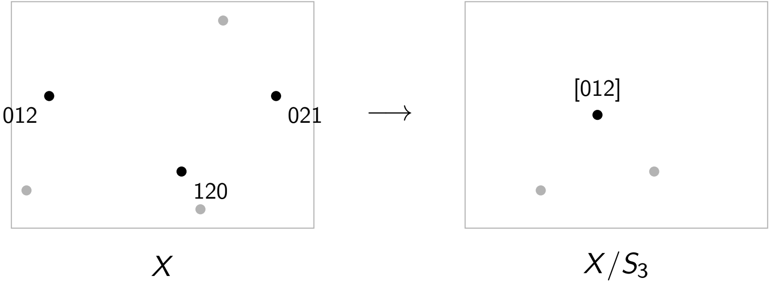
- Example: interpretation of identity types in intensional type theory (from Hofmann-Streicher to univalent foundations)
- But there are simpler examples

A combinatorial example

Finite set with a group action (example: triples of numbers in $\{0, \ldots, 4\}$, with action of permutations on $\{0, 1, 2\}$)



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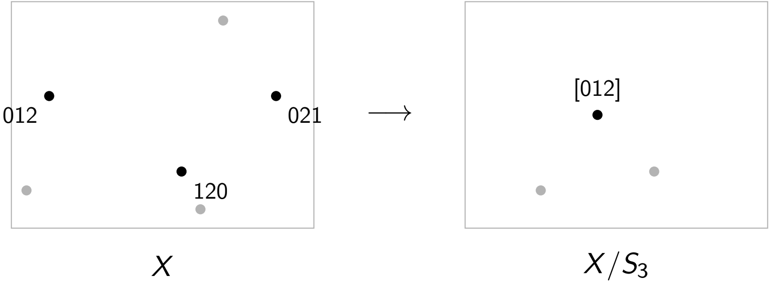

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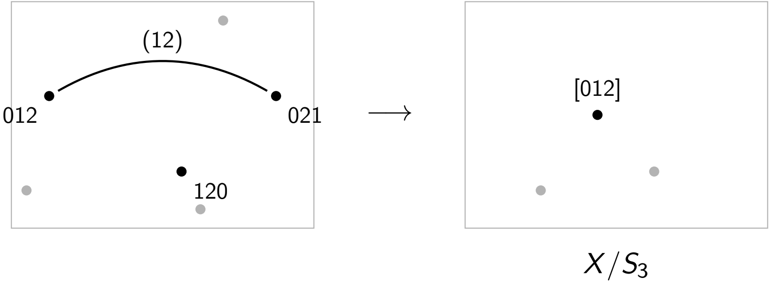

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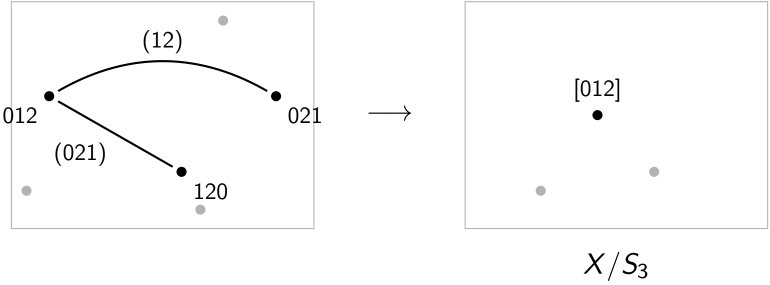

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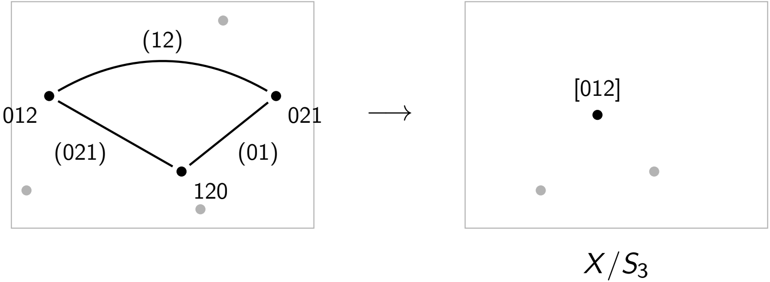

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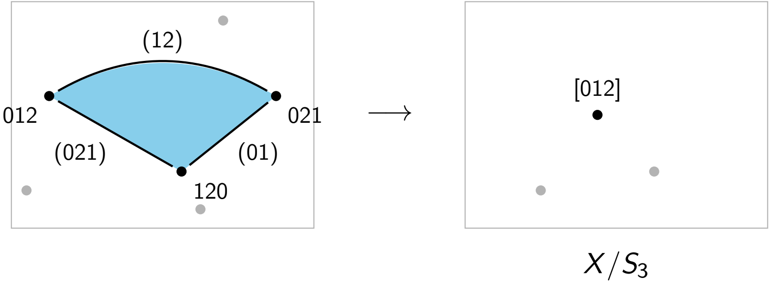

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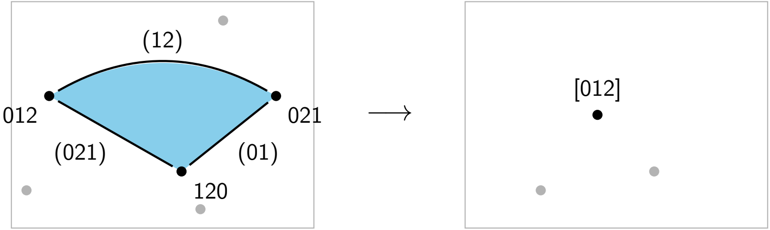

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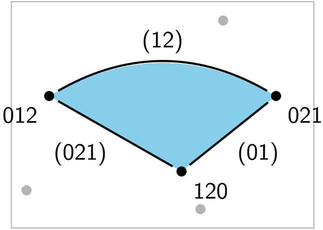
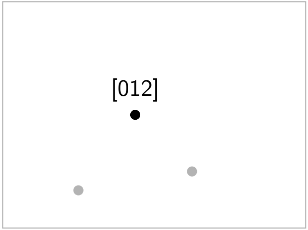


$\longrightarrow$ **truncation**



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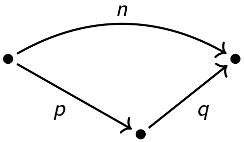

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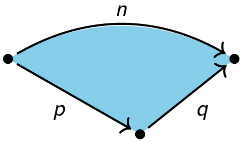

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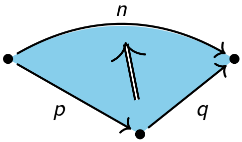

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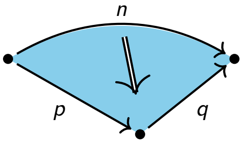

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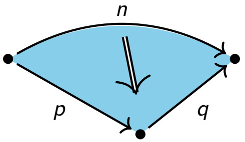

Higher-dimensional rewriting theory studies "spaces of directed cells"

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- **Polygraphs** (Street, Burroni): the combinatorics are given by strict higher categories

Standard in HDR (Lafont, Métayer, Mimram, Malbos, ...)

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- 1-dimensional polygraph $\sim$ (labelled) abstract rewriting system

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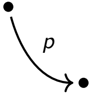

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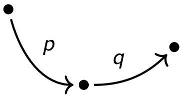

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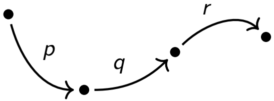

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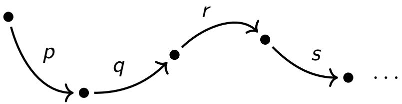

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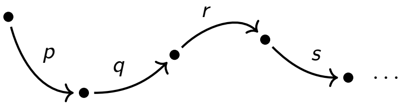

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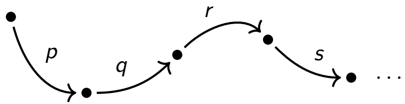

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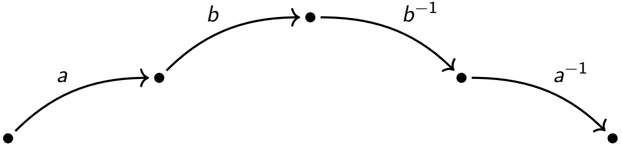

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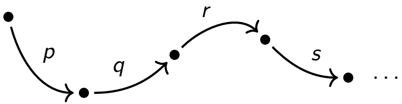

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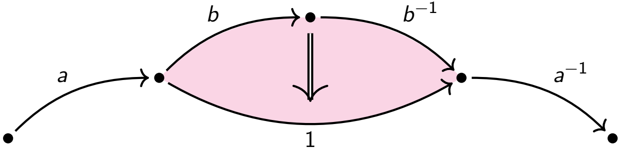

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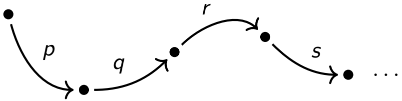

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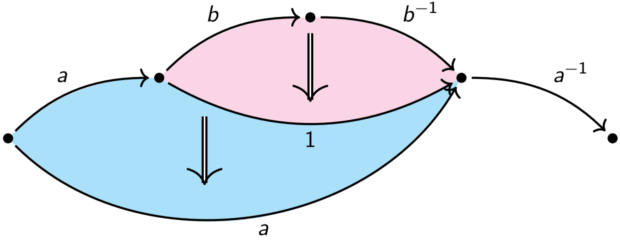

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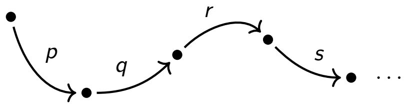

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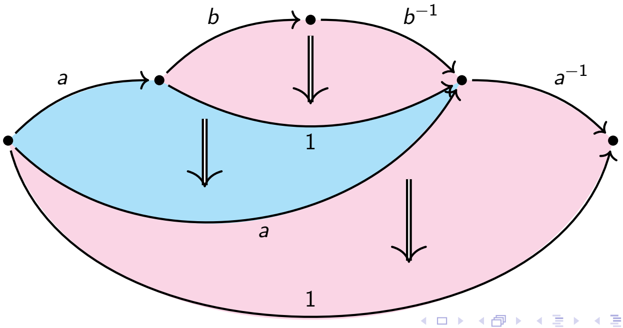

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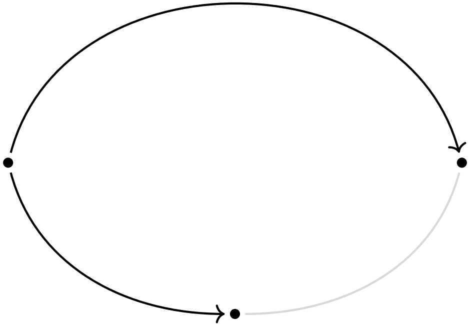

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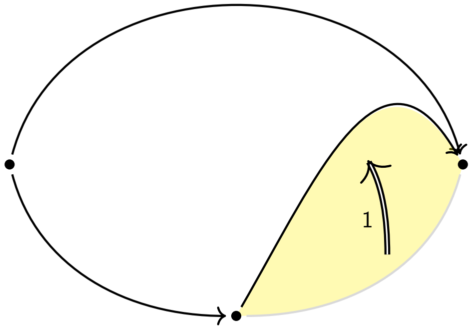

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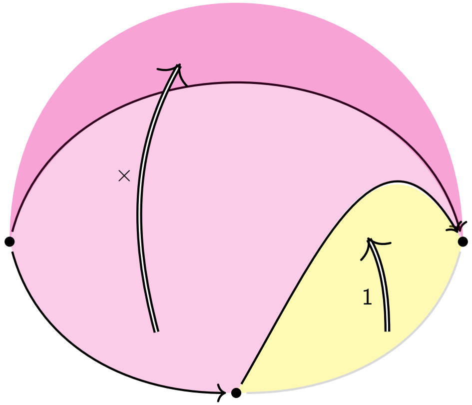

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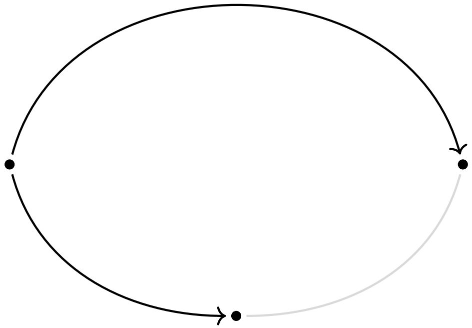

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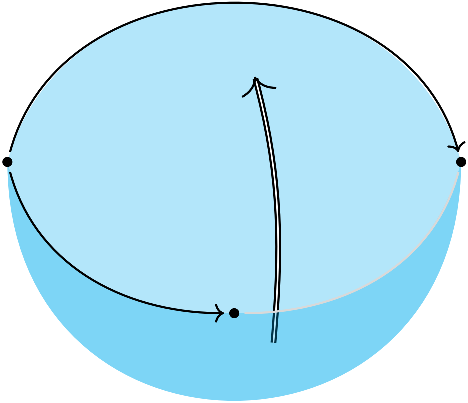

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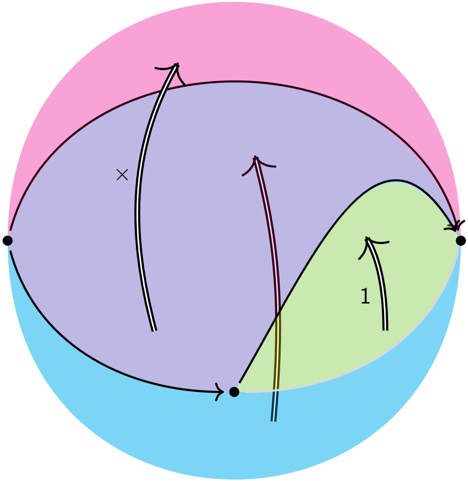

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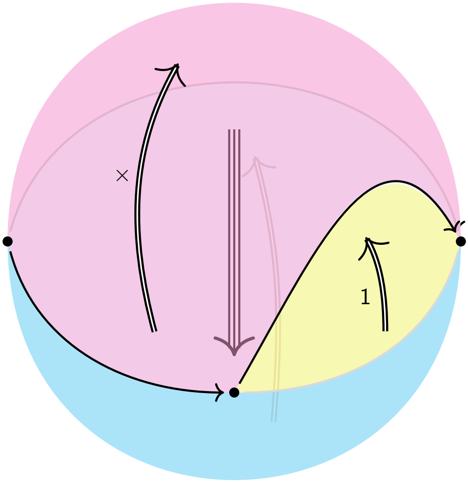

Polygraphic resolutions

Resolution of a higher category X: surjective map $f : P \to X$, equalities lift to proper cells

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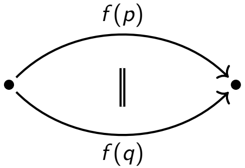

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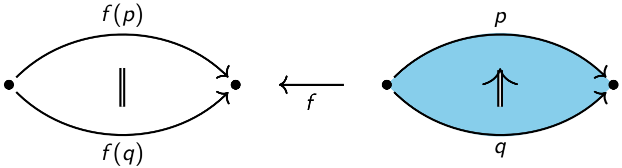

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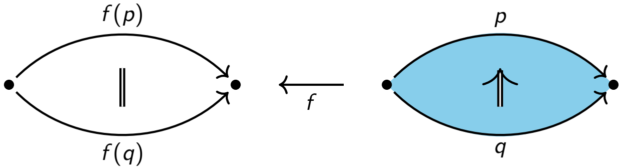

Example: (linear) λ-calculus $\rightsquigarrow$ monoidal closed categories

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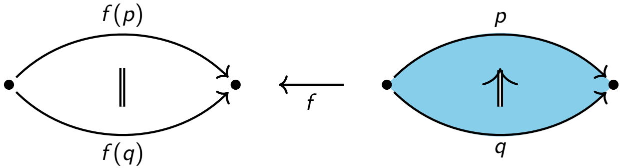

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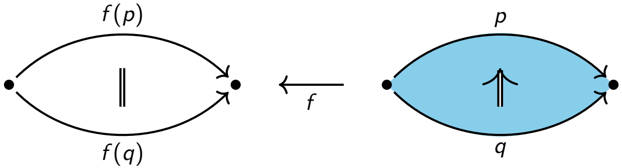


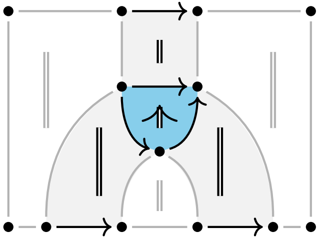
Example: (linear) λ-calculus $\rightsquigarrow$ monoidal closed categories $\simeq$ closed representable multicategories (special 2-polygraphs with algebraic composition of 2-cells) $\rightsquigarrow$ resolutions

To briefly mention

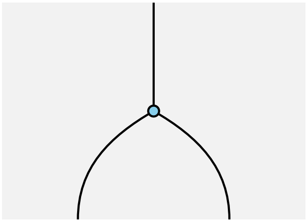
- Better understanding and expansion of classic results linking rewriting theory and homological algebra, e.g. Squier's criterion for the existence of finite convergent presentations of a monoid (Guiraud, Malbos 2016)
- Refined analysis of confluence, convergence etc. by keeping direction in higher dimensions, e.g. directing confluence squares ("rewrites of rewrites")

From directed cells to string diagrams

"Expand" lower-dimensional cells by filling the space with identities



$\rightsquigarrow$



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1. Do these operations have directed analogues?
2. If so, do they make sense for rewriting and universal algebra?

Tensor product of polygraphs

Do these operations have directed analogues?

Yes. Disjoint unions and quotients are basically unvaried.

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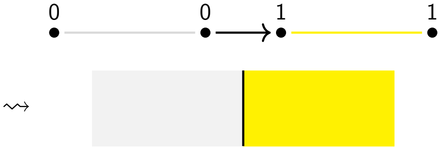
We can replace the cartesian product of spaces with the (noncommutative) *tensor product* of polygraphs:

- for each n-dimensional x in X, and m-dimensional y in Y, the polygraph $X \otimes Y$ has an $(n+m)$-dimensional $x \otimes y$ with $\partial(x \otimes y) = (\partial x \otimes y) \cup (x \otimes \partial y)$;
- there is only one division of $\partial(x \otimes y)$ into input and output that makes sense combinatorially

Tensor product in pictures

$\vec{I} := \; 0 \bullet \longrightarrow \bullet \, 1$; as a "one-dimensional string diagram":



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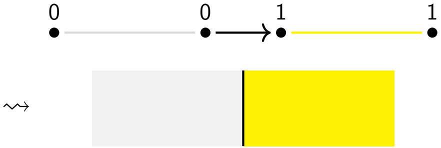
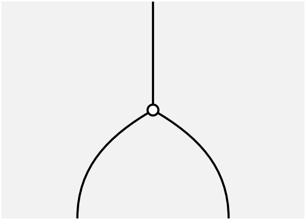
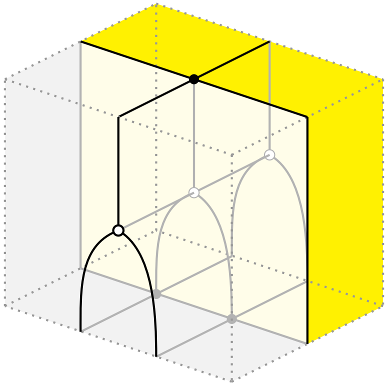


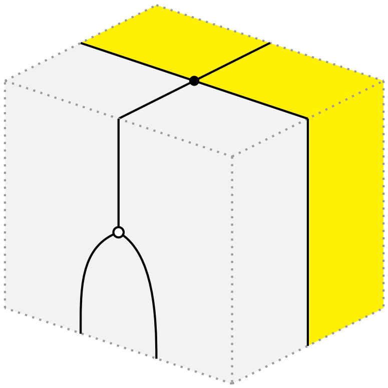
Tensor product with the binary operation cell



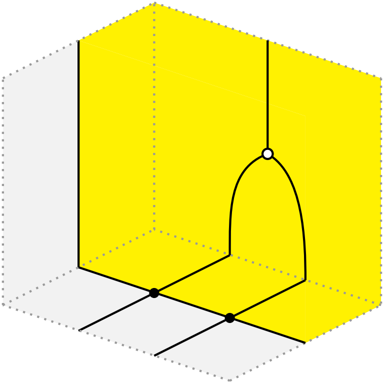
The cube



The cube: input boundary

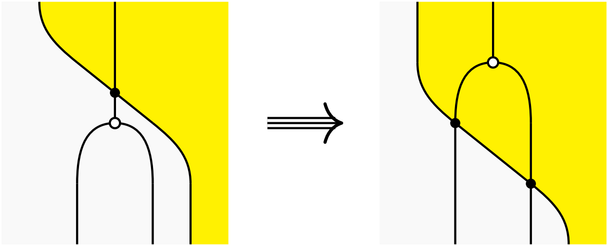


The cube: output boundary



Ironing it out

Do these operations make sense for rewriting and universal algebra?



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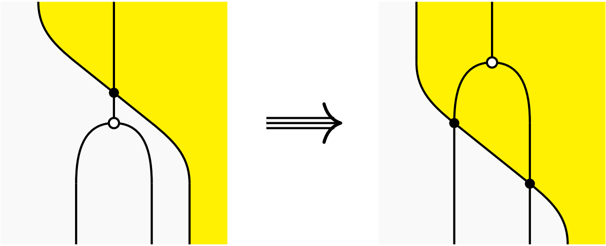
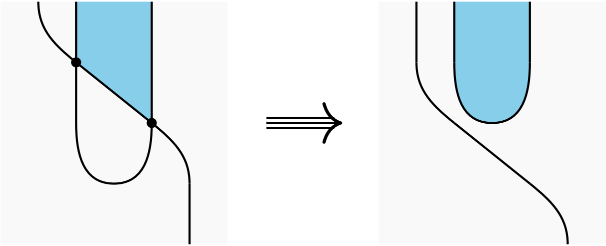


(Homomorphism of monads)

Tensor products are everywhere



Instantiates to quantum teleportation protocol **and** encrypted communication with one-time pads (Stay, Vicary 2013)

Tensor products are everywhere

"Monads are just monoids in the category of endofunctors"

$$T : Mon \to \mathbf{Cat}$$

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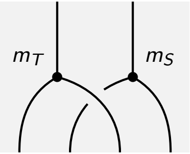
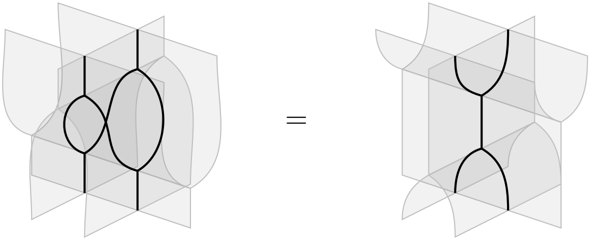


Distributive laws of monads are functors

$$D : Mon \otimes Mon \to \mathbf{Cat}$$

Tensor products are everywhere

Theory of bialgebras: a quotient* of $Mon \otimes Mon$ (smash product)



- Plus everything comes with higher-dimensional coherence/confluence cells

More than an analogy?

The analogy is quite strong, it has led to transmission of ideas (e.g. Lafont, Métayer, Worytkiewicz 2010)...

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But is not quite a functor.

There is no "good" direction-forgetting functor $|-|$: **Pol** $\to$ **Top** with $|X \otimes Y| \simeq |X| \times |Y|$

(And the reason why it doesn't exist is linked to several technical problems)

A solution

Regular polygraphs: sub-class closed under tensor products, good technical properties, and a nice functor $|-| : \mathbf{RPol} \to \mathbf{Top}$ exists

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- Capture more aspects of topological spaces in the theory of regular polygraphs
- Transport and generalise — a lever for compositional rewriting theory, beyond example-collection

A.H., *The algebra of entanglement and the geometry of composition*, PhD thesis, 2017
Work in progress: *A combinatorial-topological shape category for polygraphs*