Category theory and diagrammatic reasoning

20th March 2019

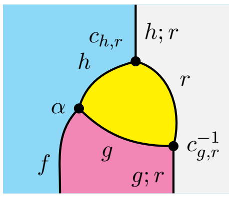
Last updated: 10th March 2019

7 Adjunctions, monads and comonads

Adjunctions in bicategories have a strong connection with Kan extensions. In **Cat**, this is the basis of a number of useful results in category theory.

Definition 1. Let $f : x \to y$, $g : y \to z$ and $h : x \to z$ be 1-cells in a bicategory, and let $\alpha : (f, g) \Rightarrow (h)$ be a right Kan extension of h along f.

Let $r : z \to w$ be another 1-cell. We say that r *preserves* the Kan extension α if



(1)

is a right Kan extension of $h;r$ along f.

We say that α is an *absolute* right Kan extension if it is preserved by every 1-cell with source z.

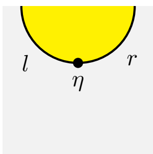
There are dual notions of preservation of left Kan extensions and absolute left Kan extensions.

Theorem 2. *Let r be a right adjoint 1-cell in a bicategory. Then r preserves right Kan extensions.*

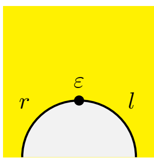
Dually, if l is a left adjoint 1-cell, then l preserves left Kan extensions.

Remark 3. The idea of the proof is that we can use the counit of an adjunction to turn a division problem for (1) into a division problem for α, and then the unit to turn a solution of the latter into a solution of the former.

Proof. Let $(l, r, \eta, \varepsilon)$ be an adjunction from w to z, with r as right adjoint, and picture η and ε as

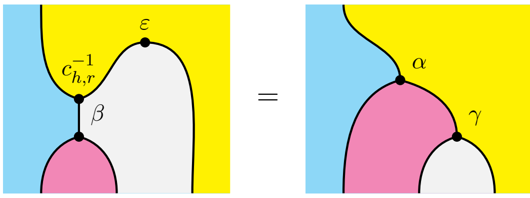


,



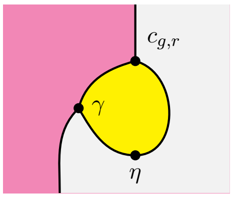
. (2)

Let $\alpha : (f, g) \Rightarrow (h)$ be a right Kan extension of h along f. Given any $\beta : (f, k) \Rightarrow (h; r)$, by the universal property of α, there exists a unique $\gamma : (k, l) \Rightarrow (h)$ such that



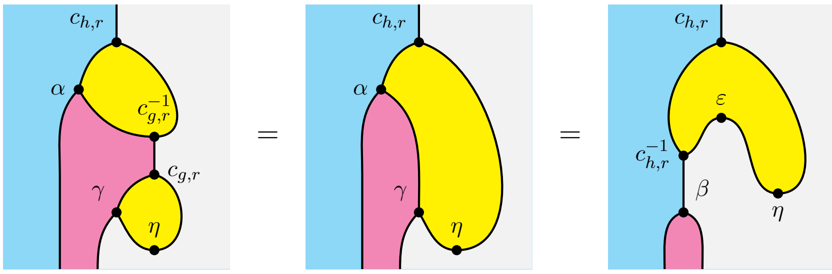
. (3)

We define $\xi : (k) \Rightarrow (g; r)$ to be the 2-cell



;

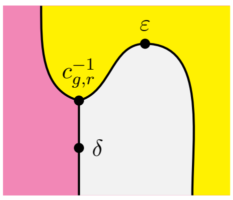
then



, (4)

which, using one zig-zag equation and eliminating $c_{h,r}$ and its inverse, reduces to β. This proves the existence part of the universal property of (1).

For uniqueness, suppose that $\delta : (k) \Rightarrow (g; r)$ is another solution. In that case,



is a solution of the division problem (3), and is therefore equal to γ (exercise). A calculation similar to (4) then shows that δ is equal to ξ. $\square$

Corollary 4. *Right adjoints preserve all limit cones, and left adjoints preserve all colimit cones.*

Instantiated in **Cat**, these dual results are very useful in pratice: often, we may have prior knowledge about limits and colimits in a category X, and little knowledge about Y. Relating X and Y with an adjunction, we can transport some limits or colimits, and avoid repeating computations.

Remark 5. By [Lecture 6, Proposition 30] every equivalence is both a left and a right adjoint. It follows that equivalences preserve all right and left Kan extensions, in particular all limits and colimits.

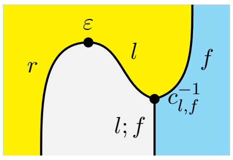
Example 6. In [Lecture 6, Example 27] we saw that the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$ is both a left and a right adjoint. It follows that the underlying set of any limit or colimit in **Top** is the limit or colimit of the diagram of underlying sets.

The next result shows that adjoint functors can be characterised by the existence and preservation of certain Kan extensions.

Theorem 7. *Let $r : x \to y$ be a 1-cell in a bicategory. The following conditions are equivalent:*

1. *r is a right adjoint;*
2. *a right Kan extension of id_x along r exists and is absolute;*
3. *a right Kan extension of id_x along r exists and is preserved by r.*

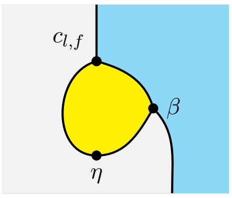
Proof. First, we prove the implication from 1 to 2. Let $(l, r, \eta, \varepsilon)$ be an adjunction: we will prove that $\varepsilon : (r, l) \Rightarrow (\mathrm{id}_x)$ is an absolute right Kan extension of id_x along r. It suffices to prove that for all $f : x \to z$



(5)

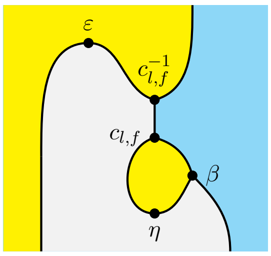
is a right Kan extension of f along r.

Let $g : y \to z$ be a 1-cell parallel to $l; f$, and $\beta : (r, g) \Rightarrow (f)$ a 2-cell. We define $\xi : (g) \Rightarrow (l; f)$ to be the 2-cell

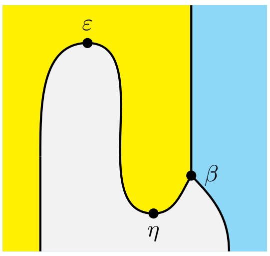


; (6)

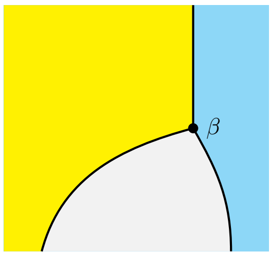
then



=



=

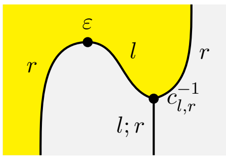


,

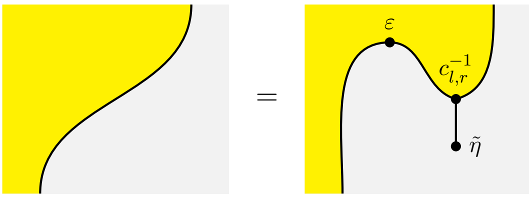
which proves the existence part of the universal property of (5).

Exercise 8. Prove the uniqueness part.

The implication from 2 to 3 is obvious, so it suffices to prove that 3 implies 1. Suppose $\varepsilon : (r, l) \Rightarrow (\mathrm{id}_x)$ is a right Kan extension of id_x along r, and that it is preserved by r, that is,



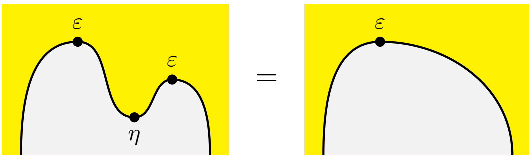
is a right Kan extension of r along r. It follows that there is a unique $\tilde{\eta} : (\mathrm{id}_y) \Rightarrow (l; r)$ such that



, (7)

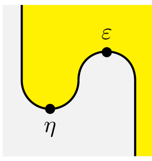
the solution of a division problem involving the unitor $(r, \mathrm{id}_y) \Rightarrow (r)$. Defining η as the composite of $\tilde{\eta}$ with $c^{-1}_{l,r}$, (7) becomes one of the zig-zag equations.

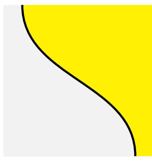
To prove the second zig-zag equation, observe that, by the first zig-zag equation,



,

so both



and


are solutions of a division problem involving ε. Because ε is a right Kan extension, they must be equal. □

Of course, a dual result also holds.

Corollary 9. *Let $l : x \to y$ be a 1-cell in a bicategory. The following conditions are equivalent:*

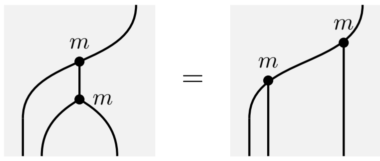
1. *l is a left adjoint;*

2. *a left Kan extension of id_x along l exists and is absolute;*

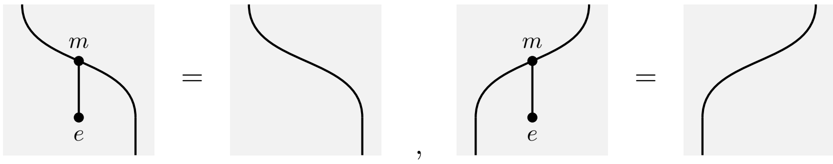
3. a left Kan extension of id_x *along* l *exists and is preserved by* l*.*

Remark 10. There is a number of results, collectively known as "adjoint functor theorems", that give sufficient conditions for a limit-preserving functor to be a right adjoint, or for a colimit-preserving functor to be a left adjoint. In particulary good cases, these conditions only involve the source and target category of a functor, so that for instance "limit-preserving functors from X to Y" are the same as right adjoint functors from X to Y. In general this is not the case.

From an adjunction involving 1-cells $l : x \to y$ and $r : y \to x$, we obtain two endo-1-cells $l;r : x \to x$ and $r;l : y \to y$. These come with additional algebraic structure. Recall the equations of the theory of monoids from [Lecture 4, Definition 17]:

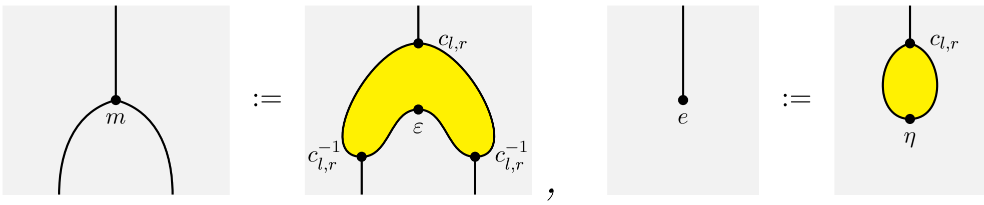


,

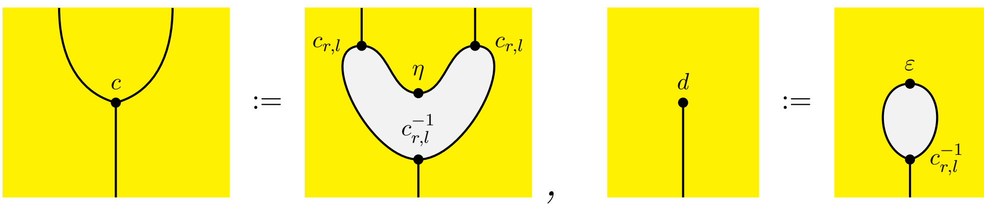


These diagrams and equations can be interpreted in any bicategory, not just a monoidal category, by also fixing an interpretation of the single 0-cell. This gives a notion of *internal monoid on a 0-cell* in a bicategory. There is a dual notion of *internal comonoid on a 0-cell.*

Proposition 11. *Let* $(l, r, \eta, \varepsilon)$ *be an adjunction from* x *to* y *in a bicategory. Then*



determines an internal monoid on x*, and*



determines an internal comonoid on y*.*

Exercise 12. Prove Proposition 11.

Remark 13. Due to the appearance of its multiplication, the monoid induced by an adjunction is often referred to as the *pair-of-pants* monoid.

Thus, in an arbitrary bicategory, an adjunction induces a pair of a monoid and a comonoid. In general there are monoids and comonoids that do not arise in this way. In $\underline{\mathbf{Cat}}$, however, a converse does hold: every internal monoid is induced by an adjunction (in a non-unique way), and so is every internal comonoid.

In the rest of this lecture, we will focus on $\underline{\mathbf{Cat}}$, where some special terminology is used.

Definition 14. A *monad* is an internal monoid in $\underline{\mathbf{Cat}}$. A *comonad* is an internal comonoid in $\underline{\mathbf{Cat}}$.

Explicitly, a monad on a category X is given by an endofunctor $T : X \to X$, together with a pair of natural transformations $m : T;T \Rightarrow T$ and $\eta : \mathrm{id}_X \Rightarrow T$, satisfying the monoid equations.

Example 15. Let P be a poset. A *closure operator* on P is an order-preserving map $T : P \to P$ such that, for all $x \in P$,

$$x \leq T(x) \qquad \text{and} \qquad T(T(x)) \leq T(x).$$

Because T is order preserving, from $x \leq T(x)$ it follows that $T(x) \leq T(T(x))$, so in fact $T(T(x)) = T(x)$, that is, T is idempotent.

If we see P as a category, then a closure operator is precisely a monad on P. There is a dual notion of *interior operator*, which is an idempotent order-preserving map $I : P \to P$ satisfying $I(x) \leq x$ for all $x \in P$: this is the same as a comonad on P.

Example 16. Let $(-)_+ : \mathbf{Set} \to \mathbf{Set}$ be the endofunctor sending a set S to the disjoint union $S_+ := S + \{*\}$, and a function $f : S \to T$ to the function $f_+ : S_+ \to T_+$ defined as $x \mapsto f(x)$ for $x \in S$, and $* \mapsto *$.

There is a natural transformation $m : (-)_+;(-)_+ \Rightarrow (-)_+$ whose component $m_S : (S_+)_+ \to S_+$ is the identity on S, and sends both the added copies of $*$ in $(S_+)_+$ to the one copy of $*$ in S_+; and a natural transformation $e : \mathrm{id}_{\mathbf{Set}} \Rightarrow (-)_+$ whose component $e_S : S \to S_+$ is the canonical inclusion of S into $S + \{*\}$. These determine a monad on $\mathbf{Set}$.

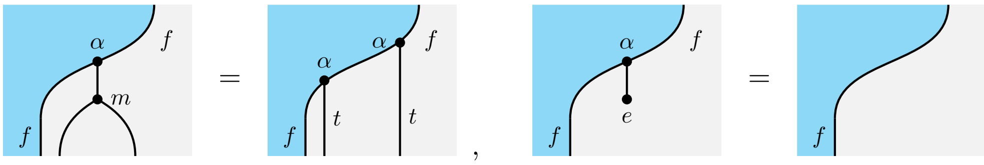
Example 17. In addition to the contravariant powerset functor, there is a *covariant* endofunctor $\mathcal{P}_* : \mathbf{Set} \to \mathbf{Set}$ sending a set S to its powerset, and a function $f : S \to T$ to the function $\mathcal{P}_* f$ defined by $U \mapsto f(U)$, for all $U \subseteq S$.

There is a natural transformation $m : \mathcal{P}_*;\mathcal{P}_* \Rightarrow \mathcal{P}_*$, whose component on S sends a set $\{U_i\}$ of subsets of S to its union $\bigcup\{U_i\} \subseteq S$, and a natural transformation $e : \mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{P}_*$, whose component on S is defined by $x \mapsto \{x\}$ for each $x \in S$. These determine a monad on **Set**.

Construction 18. Recall that, for all 0-cells x, y in a bicategory X, there is a category $\mathrm{Hom}_X(x, y)$ whose objects are 1-cells $x \to y$, and morphisms are 2-cells between them.

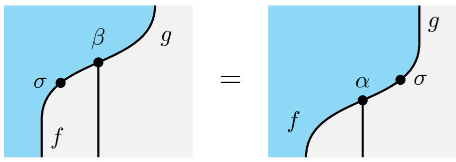
Let $(t : y \to y, m, e)$ be a monoid on y. There is a category $\mathrm{Mod}_t(x, y)$ whose

- objects are 1-cells $f : x \to y$ together with a *right action* of t on f, that is, a 2-cell $\alpha : (f, t) \Rightarrow (f)$ satisfying



,

- morphisms $\sigma : (f, \alpha) \to (g, \beta)$ are 2-cells $\sigma : (f) \Rightarrow (g)$ which are compatible with the right actions, that is,

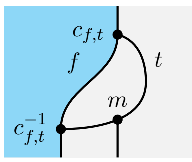


.

Composition is composition of 2-cells, and the identity on f is compatible with any right action, so it becomes an identity on (f, α) for any right action α of t.

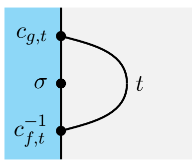
There is a functor $U : \mathrm{Mod}_t(x, y) \to \mathrm{Hom}_X(x, y)$ which simply "forgets" the right actions: it sends (f, α) to f, and $\sigma : (f, \alpha) \to (g, \beta)$ to $\sigma : f \to g$.

There is also a functor $F : \mathrm{Hom}_X(x, y) \to \mathrm{Mod}_t(x, y)$, defined as follows: the 1-cell $f : x \to y$ is sent to $f; t$ with the *free* action $\varphi_f : (f; t, t) \Rightarrow (f; t)$ given by



,

which is a right action by the equations of monoids, and a 2-cell $\sigma : (f) \Rightarrow (g)$ is sent to the 2-cell $\sigma; t : (f; t) \Rightarrow (g; t)$ defined by



,

which is trivially compatible with the free actions on $f; t$ and $g; t$.

Example 19. An internal monoid (M, m, e) in $\mathbf{Set}_\times$ is a monoid in the usual sense, and a right action $\alpha : (S, M) \Rightarrow (S)$ of M on S is a right action in the usual sense, that is, a function $\alpha : S \times M \to S$ such that, writing $x \cdot a := \alpha(x, a)$ for $x \in S$ and $a \in M$,

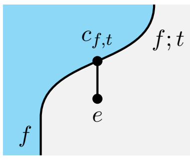
$$(x \cdot a) \cdot b = x \cdot (ab), \qquad x \cdot e = x$$

for all $x \in S$ and $a, b \in M$.

The category $\mathbf{Mod}_M := \mathrm{Mod}_M(*, *)$ is the category of right M-modules, that is, sets equipped with a right action of M, together with functions that are equivariant with respect to the actions. The category $\mathrm{Hom}_{\mathbf{Set}_\times}(*, *)$ is isomorphic to $\mathbf{Set}$, and the functor $F : \mathbf{Set} \to \mathbf{Mod}_M$ sends the set S to the set $S \times M$ equipped with the free right action of M, defined by $(x, a) \cdot b \mapsto (x, ab)$.

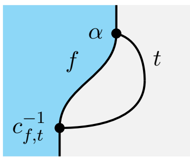
Proposition 20. *The functor* $F : \mathrm{Hom}_X(x, y) \to \mathrm{Mod}_t(x, y)$ *is left adjoint to* $U : \mathrm{Mod}_t(x, y) \to \mathrm{Hom}_X(x, y)$.

Proof. We exhibit an explicit adjunction. The unit $\eta : \mathrm{id}_{\mathrm{Hom}_X(x,y)} \Rightarrow F; U$ has components $\eta_f : f \to f; t$ given by the 2-cells



.

The counit $\varepsilon : U; F \Rightarrow \mathrm{id}_{\mathrm{Mod}_t(x,y)}$ has components $\varepsilon_{(f,\alpha)} : (f; t, \varphi_f) \to (f, \alpha)$ given by



,

which are compatible with φ_f and α essentially by definition of a right action. The verification of the zig-zag equations is straightforward. □

From the adjunction, we obtain an endofunctor $F; U$ on the category $\mathrm{Hom}_X(x, y)$, which has the structure of a monad, and whose effect is essentially post-composition with t.

Now, let us specialise to $\underline{\mathbf{Cat}}$. Given a small category X, by the correspondence between objects of X and functors $1 \to X$, and between morphisms of X and their natural transformations, $\mathrm{Hom}_{\underline{\mathbf{Cat}}}(1, X)$ is isomorphic to X.

Definition 21. Let (T, m, e) be a monad on a category X. We write X^T for $\mathrm{Mod}_T(1, X)$, and call it the *Eilenberg-Moore category* of T.

By Proposition 20, for every monad (T, m, e) on X, we have an adjunction involving a pair of functors

$$F : X \to X^T, \qquad U : X^T \to X.$$

Moreover, the functor $F;U$ sends an object x, corresponding to a functor $x : 1 \to X$, to the composite $x;T$, which is just $T(x)$. Similarly, $F;U$ sends a morphism $f : x \to y$ to $T(f) : T(x) \to T(y)$. Therefore $F;U = T$.

Exercise 22. Check that the monad on X induced by the adjunction is the original monad (T, m, e).

This proves our initial claim.

Theorem 23. *Every monad is induced by an adjunction.*

Beyond their role in this result, decomposing monads into adjunctions, Eilenberg-Moore categories are interesting in their own right, as a generalisation of categories of *algebras* for an algebraic theory.

Given a monad (T, m, e) on a category X, a right action of T on $x : 1 \to X$ is a morphism $\alpha : T(x) \to x$ in X which interacts appropriately with the natural transformations m and e. The idea is that $T(x)$ is the object x *freely endowed* with some algebraic structure, and $\alpha : T(x) \to x$ gives a way of "internalising" that algebraic structure in x, that is, making x closed under the operations encoded by T.

Definition 24. An object $\alpha : T(x) \to x$ of the Eilenberg-Moore category X^T is called an *algebra* on x for the monad T, or a T-*algebra*. Morphisms in X^T are called *homomorphisms* of T-algebras.

Example 25. The forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$ has a left adjoint $F : \mathbf{Set} \to \mathbf{Grp}$, which sends a set S to the *free group* generated by the elements of S, that is, the set of finite (possibly empty) sequences $x_1 \cdot \ldots \cdot x_n$ of elements in the set $\{x, x^{-1} \,|\, x \in S\}$, quotiented by $x \cdot x^{-1} = e = x^{-1} \cdot x$, where e denotes the empty sequence.

This induces a monad $T = F;U$ on $\mathbf{Set}$. An algebra $\alpha : T(S) \to S$ for this monad assigns an element of S to every element $x_1 \cdot \ldots \cdot x_n$ of $T(S)$; compatibility with the multiplication and unit of the monad impose that this assignment is associative and unital. Therefore, the algebra gives a group structure on the set S. A homomorphism of T-algebras is then a homomorphism of groups. We can conclude that the Eilenberg-Moore category $\mathbf{Set}^T$ is equivalent to the category $\mathbf{Grp}$.

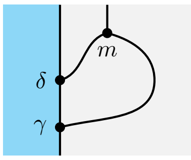
This is an instance of a general pattern of "free-forgetful" adjunctions between $\mathbf{Set}$ and a category of algebras, after which the latter can be identified with the Eilenberg-Moore category of the resulting monad.

Exercise 26. Let T be a closure operator on a poset P. What is the Eilenberg-Moore category X^T?

As we mentioned, the decomposition of T as an adjunction is not generally unique, not even up to equivalence. In particular, we can "restrict" the adjunction between X and X^T to an adjunction between X and the full subcategory of X^T on the free algebras, that is, equivalently, the full image $\overline{\text{im}}F$. This category has an equivalent, more convenient description.

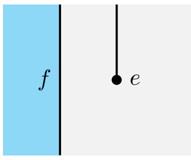
Construction 27. Let $(t : y \to y, m, e)$ be a monoid on y in a bicategory X. There is a category $\mathrm{Kl}_t(x, y)$ whose

- objects are 1-cells $f : x \to y$,
- morphisms from f to g are 2-cells $\gamma : (f) \Rightarrow (g, t)$,
- the composition of $\gamma : (f) \Rightarrow (g, t)$ and $\delta : (g) \Rightarrow (h, t)$ is given by the 2-cell



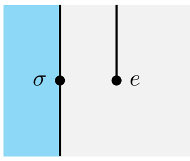
,

- the identity on $f : x \to y$ is given by the 2-cell



.

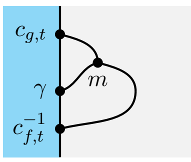
There is a functor $J : \mathrm{Hom}_X(x, y) \to \mathrm{Kl}_t(x, y)$, which is the identity on objects, and sends a 2-cell $\sigma : (f) \Rightarrow (g)$ to the 2-cell



.

Moreover, there is a functor $K : \mathrm{Kl}_t(x, y) \to \mathrm{Mod}_t(x, y)$, defined as follows:

- the object $f : x \to y$ is sent to $f; t$ with the free right action $\varphi_f : (f; t, t) \Rightarrow (f; t)$;
- a morphism $\gamma : (f) \Rightarrow (g, t)$ is sent to the 2-cell $\sigma : (f; t) \Rightarrow (g; t)$ defined by

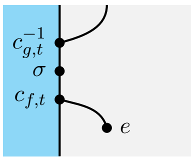


.

This is compatible with the free actions φ_f and φ_g. It is easy to see that the composite $J; K$ is equal to F.

Proposition 28. *The functor $K : \mathrm{Kl}_t(x, y) \to \mathrm{Mod}_t(x, y)$ is full and faithful.*

Proof. Let $\sigma : (f; t, \varphi_f) \Rightarrow (g; t, \varphi_g)$ be a morphism in $\mathrm{Mod}_t(x, y)$, that is, a 2-cell $\sigma : (f; t) \Rightarrow (g; t)$ compatible with the free actions of t on $f; t$ and $g; t$. Define $\gamma : (f) \Rightarrow (g, t)$ to be the 2-cell



.

It is an exercise to check that $K(\gamma) = \sigma$. Faithfulness is straightforward, and left as an exercise. □

Corollary 29. *The category $\mathrm{Kl}_t(x, y)$ is equivalent to the full image of $F : \mathrm{Hom}_X(x, y) \to \mathrm{Mod}_t(x, y)$.*

Proof. The functor F factors as $J; K$, where J is bijective on objects, and K is full and faithful. The claim follows from the essential uniqueness of such a factorisation. □

Definition 30. Let (T, m, e) be a monad on a category X. We write X_T for $\mathrm{Kl}_T(1, X)$, and call it the *Kleisli category* of T.

By the general result we proved, there is a full and faithful functor $K : X_T \to X^T$, exhibiting X_T as the full subcategory of X^T on the free T-algebras.

Example 31. Consider the monad $(-)_+$ of Example 16. The morphisms from S to T in the Kleisli category $\mathbf{Set}_{(-)_+}$ are functions $f : S \to T_+$. These can be identified with partial functions $f : S \rightharpoonup T$: a function $f : S \to T_+$ corresponds to the partial function sending x to $f(x)$ when $f(x) \in T$, and undefined when $f(x) = *$. Thus $\mathbf{Set}_{(-)_+}$ is equivalent to the category **PFun** of sets and partial functions.

Remark 32. Kleisli categories are the mathematical foundation for the use of monads in functional programming languages, such as Haskell. The monad $(-)_+$ of the previous example is what is known as the *Maybe monad*: the morphisms $S \to T_+$ correspond to computations which take an input of type S and return either a value of type T, or an *undefined* value.

Example 33. Consider the covariant powerset monad $\mathcal{P}_*$ of Example 17. Morphisms from S to T in $\mathbf{Set}_{\mathcal{P}_*}$ are functions $f : S \to \mathcal{P}T$, sending $x \in S$ to a subset $f(x) \subseteq T$. These can be identified with *relations* between S and T: namely, f corresponds to the relation $R_f(x, y)$ if and only if $y \in f(x)$.

The composite of $f : S \to \mathcal{P}T$ and $g : T \to \mathcal{P}U$ is the function $h : S \to \mathcal{P}U$ defined by

$$x \mapsto f(x) \mapsto \{g(y) \mid y \in f(x)\} \mapsto \bigcup_{y \in f(x)} g(y).$$

This corresponds to the relation $R_h(x, z)$ if and only if there exists $y \in T$ such that $y \in f(x)$ and $z \in g(y)$, or equivalently, $R_f(x, y)$ and $R_g(y, z)$; this is the *relational composite* of R_f and R_g. We conclude that $\mathbf{Set}_{\mathcal{P}_*}$ is equivalent to the category $\mathbf{Rel}$ of sets and relations.

Example 34. There are cases in which the Eilenberg-Moore category and the Kleisli category of a monad coincide.

Consider the monad T on $\mathbf{Set}$ induced by the free-forgetful adjunction between $F : \mathbf{Set} \to \mathbf{Vec}_k$, $U : \mathbf{Vec}_k \to \mathbf{Set}$; this exhibits $\mathbf{Vec}_k$ as the Eilenberg-Moore category $\mathbf{Set}^T$. The Kleisli category of T is equivalent to the full subcategory on the free algebras; but every vector space is isomorphic to the free vector space on a set. It follows that $\mathbf{Set}^T$ and $\mathbf{Set}_T$ are equivalent.

To conclude, it goes without saying that all the constructions can be dualised.

1. For all 0-cells x, y in a bicategory X, and internal comonoids $(t : y \to y, c, d)$, we can construct a category $\mathrm{coMod}_t(x, y)$ of right *coactions* of t, with a functor $F : \mathrm{Hom}_X(x, y) \to \mathrm{coMod}_t(x, y)$ which is right adjoint to the forgetful functor.

2. In particular, when (T, c, d) is a comonad on a category X, we obtain an adjunction between $\mathrm{coMod}_T(1, X)$ and X, whose induced comonad on X is none other than T. We deduce that every comonad is induced by an adjunction. The category $\mathrm{coMod}_T(1, X)$ is called the Eilenberg-Moore category of the comonad T, and its objects are called *T-coalgebras*.

3. There is a category $\mathrm{coKl}_t(x, y)$, whose definition is dual to Construction 27, and which is equivalent to the full image of F. For (T, c, d) a comonad on a category X, we then obtain a notion of Kleisli category of a comonad.

Monads on $\mathbf{Set}$ and their Eilenberg-Moore categories generalise familiar notions of algebra. Comonads on $\mathbf{Set}$ are a foundation for *coalgebra*, which has a very distinct flavour in practice, centred on ideas of “state” and “observation”.