Category theory and diagrammatic reasoning

13th March 2019

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6 Equivalences, embeddings, adjunctions

Many mathematical objects are considered to be "essentially the same" if and only if they are isomorphic in a suitable sense. This is true of sets, groups, vector spaces, manifolds, and many others.

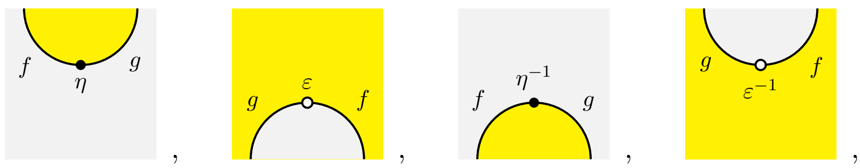
For categories, however, this is too restrictive. When we speak of "the category of finite-dimensional vector spaces and linear maps", what do we mean? As objects, do we choose all small sets with the structure of a finite-dimensional vector space, or do we choose only one vector space for each dimension, since any two vector spaces with the same dimension are isomorphic? The two choices lead to non-isomorphic categories — there is no bijection between their sets of objects — yet "in practice" they should be the same.

In fact, the correct notion of "sameness" of categories is weaker than isomorphism: this can be seen as a consequence of the fact that categories properly belong in the bicategory **Cat**, rather than a category. The same can be said of "sameness" of 0-cells in any other bicategory.

Definition 1. Let x, y be 0-cells in a bicategory X. A 1-cell $f : x \to y$ is an *equivalence* if there exist a 1-cell $g : y \to x$ and invertible 2-cells $\eta : (\mathrm{id}_x) \Rightarrow (f, g)$ and $\varepsilon : (g, f) \Rightarrow (\mathrm{id}_y)$. In this case, g is called a *weak inverse* of f.

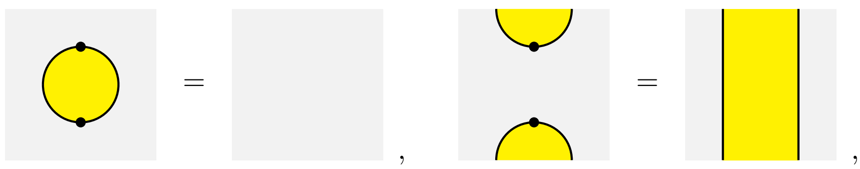
An equivalence is an *isomorphism* if g can be picked so that η^{-1} and ε are the compositors $c_{f,g}$ and $c_{g,f}$.

In colour-coded string diagrams, we need 2-cells



(1)

satisfying the equations (we omit labels, since there is only one labelling compatible with the colouring)



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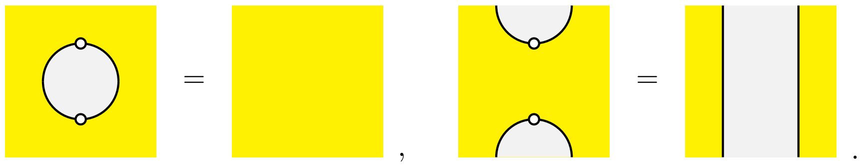


(3)

Instantiated in $\underline{\mathbf{Cat}}$, this gives the notion of *equivalence of categories*. We say that two categories are *equivalent* if there is an equivalence between them. You can check that isomorphisms of categories correspond to isomorphisms in $\underline{\mathbf{Cat}}$.

Example 2. Let $\mathbf{fVec}_{\mathbb{R}}$ be the category of finite-dimensional real vector spaces and linear maps, and let $\mathbf{Mat}_{\mathbb{R}}$ be the category whose

- objects are natural numbers, and
- morphisms $A : n \to m$ are matrices with m rows and n columns,

with matrix multiplication as composition, and the identity $n \times n$ matrix as the identity on n. There is a functor $F : \mathbf{Mat}_{\mathbb{R}} \to \mathbf{fVec}_{\mathbb{R}}$ sending n to $\mathbb{R}^n$ as a vector space, and an $m \times n$ matrix A to the linear map $\mathbb{R}^n \to \mathbb{R}^m$ represented by A in the standard basis.

We claim that, as long as we accept that every finite-dimensional vector space has a choice of basis, F is an equivalence of categories. We fix an ordered basis on every finite-dimensional vector space; on $\mathbb{R}^n$, we pick the standard basis. We let $G : \mathbf{fVec}_{\mathbb{R}} \to \mathbf{Mat}_{\mathbb{R}}$ be the functor sending a vector space V to its dimension $n = \dim(V)$, and a linear map $V \to W$ to the matrix representing it in the chosen bases of V and W.

Then, $F; G = \mathrm{id}_{\mathbf{Mat}_{\mathbb{R}}}$, and as $\eta : (\mathrm{id}_{\mathbf{Mat}_{\mathbb{R}}}) \Rightarrow (F, G)$, we take the identity natural transformation. As $\varepsilon : (G, F) \Rightarrow (\mathrm{id}_{\mathbf{fVec}_{\mathbb{R}}})$, we take the natural transformation whose component $\varepsilon_V : \mathbb{R}^n \to V$ at each n-dimensional vector space V is the unique isomorphism between $\mathbb{R}^n$ and V induced by the choice of ordered basis on V. These are both clearly invertible.

Example 3. While the previous example is an "intuitively obvious" equivalence, some celebrated results in mathematics are statements of non-obvious equivalences of categories.

For example, *Stone duality* most commonly refers to an equivalence between a subcategory **Stone** of **Top** whose objects are *Stone spaces*, that is, compact, totally disconnected, Hausdorff topological spaces, and morphisms are continuous maps, and the opposite $\mathbf{Bool}^{\mathrm{op}}$ of the category **Bool** whose objects are Boolean algebras and morphisms are algebra homomorphisms.

Similarly, *Gelfand duality* is the equivalence between the category $\mathbf{Top}_{\mathrm{cpt}}$ of compact Hausdorff topological spaces and continuous maps, and the opposite $C^*\mathbf{Alg}^{\mathrm{op}}_{\mathrm{com}}$ of the category of commutative C^*-algebras and C^*-algebra homomorphisms.

Both of these are foundational results for entire fields: pointless topology and non-commutative topology, respectively.

Exercise 4. Let P be a set with an equivalence relation $\sim$. Prove that P, seen as a category, is equivalent to its quotient $P/\sim$, seen as a discrete category.

Exercise 5. Let **PFun** be the category of sets and *partial functions*, and let $\mathbf{Set}_*$ be the category whose objects are *pointed sets* (X, x), that is, sets with a chosen "basepoint" $x \in X$, and morphisms $f : (X, x) \to (Y, y)$ are functions $f : X \to Y$ such that $f(x) = y$.

There is a functor $(-)_+ : \mathbf{PFun} \to \mathbf{Set}_*$ which sends a set X to the disjoint union $X + \{*\}$ of X with a one-element set, and a partial function $f : X \rightharpoonup Y$ to the function $f_+ : X + \{*\} \to Y + \{*\}$ defined by

$$x \in X \mapsto \begin{cases} f(x) & \text{if } f(x) \text{ is defined,} \\ * & \text{otherwise,} \end{cases} \qquad * \mapsto *.$$

Prove that $(-)_+$ is an equivalence of categories.

For functions of sets, homomorphisms of groups, and morphisms in many other categories, there are ways to recognise isomorphisms without exhibiting an inverse: for example, we know that a function is an isomorphism whenever it is both injective and surjective. The same is true of isomorphisms of categories, since they correspond to functors whose components are bijective functions.

In fact, such a characterisation also exists for equivalences (at least if we accept some form of choice principle).

Definition 6. Let X, Y be categories, and $F : X \to Y$ a functor.

We say that F is *essentially surjective* if for all $y \in Y_0$, there exists $x \in X_0$ and an isomorphism $e : F(x) \to y$ in Y.

We say that F is *full* if for all $x, x' \in X_0$, and all morphisms $g : F(x) \to F(x')$ in Y, there exists a morphism $f : x \to x'$ in X such that $F(f) = g$.

We say that F is *faithful* if, for all parallel pairs of morphisms $f, f' : x \to x'$ in X, if $F(f) = F(f')$ in Y, then $f = f'$.

Proposition 7. *A functor is an equivalence of categories if and only if it is essentially surjective, full, and faithful.*

Example 8. Let us reconsider the functor $F : \mathbf{Mat}_{\mathbb{R}} \to \mathbf{fVec}_{\mathbb{R}}$ of Example 2.

For each finite-dimensional vector space V, the existence of a basis on V is equivalent to the existence of a linear isomorphism between V and $\mathbb{R}^n = F(n)$ for some n: this gives essential surjectivity of F.

For any linear map $f : \mathbb{R}^n \to \mathbb{R}^m$, there is a matrix A which represents it in the standard basis, so $f = F(A)$: this gives fullness of F. Moreover, this matrix is unique: this gives faithfulness of F.

Remark 9. Isomorphisms of sets are characterised by one injectivity property, and one surjectivity property. Thus, it may seem strange that equivalences of categories are characterised by *two* "surjectivity-like" properties (essential surjectivity and fullness), and *one* "injectivity-like" property (faithfulness). Where does the asymmetry come from?

Consider the alternative definition of category we gave in Lecture 2, as the special case of a *bicategory* whose 2-cells are a particular kind of equivalence relation on finite paths of 1-cells. With this definition, faithfulness becomes the property that for every parallel pair of 1-cells $f, f' : x \to x'$, and for every *2-cell* $\beta : F(f) \Rightarrow F(f')$ in Y, there exists a 2-cell $\alpha : f \Rightarrow f'$ in X such that $\beta = F(\alpha)$. This is another "surjectivity"-like property!

This is one of many notions which are revealed in their regularity only in the theory of *higher categories*: in categories and bicategories, we only see their "shadow", and may fail to spot the real pattern.

A basic property of functions of sets is the existence of a unique up-to-isomorphism factorisation as a surjective function followed by an injective function; the codomain of the surjection is the *image* of the function. For functors of categories, there are different possible factorisations, with fullness grouped with essential surjectivity or with faithfulness.

Definition 10. The *image* $\mathrm{im}F$ of a functor $F : X \to Y$ is the category whose set of objects is X_0, and whose set of morphisms is $X_1/\sim$, where $\sim$ is the equivalence relation $f \sim g$ if and only if $F(f) = F(g)$. If $f_\sim$ and $g_\sim$ are the equivalence classes of $f : x \to y$ and $g : y \to z$, we define $f_\sim ; g_\sim := (f; g)_\sim$; this is independent of the choice of representatives.

The *full image* $\overline{\mathrm{im}}F$ of a functor $F : X \to Y$ is the category whose set of objects is X_0, and whose set of morphisms is $\{g : x \to y \,|\, g : F(x) \to F(y) \in Y_1\}$, with composition and identities as in Y.

Proposition 11. *Every functor $F : X \to Y$ factors as*

1. *an essentially surjective and full functor $F_{esf} : X \to \mathrm{im}F$ followed by a faithful functor $F_f : \mathrm{im}F \to Y$, and*
2. *an essentially surjective functor $F_{es} : X \to \overline{\mathrm{im}}F$ followed by a full and faithful functor $F_{ff} : \overline{\mathrm{im}}F \to Y$.*

The factorisations are unique up to equivalence.

Proof. We prove the second factorisation, and leave the first as an exercise. Define F_{es} to be the identity on objects, and $(f : x \to y) \mapsto (F(f) : x \to y)$ on morphisms; this is clearly essentially surjective (in fact, it is bijective on objects). Then, define F_{ff} to be $x \mapsto F(x)$ on objects, and $(g : x \to y) \mapsto (g : F(x) \to F(y))$ on morphisms; this is clearly full and faithful.

To prove uniqueness, suppose there is another factorisation of F as an essentially surjective functor $F' : X \to Z$, followed by a full and faithful functor $F'' : Z \to Y$. We define a functor $E : \overline{\mathrm{im}F} \to Z$, sending an object x to $E(x) := F'(x)$, and a morphism $g : x \to y$ to the unique morphism $E(g) : F'(x) \to F'(y)$ such that

$$F''(E(g)) = g : F(x) \to F(y)$$

in Y, which is well-defined because F'' induces a bijection between $\mathrm{Hom}_Z(F'(x), F'(y))$ and $\mathrm{Hom}_Y(F(x), F(y))$.

Then E is essentially surjective because F' is, and full and faithful by construction; by Proposition 7, it is an equivalence. □

Remark 12. A consequence of the uniqueness is that whenever a functor $F : X \to Y$ already satisfies some of these properties, it induces an equivalence between its source or target and its image or full image, by comparison with the factorisations $\mathrm{id}_X; F$ or $F; \mathrm{id}_Y$: for example, if F is full and faithful, then $F_{es} : X \to \overline{\mathrm{im}}F$ is an equivalence.

Remark 13. We can "amalgamate" the two factorisations into a single *ternary* factorisation: a functor $X \to Y$ factors as an essentially surjective, full functor $X \to \mathrm{im}F$, followed by an essentially surjective, faithful functor $\mathrm{im}F \to \overline{\mathrm{im}}F$, followed by a full and faithful functor $\overline{\mathrm{im}F} \to Y$.

A common situation in dealing with some category is that we want to restrict to objects satisfying some property, while keeping all morphisms between them. For example, we may want to pass from sets to finite sets; from topological spaces to Hausdorff spaces; from groups to abelian groups, and so on. In category theory, this is called passing to a *full* subcategory.

Definition 14. Let X be a subcategory of Y. We say that X is a *full subcategory* if the inclusion of X into Y is full.

Example 15. Let $F : X \to Y$ be a functor. There is a full subcategory Y' of Y whose objects are exactly those of the form $F(x)$ for some object $x \in X_0$. Then F factors as an essentially surjective functor $X \to Y'$, followed by the inclusion of Y' into Y, which is full and faithful. It follows that Y' is equivalent to $\overline{\mathrm{im}}F$.

If F is full and faithful, by Remark 12, F is an equivalence between X and Y'. So up to equivalence, we can think that any full and faithful functor exhibits X as a full subcategory of Y. For this reason, a full and faithful functor is also called an *embedding* of categories.

In general, there is a preference in category theory for definitions that are "stable under equivalence", hence do not rely on special equalities between objects and morphisms.

For example, it may be preferable to work with full and faithful functors $F : X \to Y$ rather than full subcategories of Y.

Construction 16. For any pair of categories X and Y, there is a category $[X, Y] := \mathrm{Hom}_{\underline{\mathbf{Cat}}}(X, Y)$, whose objects are functors $F : X \to Y$ and morphisms from F to G are natural transformations $\alpha : F \Rightarrow G$. This is called the *category of functors* from X to Y.

In particular, a functor $F : X^{\mathrm{op}} \to \mathbf{Set}$ is called a *presheaf* on X, and $[X^{\mathrm{op}}, \mathbf{Set}]$ the *category of presheaves.* Any object $x \in X_0$ determines a presheaf on X, namely, the contravariant homset functor $\mathrm{Hom}_X(-, x)$. In [Lecture 2, Example 18] we saw that each morphism $f : x \to y$ determines a natural transformation $f_* : \mathrm{Hom}_X(-, x) \Rightarrow \mathrm{Hom}_X(-, y)$, in a way that is compatible with identities and compositions.

Together, these assignments define a functor

$$\mathsf{y} : X \to [X^{\mathrm{op}}, \mathbf{Set}],$$

called the *Yoneda embedding* for X.

Proposition 17. *The Yoneda embedding is full and faithful.*

Proof. Let $\alpha : \mathrm{Hom}_X(-, x) \Rightarrow \mathrm{Hom}_X(-, y)$ be a natural transformation; we want to show that it is of the form f_* for a unique $f : x \to y$. The component of α at x is a function $\alpha_x : \mathrm{Hom}_X(x, x) \to \mathrm{Hom}_X(x, y)$, and we define $f := \alpha_x(\mathrm{id}_x)$.

We need to show that $\alpha_z = (f_*)_z$ for each $z \in X_0$. For each $g : z \to x$, the naturality square

$$\begin{tikzcd}
& \mathrm{Hom}_X(x,y) \arrow[dr, "g_y^*"] \arrow[dd, phantom, "\|"] & \\
\mathrm{Hom}_X(x,x) \arrow[ur, "\alpha_x"] \arrow[dr, "g_x^*"'] & & \mathrm{Hom}_X(z,y) \\
& \mathrm{Hom}_X(z,x) \arrow[ur, "\alpha_z"'] &
\end{tikzcd}$$

where $g^* := \mathrm{Hom}_X(g, -)$ implies that

$$\alpha_z(g) = \alpha_z(g_x^*(\mathrm{id}_x)) = g_y^*(\alpha_x(\mathrm{id}_x)) = g_y^*(f) = g; f = (f_*)_z(g).$$

This proves fullness of the Yoneda embedding.

Suppose $f, f' : x \to y$ are such that $f_* = f'_*$. Then

$$f = \mathrm{id}_x; f = (f_*)_x(\mathrm{id}_x) = (f'_*)_x(\mathrm{id}_x) = \mathrm{id}_x; f' = f'.$$

This proves faithfulness. $\square$

In general, embeddings have the following property.

Lemma 18. *Let $F : X \to Y$ be a full and faithful functor, $f : x \to x'$ a morphism in X. If $F(f)$ is an isomorphism in Y, then f is an isomorphism in X.*

Proof. Let $h : F(x') \to F(x)$ be the inverse of $F(f)$. By fullness of F, there exists $g : x' \to x$ such that $h = F(g)$. Then, $F(f; g) = F(f); F(g) = \mathrm{id}_{F(x)} = F(\mathrm{id}_x)$, and by faithfulness $f; g = \mathrm{id}_x$; similarly, $g; f = \mathrm{id}_{x'}$. □

Because of the lemma, using Proposition 17 we can deduce isomorphisms of objects of any category from natural isomorphisms of homsets into those objects; the latter are "concrete", coherent families of bijections, and are often easier to manipulate than "abstract" morphisms.

Example 19. There are some isomorphisms that hold in any cartesian closed category X: for example,

$$(c \to (a \times b)) \simeq ((c \to a) \times (c \to b)), \qquad (b \to (a \to c)) \simeq ((a \times b) \to c),$$

for each triple of 1-cells a, b, c. Proving them with elementary reasoning gets quite complicated. An indirect way is to use the simply-typed λ-calculus to construct terms for the two sides of the isomorphism, and show that the application of one to the other β-reduces to the identity.

Another way is to use the Yoneda embedding: thinking of a, b, c as objects of the category $\tilde{X} := \mathrm{Hom}_X(*, *)$, we have

$$\begin{aligned}\mathrm{Hom}_{\tilde{X}}(d, c \to (a \times b)) &\simeq \mathrm{Hom}_{\tilde{X}}(c \times d, a \times b) \simeq \\ &\simeq \mathrm{Hom}_{\tilde{X}}(c \times d, a) \times \mathrm{Hom}_{\tilde{X}}(c \times d, b) \simeq \\ &\simeq \mathrm{Hom}_{\tilde{X}}(d, c \to a) \times \mathrm{Hom}_{\tilde{X}}(d, c \to b) \simeq \\ &\simeq \mathrm{Hom}_{\tilde{X}}(d, (c \to a) \times (c \to b)),\end{aligned}$$

all from simple considerations on the universal properties of products and Kan extensions.

All the isomorphisms are independent of ("natural in") d, so they give an isomorphism between the Yoneda embeddings of $c \to (a \times b)$ and of $(c \to a) \times (c \to b)$, which we can bring back to X.

Proposition 17 can be seen as a consequence of the following fact. There is an evaluation functor

$$\mathrm{ev} : X^{\mathrm{op}} \times [X^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set},$$

sending a pair (x, F) of an object of x and a presheaf F on X to the set Fx, and a pair $(f : x \to y, \alpha : F \Rightarrow G)$ to the function $\alpha_y; G(f) = F(f); \alpha_x : F(y) \to G(x)$.

We can obtain another functor of the same type, as follows: compose the functor

$$\mathsf{y} \times \mathrm{id} : X^{\mathrm{op}} \times [X^{\mathrm{op}}, \mathbf{Set}] \to [X^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} \times [X^{\mathrm{op}}, \mathbf{Set}]$$

which is the Yoneda embedding on the first component and the identity on the second component, with the homset functor for $[X^{\mathrm{op}}, \mathbf{Set}]$, that is,

$$\mathrm{Hom}_{[X^{\mathrm{op}},\mathbf{Set}]} : [X^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} \times [X^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set},$$

to obtain a functor $\mathrm{Hom}_{[X^{\mathrm{op}},\mathbf{Set}]}(\mathsf{y}-, -)$. The following holds.

Proposition 20 (Yoneda lemma)**.** *There is a natural isomorphism between* ev *and* $\mathrm{Hom}_{[X^{\mathrm{op}},\mathbf{Set}]}(\mathsf{y}-, -)$.

Remark 21. The Yoneda lemma is considered one of the fundamental theorems of category theory, partly because of its generalisations in *enriched* category theory, where instead of hom*sets* one has hom-*objects* in a different category. In fairness, though, these belong to a "style" of category theory in which computations with homsets are fundamental, different from the style we have adopted in this course. We state it because it is important and you will encounter it again, but we will not make any use of it.

If $F : X \to Y$ is an equivalence, for each object y of Y we can find an object $G(y)$ of X such that $FG(y)$ is *isomorphic* to y, via the isomorphism $\varepsilon_y : FG(y) \to y$. Moreover, to each morphism $g : y \to y'$, we can associate a morphism $G(g) : G(y) \to G(y')$, in such a way that we can "transport" g onto $FG(g)$ using ε and its inverse. In words, ε allows us to "approximate" Y with the image of F in a way that is reversible.

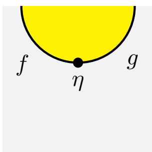
If F is not an equivalence, we may settle for less: we may ask that, for each y, there is a *best approximation* of y in the image of F, in the sense of [Lecture 3, Example 6] or [Lecture 3, Example 12].

Definition 22. Let $F : X \to Y$ be a functor. We say that F is a *left adjoint* if, for all $y \in Y_0$, there exist $x \in X_0$ and a morphism $\varepsilon_y : F(x) \to y$ which is universal from F to y.

Dually, we say that F is a *right adjoint* if, for all $y \in Y_0$, there exist $x \in X_0$ and a morphism $\eta_y : y \to F(x)$ which is universal from y to F.

Similarly to equivalences, there is an algebraic characterisation of left and right adjoints.

Definition 23. Let x, y be 0-cells in a bicategory X. An *adjunction* from x to y is a pair of 1-cells $f : x \to y$, $g : y \to x$, together with 2-cells $\eta : (\mathrm{id}_x) \Rightarrow (f, g)$ and $\varepsilon : (g, f) \Rightarrow (\mathrm{id}_y)$, pictured as



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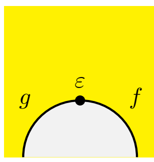

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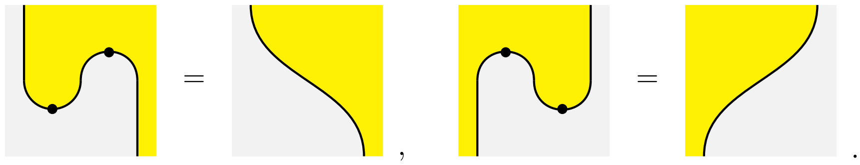
and satisfying the following equations:



(4)

Given an adjunction $(f, g, \eta, \varepsilon)$, we say that f is *left adjoint* to g, and g is *right adjoint* to f. The 2-cell η is called the *unit*, and the 2-cell ε is called the *counit* of the adjunction.

Proposition 24. *Let $F : X \to Y$ be a functor. The following conditions are equivalent:*

1. *F is a left adjoint;*
2. *there exist a functor $G : Y \to X$ and natural transformations $\eta : (\mathrm{id}_X) \Rightarrow (F, G)$ and $\varepsilon : (G, F) \Rightarrow (\mathrm{id}_Y)$ forming an adjunction in* $\underline{\mathbf{Cat}}$.

Proof. We prove one implication, and leave the converse as an exercise.

Suppose that F is a left adjoint. For each $y \in Y_0$, we choose an object $G(y) := x$ in X and a universal morphism $\varepsilon_y : FG(y) \to y$. Let $g : y \to y'$ be a morphism in Y, and consider the division problem

$$\begin{tikzcd} & FG(y') \arrow[dr, "\varepsilon_{y'}"] & \\ FG(y) \arrow[rr, bend right, "\varepsilon_y; g"'] & & y' \end{tikzcd} ;$$

we define $G(g) : G(y) \to G(y')$ to be the unique morphism in X satisfying

$$\begin{tikzcd} & FG(y') \arrow[dr, "\varepsilon_{y'}"] \arrow[d, phantom, "\Vert"] & \\ FG(y) \arrow[ur, "FG(g)"] \arrow[rr, bend right, "\varepsilon_y; g"'] & {} & y' \end{tikzcd} . \quad (5)$$

Using universality, it is easy to check that this assignment is compatible with composition and identities, hence extends G to a functor $G : Y \to X$. The (5) are in essence naturality squares, proving that the ε_y are components of a natural transformation $\varepsilon : (G, F) \Rightarrow (\mathrm{id}_Y)$.

Next, for each object x of X, consider the division problem

$$\begin{tikzcd} & FGF(x) \arrow[dr, "\varepsilon_{F(x)}"] & \\ F(x) \arrow[rr, bend right, "\mathrm{id}_{F(x)}"'] & & F(x) \end{tikzcd} ;$$

we define $\eta_x : x \to GF(x)$ to be the unique morphism in X satisfying

$$\begin{tikzcd} & FGF(x) \arrow[dr, "\varepsilon_{F(x)}"] \arrow[d, phantom, "\Vert"] & \\ F(x) \arrow[ur, "F(\eta_x)"] \arrow[rr, bend right, "\mathrm{id}_{F(x)}"'] & {} & F(x) \end{tikzcd} . \tag{6}$$

Another argument from universality shows that the η_x are components of a natural transformation $\eta : (\mathrm{id}_X) \Rightarrow (F, G)$. Given $f : x \to x'$, the division problem

$$\begin{tikzcd} & FGF(x') \arrow[dr, "\varepsilon_{F(x')}"] & \\ F(x) \arrow[rr, bend right, "F(f)"'] & & F(x') \end{tikzcd}$$

has a unique solution; but

$$F(f; \eta_{x'}); \varepsilon_{F(x')} = F(f); F(\eta_{x'}); \varepsilon_{F(x')} = F(f),$$

so $f; \eta_{x'}$ is a solution, and

$$F(\eta_x; GF(f)); \varepsilon_{F(x')} = F(\eta_x); FGF(f); \varepsilon_{F(x')} = F(\eta_x); \varepsilon_{F(x)}; F(f) = F(f),$$

so $\eta_x; GF(f)$ is also a solution, which gives a naturality square $\eta_x; GF(f) = f; \eta_{x'}$.

Now, commutativity of (6) proves the left-hand side of (4). For the right-hand side, we need to prove that the following diagram commutes in X:

$$\begin{tikzcd} & GFG(y) \arrow[dr, "G(\varepsilon_y)"] & \\ G(y) \arrow[ur, "\eta_{G(y)}"] \arrow[rr, bend right, "\mathrm{id}_{G(y)}"'] & & G(y) \end{tikzcd} .$$

We apply F to obtain

$$\begin{tikzcd} & FGFG(y) \arrow[dr, "FG(\varepsilon_y)"] & \\ FG(y) \arrow[ur, "F(\eta_{G(y)})"] \arrow[rr, bend right, "\mathrm{id}_{FG(y)}"'] & & FG(y) \end{tikzcd} ;$$

post-composing with $\varepsilon_y : FG(y) \to y$, we have

$$F(\eta_{G(y)}); FG(\varepsilon_y); \varepsilon_y = F(\eta_{G(y)}); \varepsilon_{FG(y)}; \varepsilon_y$$

by naturality of ε. By (6), this is equal to ε_y, and by the universal property of ε_y

$$F(\eta_{G(y)}; G(\varepsilon_y)); \varepsilon_y = \varepsilon_y = F(\mathrm{id}_{G(y)}); \varepsilon_y$$

implies that $\eta_{G(y)}; G(\varepsilon_y) = \mathrm{id}_{G(y)}$. This completes the proof. $\square$

The dual proposition also holds.

Proposition 25. *Let $G : Y \to X$ be a functor. The following conditions are equivalent:*

1. *G is a right adjoint;*
2. *there exist a functor $F : X \to Y$ and natural transformations $\eta : (\mathrm{id}_X) \Rightarrow (F, G)$ and $\varepsilon : (G, F) \Rightarrow (\mathrm{id}_Y)$ forming an adjunction in $\underline{\mathbf{Cat}}$.*

Example 26. By [Lecture 3, Example 6] and [Lecture 3, Example 12], the inclusion $\imath : \mathbb{Z} \to \mathbb{R}$ is both a left and a right adjoint. The floor function $\lfloor - \rfloor : \mathbb{R} \to \mathbb{Z}$ is right adjoint to $\imath$, with counit given by the family of inequalities $\imath(\lfloor r \rfloor) \leq r$ for $r \in \mathbb{R}$, and unit given by the family of inequalities (in fact, equalities) $k \leq \lfloor \imath(k) \rfloor$ for $k \in \mathbb{Z}$.

Dually, the ceiling function $\lceil - \rceil : \mathbb{R} \to \mathbb{Z}$ is left adjoint to $\imath$. The unit is given by the family of inequalities $r \leq \imath(\lceil r \rceil)$ for $r \in \mathbb{R}$, and the counit by the family of inequalities $\lceil \imath(k) \rceil \leq k$ for $k \in \mathbb{Z}$.

Example 27. The forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is both a left and a right adjoint. For each set S, let $I(S)$ and $D(S)$ be the topological spaces with S as underlying set and, respectively, the *indiscrete* topology (only S and $\emptyset$ are open) and the *discrete* topology (every subset is open). Clearly, $UI(S) = UD(S) = S$, and the identity function on S is both universal from U to S (which induces a functor $I : \mathbf{Set} \to \mathbf{Top}$ right adjoint to U) and from S to U (which induces a functor $D : \mathbf{Set} \to \mathbf{Top}$ left adjoint to U).

Exercise 28. The powerset functor $\mathcal{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ is, equivalently, a functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$. Show that it is *self-adjoint*, that is, $\mathcal{P}$ is both left and right adjoint to itself.

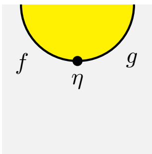
The equations (4), variably called "zig-zag", "triangle", or "snake" equations, are among the most striking diagrammatic equations in category theory, because of their intuitive interpretation as the topological move of "straightening" of a wire. In this and the following lecture, we will see how they can be used to prove rigorously yet intuitively a number of non-obvious results in category theory, which in addition will have an interpretation in any bicategory other than $\underline{\mathbf{Cat}}$.

The following shows that, in any bicategory, a left adjoint determines a right adjoint up to isomorphism, and vice versa.

Proposition 29. *Suppose $f : x \to y$ is left adjoint both to g and to $g' : y \to x$. Then g and g' are isomorphic.*

Dually, if $g : y \to x$ is right adjoint both to f and to $f' : x \to y$, then f and f' are isomorphic.

Proof. Let $(f, g, \eta, \varepsilon)$ and $(f, g', \eta', \varepsilon')$ be the two adjunctions, whose units and counits we picture as



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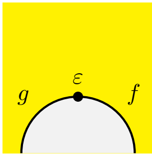

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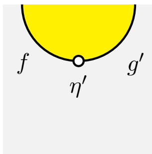

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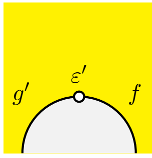

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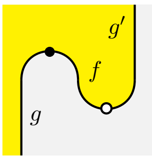
We construct 2-cells $\alpha : (g) \Rightarrow (g')$ and $\alpha' : (g') \Rightarrow (g)$ as follows:



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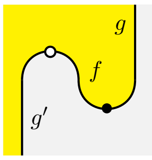

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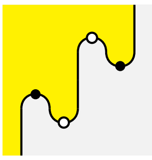
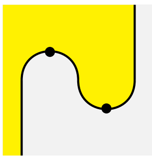
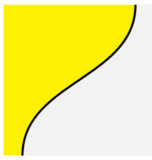
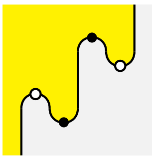
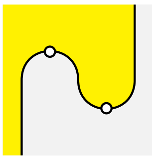
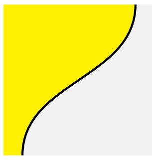
Then

 =  =  ,

and similarly

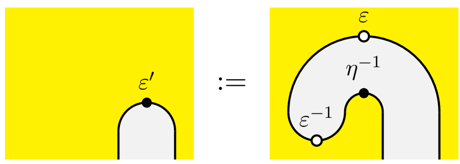
 =  =  ,

which proves that α and α' are each other's inverse. □

The following shows that, as we intuitively introduced them, left adjoints and right adjoints are generalisations of equivalences.

Proposition 30. *Suppose $f : x \to y$ is an equivalence with weak inverse $g : y \to x$. Then f is both left and right adjoint to g.*

Proof. Suppose invertible 2-cells η and ε are given as in (1). In general, these will not be the unit and counit of an adjunction; however, η is the unit of an adjunction together with the following “modified ” counit:



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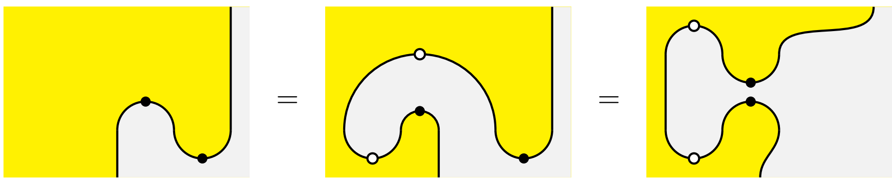
We have



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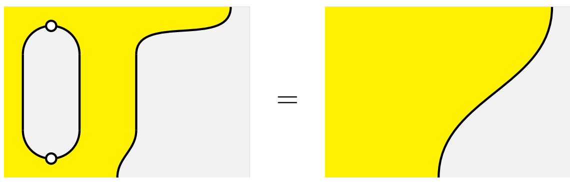
and by the rightmost equation in (2), this is equal to



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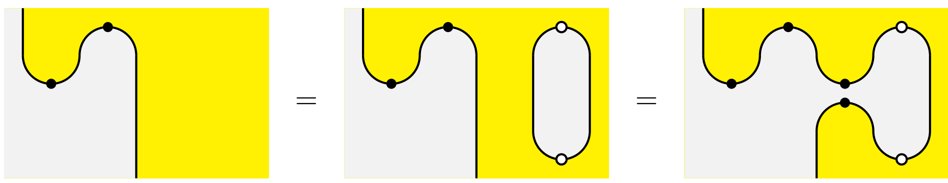
where we used the leftmost equation in (3). This proves one of the zig-zag equations.

For the other,



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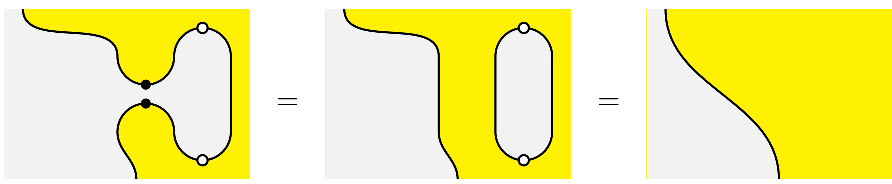
again using equations (2) and (3). By the first zig-zag equation, this is equal to



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which completes the proof that f is left adjoint to g. To prove that f is also right adjoint, invert both η and ε'. $\square$

Exercise 31. Using string diagrams, prove that if $f : x \to y$ is left adjoint to $g : y \to x$ and $h : y \to z$ is left adjoint to $k : z \to y$, then $f; h : x \to z$ is left adjoint to $k; g : z \to x$.