

Analogies in the bilax world

① Workshop on Process
Theory in Tallinn

by John Bourke

Plan

Discuss

① skew monoidal cats

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- ① skew monoidal cats
- ② algebraic weak factorisation systems

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- ① skew monoidal cats
 - ② algebraic weak factorisation systems
- Describe some analogies between them.
 - Some Thoughts & Questions

① Skew monoidal cats - prelim

- Strict monoidal cats $(\mathcal{C}, \otimes, i)$
 $(ab)c = a(bc), ia = a, a = ai$

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- Algebras for monoid 2-monad —
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- Lax T -algebras

The diagram illustrates the relationship between a category $\mathcal{C}$ and its T -algebra structure. It consists of two squares. The left square has $\mathcal{C}$ at the bottom-left, $T\mathcal{C}$ at the top-right, $\mathcal{C}$ at the bottom-right, and $T\mathcal{C}$ at the top-left. Arrows are: $\mathcal{C} \xrightarrow{\eta_{\mathcal{C}}} T\mathcal{C}$ (top), $\mathcal{C} \xrightarrow{i} \mathcal{C}$ (bottom), $T\mathcal{C} \xrightarrow{\mu_{\mathcal{C}}} \mathcal{C}$ (right), and $T\mathcal{C} \xrightarrow{\eta_{\mathcal{C}}} \mathcal{C}$ (left, red arrow). The right square has $T\mathcal{C}$ at the top-left, $T(T\mathcal{C})$ at the top-right, $T\mathcal{C}$ at the bottom-left, and $\mathcal{C}$ at the bottom-right. Arrows are: $T\mathcal{C} \xrightarrow{T\mu} T(T\mathcal{C})$ (top), $T\mathcal{C} \xrightarrow{\mu_{T\mathcal{C}}} T\mathcal{C}$ (right), $T(T\mathcal{C}) \xrightarrow{\mu_{T\mathcal{C}}} T\mathcal{C}$ (bottom), and $T(T\mathcal{C}) \xrightarrow{\eta_{T\mathcal{C}}} T\mathcal{C}$ (left, red arrow).

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$$\begin{array}{ccccc} & & TC & & TC \\ & \nearrow \eta_C & \downarrow m & & \downarrow m \\ C & \xrightarrow{\alpha_C} & C & & C \\ & & \downarrow m & & \downarrow m \\ & & TC & \xrightarrow{m} & C \end{array} \quad \begin{array}{ccc} & TC & \\ & \downarrow \mu_x & \\ & TC & \xrightarrow{m} C \end{array} \quad \begin{array}{ccc} & TC & \\ & \downarrow \mu_x & \\ & TC & \xrightarrow{m} C \end{array}$$

are lax monoidal cats :

$$m : C^n \rightarrow C : (x_1, \dots, x_n) \mapsto m(x_1, \dots, x_n) +$$

maps like $m(m(a,b), m(c,d)) \xrightarrow{\alpha} m(a,b,c,d)$
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- Colax T-algebras (reverse α 's) = colax monoidal cats
- Pseudo T-algebras (reverse α 's) = unbiased monoidal cats

monoidal cats (coherence thm)

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$$\odot: \mathcal{C}^2 \rightarrow \mathcal{C}, i \in \mathcal{C} +$$

$$(ab)c \xrightarrow{\alpha} a(bc), ia \xrightarrow{l} a, a \xrightarrow{r} ai$$

sat 5 equations.

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- Include monoidal cats & lots of examples.
- Internet joke
"A monad is just a monoid in the (monoidal) cat of endofunctors."

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- Hustalu, Chapman example:
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- Include monoidal cats & lots of examples.
- Hustalu, Chapman example:
"A (relative) monad is a monoid in a (skew) monoidal cat of ~~endofunctors~~."
- Not clear in what sense they are weakenings of the notion of monoid.

① Skew monoidal cats - nice props

- $L = i^{-1} : \mathcal{C} \rightarrow \mathcal{C}$ comonad, $ia \xrightarrow{L} a$

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- $L = i^- : \mathcal{C} \rightarrow \mathcal{C}$ comonad, $ia \xrightarrow{L} a$,
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& distributive law of L over R .

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- Also Skew Mon cats



Lax mon cats

$$(a, b, c, d) \mapsto$$

$$((ia)b)c)d$$

Colax mon cats

$$(a, b, c, d) \mapsto$$

$$a(b(c(di)))$$

② Awfs prelims

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$\rightarrow S : (-)^\downarrow : \mathbf{Cat} \rightarrow \mathbf{Cat}$ a 2-monad
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Pseudo S-alg $\equiv$ orthogonal Fact. systems
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$$a \xrightarrow{f} b$$

$$\downarrow \text{LEK} \quad \quad \quad \uparrow \text{MEK}$$
$$a \rightarrow c \rightarrow b$$

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Lax ... = ^{up} lax Factorisation algebras

Colax ... = colax Factorisation algs

Rosicky-Tholen (not studied much)

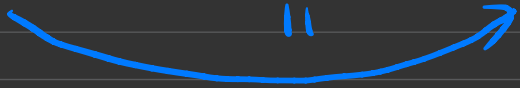
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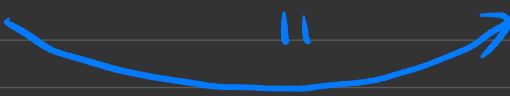
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- Capture maps with structure.
 - eg. awfs on Cat with R-algebras
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- Arise in models of homotopy type
tn.

Awfs - nice properties

- Awfs on $\mathcal{C}$ involves distrib
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Aufs - nice properties

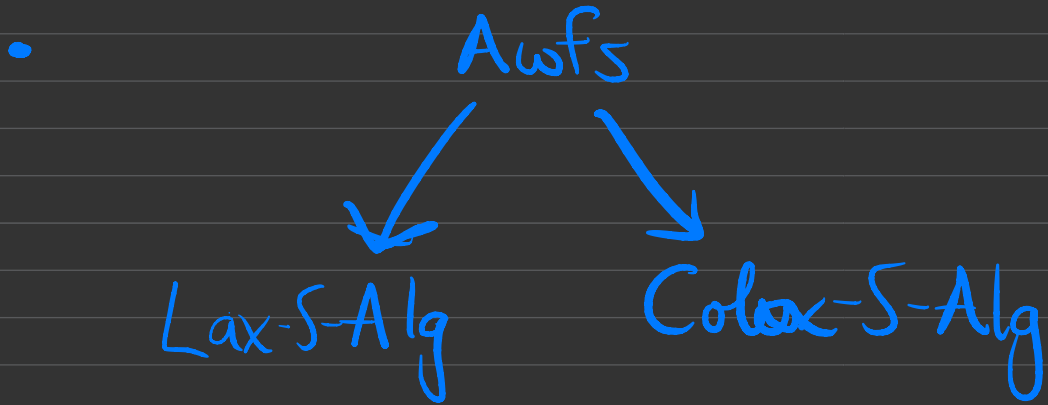
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Thanks!

