On the axioms of $\mathcal{M}, \mathcal{N}$-adhesive categories

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Università degli Studi di Padova

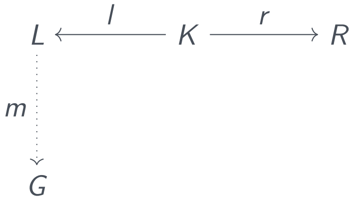
Double pushout

Let us begin introducing the *double pushout approach* to graph rewriting (cfr. Ehrig et al. 2006).

A *rewriting rule* is a pair of arrows $l : K \to L$ and $r : K \to R$ with the same domain.

To rewrite a graph G according to a rule we proceed in three steps.

First: find a *match* $m : L \to G$.



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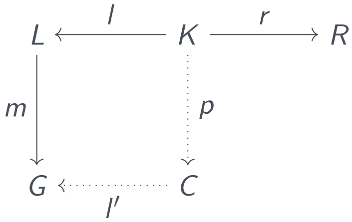
Second: remove the image of m building a pushout square.



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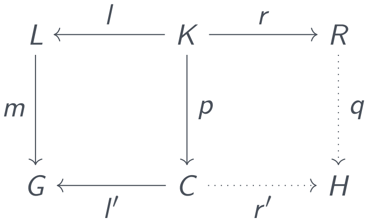
Third: fill the hole so obtained, with R taking a pushout.



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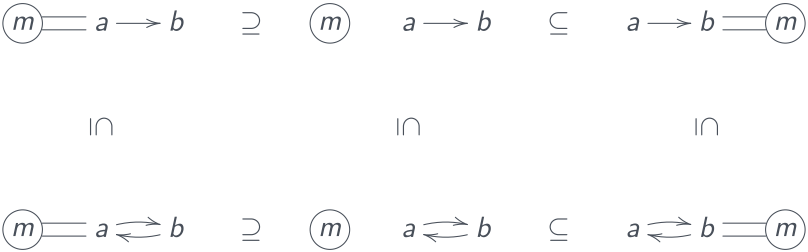


Double pushout

This construction involves only categorical notions, so it can be done in any category.

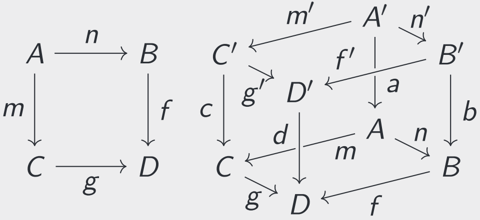
Definition

A *DPO*-rewriting system is a pair $(\mathbf{X}, R)$ where $\mathbf{X}$ is a category and R is a set of rewriting rules.

Adhesivity and *quasiadhesivity* has been introduced in Lack and Sobociński 2005 to guarantee good properties, like the Church-Rosser Theorem, of DPO-rewriting system.

Definition: Van Kampen square

A pushout square as the one on the left is *Van Kampen* if in any cube constructed upon it, having pullbacks as back faces, the top face is a pushout if and only if the front faces are pullbacks. If it enjoys only the "if" it is called a *stable square*.



$\mathcal{M}, \mathcal{N}$-adhesive categories

Now, take a category $\mathbf{X}$, and fix a class of monos $\mathcal{M}$ and a class of arrows $\mathcal{N}$. Suppose also that they interact "nicely", i.e. they enjoy some composition and decomposition property. We will refer to such a pair as a *preadhesive structure*.

Meaning of $\mathcal{M}$ and $\mathcal{N}$

The intended meaning of the classes $\mathcal{M}$ and $\mathcal{N}$ is the following:

- $\mathcal{M}$ is the class to which the two legs (or at least one) of the rules we want to use belong;
- $\mathcal{N}$ is the class to which the allowed matches belong.

$\mathcal{M},\mathcal{N}$-adhesive categories

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Definition (Habel and Plump 2012)

$\mathbf{X}$ is $\mathcal{M},\mathcal{N}$*-adhesive* if

1. every cospan $C \xrightarrow{g} D \xleftarrow{m} B$ with $m \in \mathcal{M}$ can be completed to a pullback ($\mathcal{M}$*-pullbacks*);
2. every span $C \xleftarrow{m} A \xrightarrow{n} B$ with $m \in \mathcal{M}$ and $n \in \mathcal{N}$ can be completed to a pushout ($\mathcal{M},\mathcal{N}$*-pushouts*);
3. $\mathcal{M},\mathcal{N}$-pushouts are Van Kampen squares.

Remark

If we take $\mathcal{M}$ to be the class of all (regular) monos and $\mathcal{N}$ to be the class of all maps we get obtain the traditional notion of (quasi)adhesivity.

It can also be shown that the class of $\mathcal{M}, \mathcal{N}$-adhesive categories is closed under the most common categorical constructions.

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4. hyerarchical hypergraphs: hypergraphs in which edges form a simple graph or a directed acyclic graph;
5. the category of *term graphs* (see Corradini and Gadducci 2005; Plump 1999) is quasiadhesive.

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Question

Can we generalize these two results to $\mathcal{M}, \mathcal{N}$-adhesive categories?

Definition

Let $\mathcal{M}$ be a class of monos and $\mathcal{N}$ another class of arrows, a monomorphism $u : U \to X$ is a *$\mathcal{M}, \mathcal{N}$-union* if there exist $m \in \mathcal{M}$ and $n \in \mathcal{M} \cap \mathcal{N}$ such that, in the poset $(\mathrm{Sub}(X), \leq)$,

$$[u] = [m] \vee [n]$$

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Definition

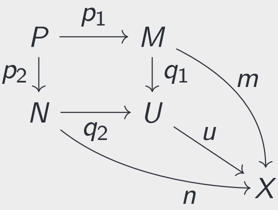
Let f be an arrow in $\mathcal{M} \cap \mathcal{N}$ and (Q_f, y_1, y_2) its cokernel pair. The *$\mathcal{M},\mathcal{N}$-codiagonal* of f is the unique arrow $\nu_f : Q_f \to Y$ such that

$$\nu_f \circ y_1 = \mathrm{id}_Y = \nu_f \circ y_2$$

The poset of subobjects: I

Theorem

Let $\mathbf{X}$ be a $\mathcal{M},\mathcal{N}$-adhesive category with all pullbacks. If $\mathcal{M} \cap \mathcal{N}$ contains all split monomorphisms and $\mathcal{N}$ contains all $\mathcal{M},\mathcal{N}$-codiagonals, then $\mathcal{M}$ is closed under $\mathcal{M},\mathcal{N}$-unions.
Moreover, such unions are *effective*: if $u : U \to X$ is the $\mathcal{M},\mathcal{N}$-union of $m : M \to X$ and $n : N \to X$, then it fits in the diagram below, in which the outer boundary is a pullback and the inner square a pushout.



The poset of subobjects: II

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The previous result has a converse. Given a preadhesive structure $(\mathcal{M}, \mathcal{N})$, the presence of $\mathcal{M}, \mathcal{N}$-unions allows us to deduce $\mathcal{M}, \mathcal{N}$-adhesivity.

Definition

A morphism $m : X \to Y$ in $\mathbf{X}$ is $\mathcal{N}$*-preadhesive* if for every $n : X \to Z$ in $\mathcal{N}$, a stable pushout square of n along m exists and it is also a pullback. m is $\mathcal{N}$*-adhesive* if all its pullbacks are $\mathcal{N}$-preadhesive.

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Theorem

Let $(\mathcal{M}, \mathcal{N})$ be a preadhesive structure on a category $\mathbf{X}$ with pullbacks. Suppose that every split mono is in $\mathcal{M} \cap \mathcal{N}$, every arrow in $\mathcal{M}$ is $\mathcal{N}$-adhesive and $\mathcal{M}$ is closed under $\mathcal{M}, \mathcal{N}$-unions, then $\mathbf{X}$ is $\mathcal{M}, \mathcal{N}$-adhesive.

An embedding theorem

Our next step is to construct an embedding of an $\mathcal{M}, \mathcal{N}$-adhesive category $\mathbf{X}$ into a topos. The strategy is to construct a suitable Grothendieck topology on $\mathbf{X}$ in such a way that the representables functors are sheaves and such that $\mathcal{M}, \mathcal{N}$-pushouts are preserved by the Yoneda embedding.

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Definition

Let $(\mathcal{M},\mathcal{N})$ be a preadhesive structure for a category $\mathbf{A}$. A $j_{\mathcal{M},\mathcal{N}}$*-covering family* for an object X is a set $\{p, q\}$ of arrows $p : Z \to X$ and $q : Y \to X$ such that there exist $m : N \to Y$ in $\mathcal{M}$ and $n : N \to Z$ in $\mathcal{N}$ making the square on the right a pushout.

$$\begin{array}{ccc} N & \xrightarrow{n} & Z \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow[q]{} & X \end{array}$$

An embedding Theorem

The next step is characterize sheaves for $j_{\mathcal{M},\mathcal{N}}$.

Lemma

If **X** is $\mathcal{M},\mathcal{N}$-adhesive, then a presheaf $F : \mathbf{X}^{op} \to \mathbf{Set}$ is a sheaf for $j_{\mathcal{M},\mathcal{N}}$ if and only if it sends $\mathcal{M},\mathcal{N}$-pushouts to pullbacks.

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Theorem

Let $\mathbf{X}$ be a $\mathcal{M},\mathcal{N}$-adhesive category, with pullbacks such that $\mathcal{M} \cap \mathcal{N}$ contains every split mono, and $\mathcal{N}$ contains all $\mathcal{M},\mathcal{N}$-codiagonals. Then the Yoneda embedding $よ_{\mathbf{X}} : \mathbf{X} \to \mathbf{Set}^{\mathbf{X}^{op}}$ factors through a full and faithful functor $よ'_{\mathbf{X}} : \mathbf{X} \to \mathbf{Sh}(\mathbf{X}, j_{\mathcal{M},\mathcal{N}})$ preserving pullbacks and $\mathcal{M},\mathcal{N}$-pushouts.

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The representable functors sends all pushouts to pullbacks and so are sheaves for $j_{\mathcal{M},\mathcal{N}}$, showing that the Yoneda embedding factors through a full and faithful functor $よ'_{\mathbf{X}} : \mathbf{X} \to \mathbf{Sh}(\mathbf{X}, j_{\mathcal{M},\mathcal{N}})$.

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Let F be any sheaf for $j_{\mathcal{M},\mathcal{N}}$ and consider the two squares, in which the left one is an $\mathcal{M},\mathcal{N}$-pushouts:

$$\begin{array}{ccc} X & \xrightarrow{n} & Z \\ m\downarrow & & \downarrow q \\ Y & \xrightarrow[p]{} & Q \end{array} \qquad \begin{array}{ccc} F(Q) & \xrightarrow{F(q)} & F(Z) \\ F(p)\downarrow & & \downarrow F(n) \\ F(Y) & \xrightarrow[F(m)]{} & F(X) \end{array}$$

Since F is a sheaf the square on the right is a pullback.

Now we can apply the Yoneda Lemma:

$$\begin{array}{ccccc}
\mathrm{nat}(\text{よ}_{\mathbf{X}}(Q), F) & & \xrightarrow{(-)\circ \text{よ}_{\mathbf{X}}(q)} & & \mathrm{nat}(\text{よ}_{\mathbf{X}}(Z), F) \\
& \searrow^{y_Q} & & \swarrow^{y_Z} & \\
& F(Q) & \xrightarrow{F(q)} & F(Z) & \\
{\scriptstyle (-)\circ \text{よ}_{\mathbf{X}}(p)}\big\downarrow & {\scriptstyle F(p)}\big\downarrow & & \big\downarrow{\scriptstyle F(n)} & \big\downarrow{\scriptstyle (-)\circ \text{よ}_{\mathbf{X}}(n)} \\
& F(Y) & \xrightarrow[F(m)]{} & F(X) & \\
& \nearrow^{y_Y} & & \nwarrow^{y_X} & \\
\mathrm{nat}(\text{よ}_{\mathbf{X}}(Y), F) & & \xrightarrow[(-)\circ \text{よ}_{\mathbf{X}}(m)]{} & & \mathrm{nat}(\text{よ}_{\mathbf{X}}(X), F)
\end{array}$$

The outer square is a pullback too. Since this holds for every sheaf F we get that $\mathbf{X}(-, Q)$ is a pushout.

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- By Cockett and Guo 2007, the category of partial maps on $\mathbf{X}$ is a join restriction category if and only if $\mathbf{X}$ is adhesive. Is there some deeper connection between $(\mathcal{M}, \mathcal{N}-)$adhesive categories and monoidal ones?

Thank you for your attention!

Bibliography I

- Cockett, Robin and X Guo (2007). "Join restriction categories and the importance of being adhesive". In: *Unpublished manuscript, slides from CT* 7.
- Corradini, A. and F. Gadducci (2005). "On term graphs as an adhesive category". In: *Proceedings of the 2nd International Workshop on Term Graph Rewriting (TERMGRAPH 2004)*. Ed. by M. Fernández. Vol. 127(5). Electronic Notes in Theoretical Computer Science. Elsevier, pp. 43–56.
- Ehrig, H. et al. (2006). *Fundamentals of Algebraic Graph Transformation*. Monographs in Theoretical Computer Science. Springer Science & Business Media.
- Habel, A. and D. Plump (2012). "$\mathcal{M}, \mathcal{N}$-Adhesive Transformation Systems". In: *6th International Conference on Graph Transformation (ICGT)*. Ed. by H. Ehrig et al. Vol. 7562. Lecture Notes in Computer Science. Springer Science & Business Media, pp. 218–233.
- Lack, S. and P. Sobociński (2005). "Adhesive and quasiadhesive categories". In: *RAIRO-Theoretical Informatics and Applications* 39.3, pp. 511–545.

Bibliography II

Lack, S. and P. Sobociński (2006). "Toposes Are Adhesive". In: *3rd International Conference on Graph Transformation (ICGT)*. Ed. by A. Corradini et al. Vol. 4178. Lecture Notes in Computer Science. Springer Science & Business Media, pp. 184–198.

Padberg, J. (2017). "Hierarchical Graph Transformation Revisited - Transformations of Coalgebraic Graphs". In: *10th International Conference on Graph Transformation (ICGT)*. Ed. by J. De Lara and D. Plump. Vol. 10373. Lecture Notes in Computer Science. Springer Science & Business Media, pp. 20–35.

Palacz, W. (2004). "Algebraic hierarchical graph transformation". In: *Journal of Computer and System Sciences* 68.3, pp. 497–520.

Plump, D. (1999). "Term Graph Rewriting". In: *Handbook of Graph Grammars and Computing by Graph Transformations, Vol. 2: Applications, Languages, and Tools*. Ed. by H. Ehrig et al. World Scientific, pp. 3–61.

Sobocinski P. nd Heindel, T. (2011). "Being Van Kampen is a universal property". In: *Logical Methods in Computer Science* 7.