Analogies in the bilax world

@ Workshop on Process Theory in Tallinn

by John Bourke

Plan

Discuss

① skew monoidal cats

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② algebraic weak factorisation systems

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- Describe some analogies between them.

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- Some Thoughts & Questions

(1) Skew monoidal cats - prelim

- Strict monoidal cats $(\mathcal{C}, \otimes, i)$

  $(ab)c = a(bc)$, $ia = a$, $a = ai$

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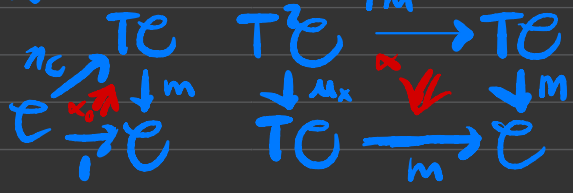

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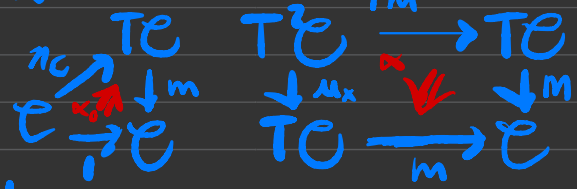

are lax monoidal cats :

$m : \mathcal{C}^n \to \mathcal{C} : (x_1, \ldots, x_n) \mapsto m(x_1, \ldots, x_n) +$

maps like $m(m(a,b), m(c,d)) \xrightarrow{\alpha} m(a,b,c,d)$

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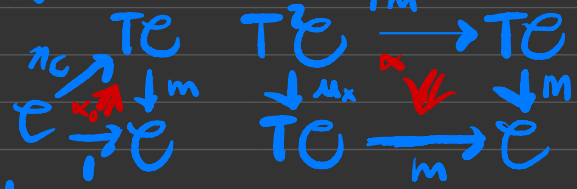

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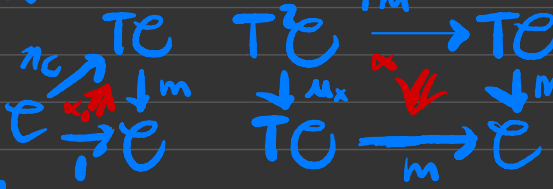
- Lax T-algebras 

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- Colax T-algebras (reverse α's) = colax monoidal cats
- Pseudo T-algebras (reverse α's) = unbiased monoidal cats

monoidal cats (coherence thm)

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$\otimes : \mathcal{C}^2 \to \mathcal{C}$, $i \in \mathcal{C}$ +

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sat 5 equations.

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- Internet joke

"A monad is just a monoid in the (monoidal) cat of endofunctors."

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- Mustalu, Chapman example:
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- Include monoidal cats & lots of examples.
- Mustafu, Chapman example:
  "A (relative) monad is a monoid in a (skew) monoidal cat of ~~endo~~functors."
- Not clear in what sense they are weakenings of the notion of monoid.

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- Also Skew Mon Cats

Lax mon cats
$(a,b,c,d) \mapsto$
$(((ia)b)c)d$

Colax mon cats
$(a,b,c,d) \mapsto$
$a(b(c(d\ i)))$

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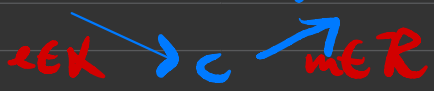

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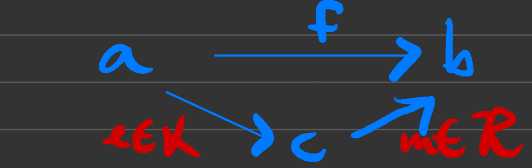

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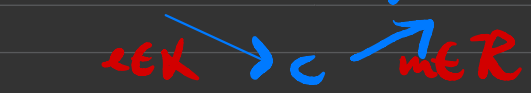

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- $\downarrow = \{0 \to 1\}$ a comonoid in Cat
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Pseudo S-alg $\equiv$ orthogonal Fact. systems (UIP)

Lax ... = lax factorisation algebras

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Rosicky-Tholen (not studied much)

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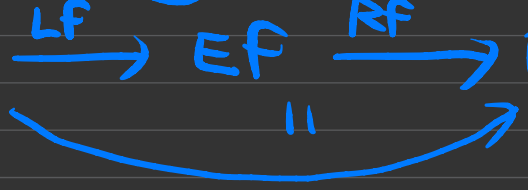

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- Arise in models of homotopy type Th.

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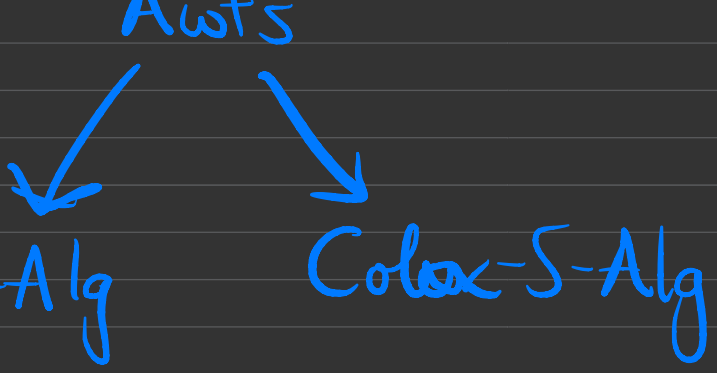

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