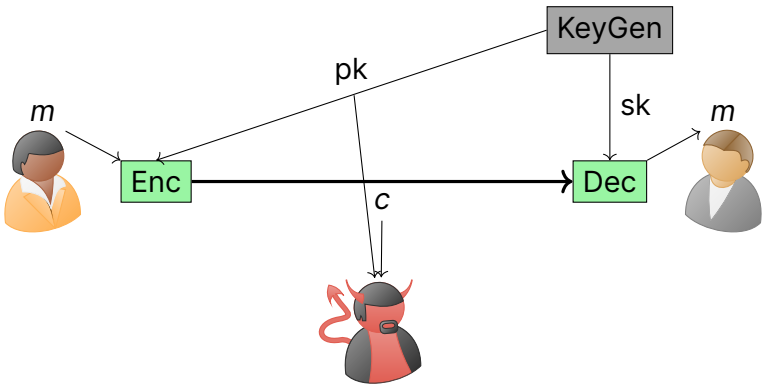
Resources in crypto proofs

Peeter Laud

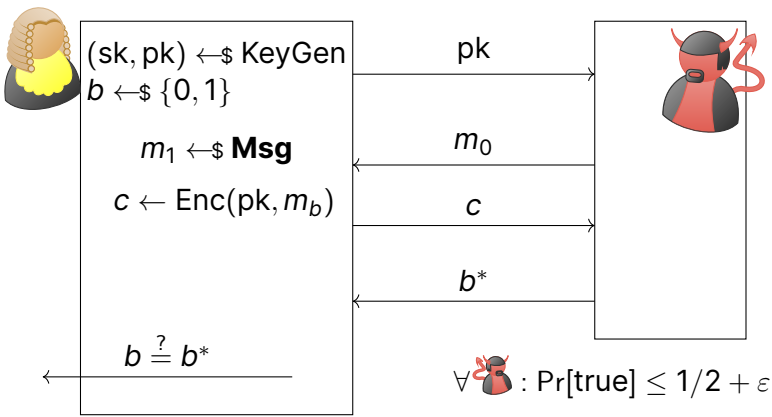
Public-key encryption



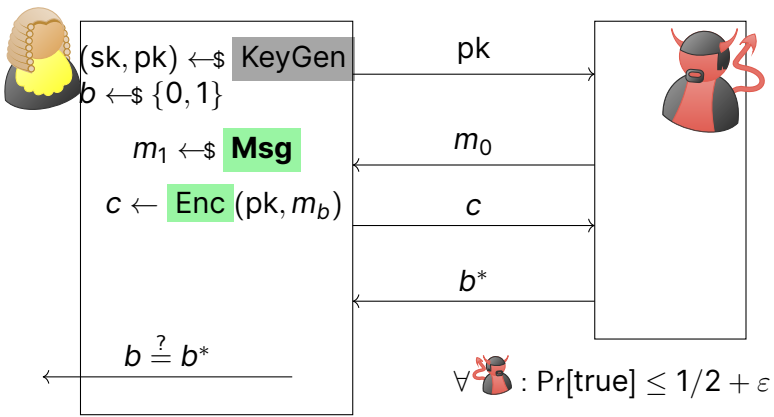
ElGamal encryption

- $\mathbb{G}$ — a large cyclic group. $\mathbb{G} = \langle g \rangle$. $|\mathbb{G}| = p$
- KeyGen works by: $\mathsf{sk} \leftarrow\!\!\$\ \mathbb{Z}_p$, $\mathsf{pk} \leftarrow g^{\mathsf{sk}}$
- Message space: $\mathbf{Msg} = \mathbb{G}$
- $\mathsf{Enc}(\mathsf{pk}, m) = (g^r, m \cdot \mathsf{pk}^r)$, where $r \leftarrow\!\!\$\ \mathbb{Z}_p$
- $\mathsf{Dec}(\mathsf{sk}, (c_1, c_2)) = c_2 / c_1^{\mathsf{sk}}$

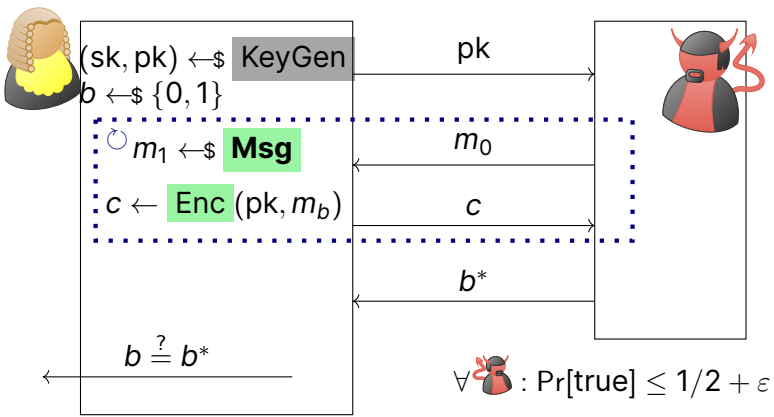
IND-CPA security



IND-CPA security

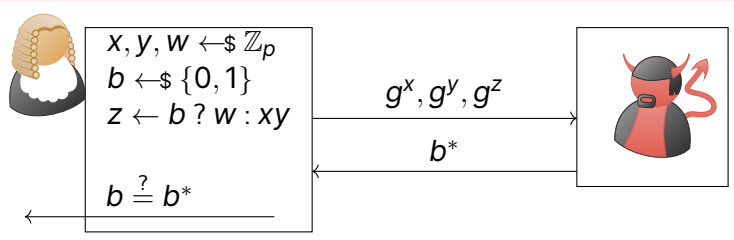


IND-CPA security



Hardness assumptions

Decisional Diffie-Hellman problem in $\mathbb{G}$



Hardness assumptions

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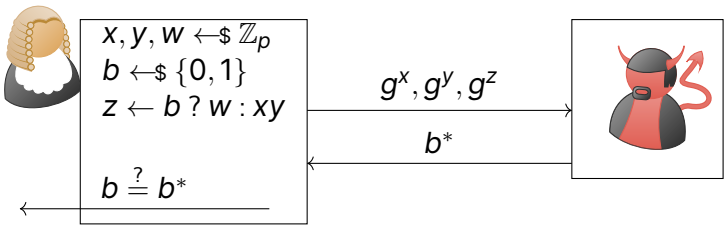

Hardness assumption

Assume that $\forall$  : $\Pr[\text{true}] \leq 1/2 + \varepsilon$

Hardness assumptions

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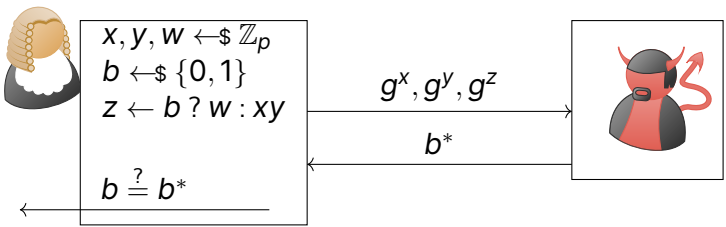


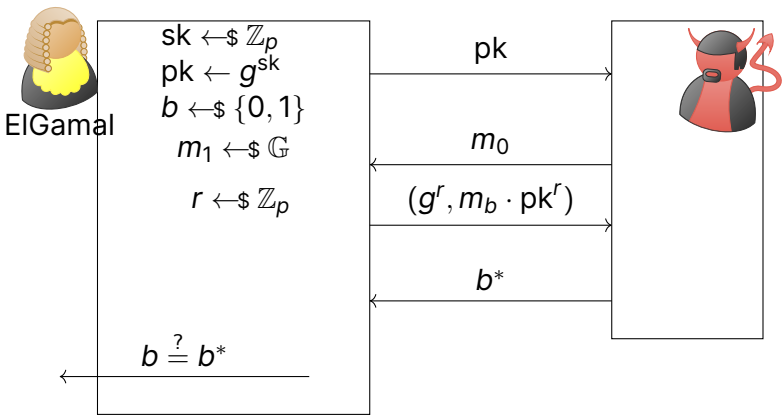
Hardness assumption

Assume that $\forall$  : $\Pr[\text{true}] \leq 1/2 + \varepsilon$

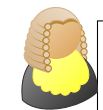
Polynomial running time?

- Yes (for )
- This presentation is a bit fast and loose with that

ElGamal is IND-CPA secure



ElGamal is IND-CPA secure



ElGamal

$\mathsf{sk} \leftarrow\$ \mathbb{Z}_p$

$\mathsf{pk} \leftarrow g^{\mathsf{sk}}$

$b \leftarrow\$ \{0,1\}$

$m_1 \leftarrow\$ \mathbb{G}$

$r \leftarrow\$ \mathbb{Z}_p$

pk

m_0

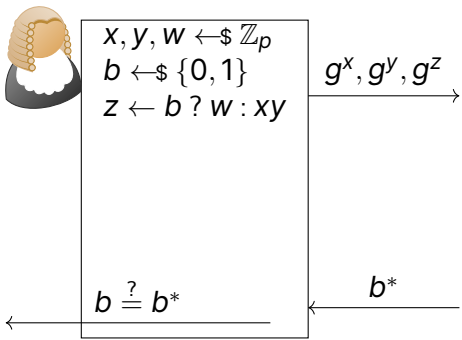
$(g^r, m_b \cdot \mathsf{pk}^r)$

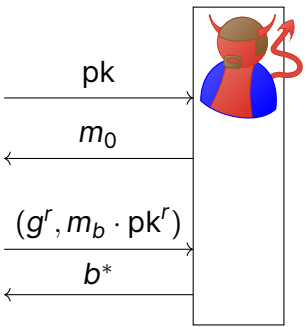
b^*

$b \stackrel{?}{=} b^*$

Suppose $\Pr[\text{true}] > 1/2 + \varepsilon$

ElGamal is IND-CPA secure





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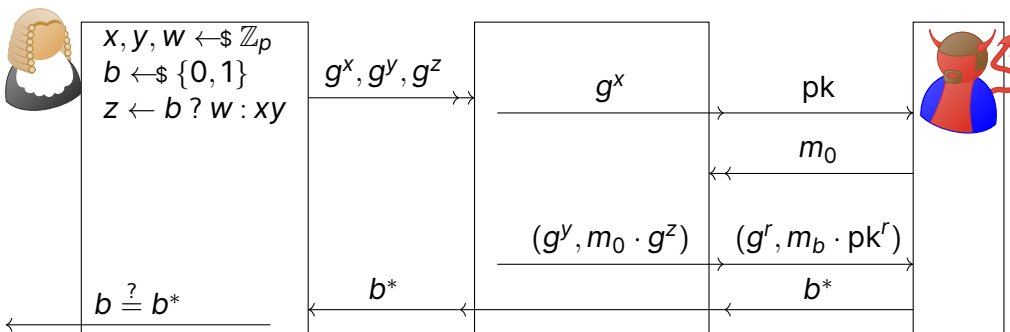

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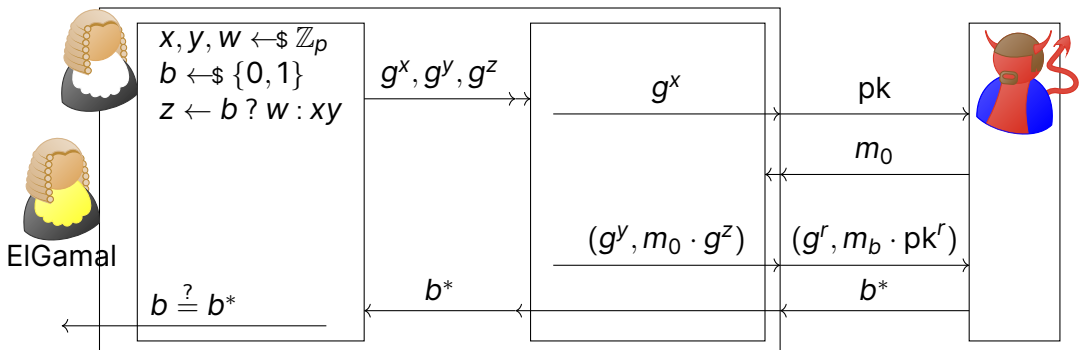

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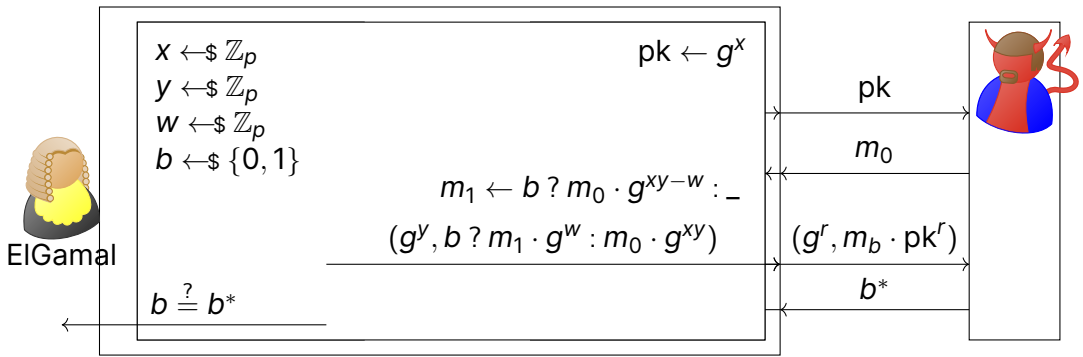

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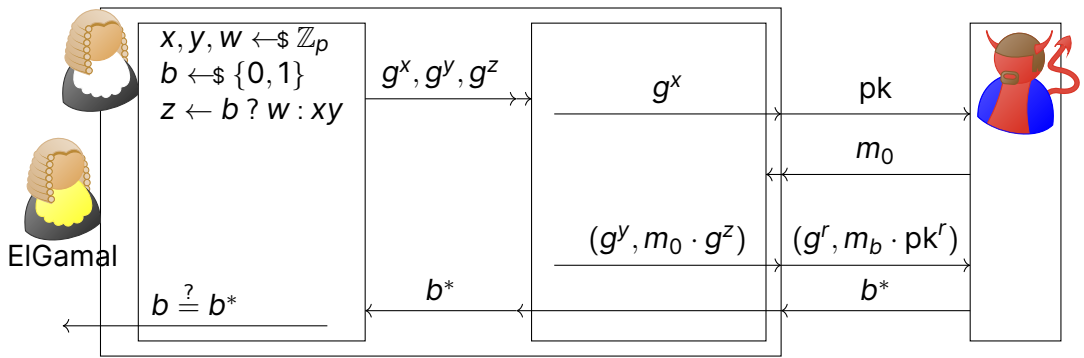

$$\Pr[\text{true}] > 1/2 + \varepsilon$$

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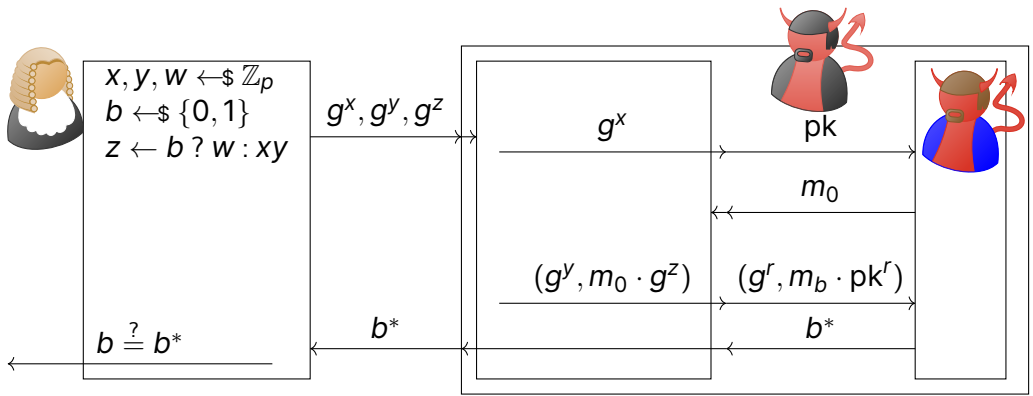

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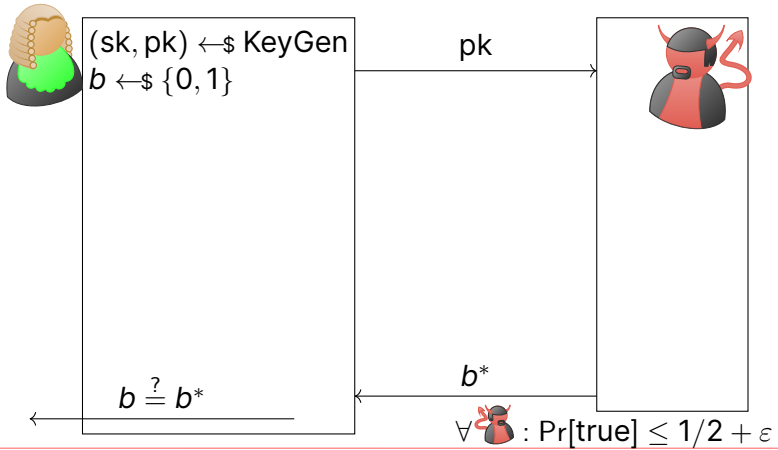
IND-CCA security



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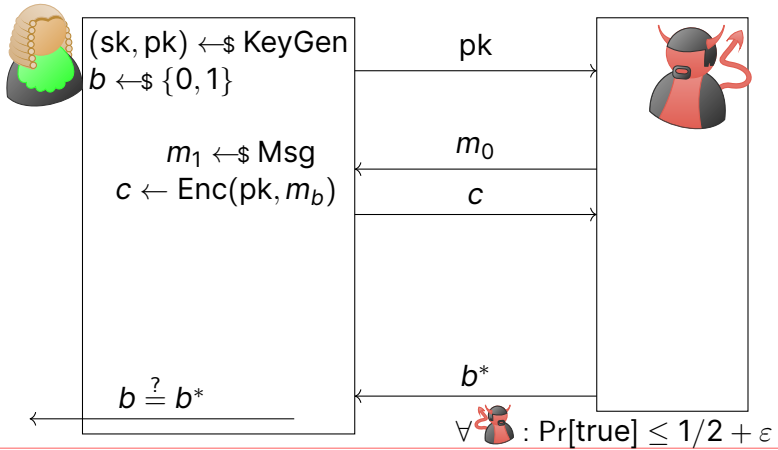

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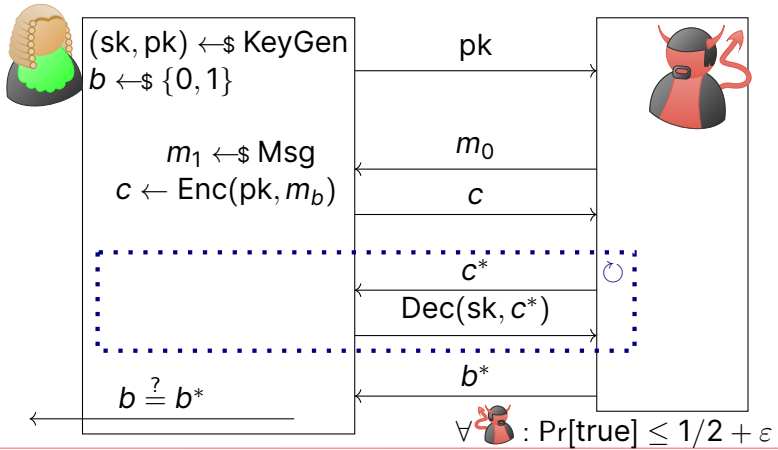

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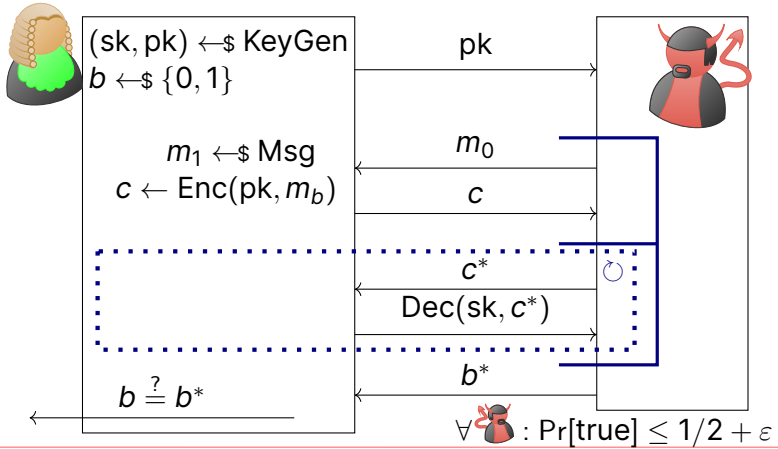

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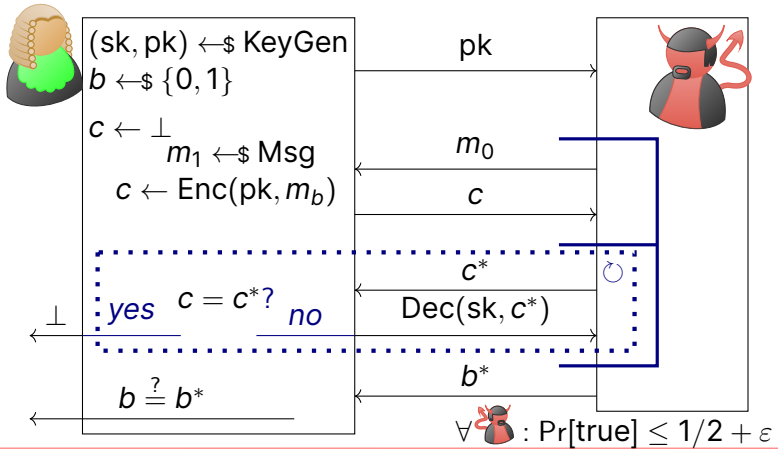
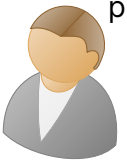


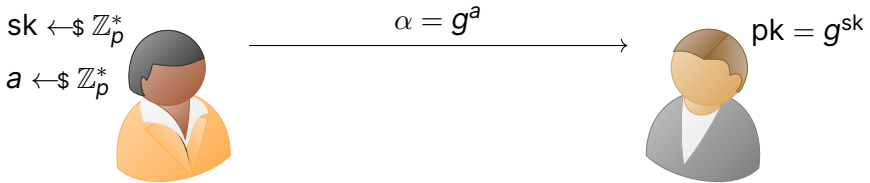
Proof of knowledge of exponent

$\mathsf{sk} \leftarrow\!\$\ \mathbb{Z}_p^*$



$\mathsf{pk} = g^{\mathsf{sk}}$

Proof of knowledge of exponent



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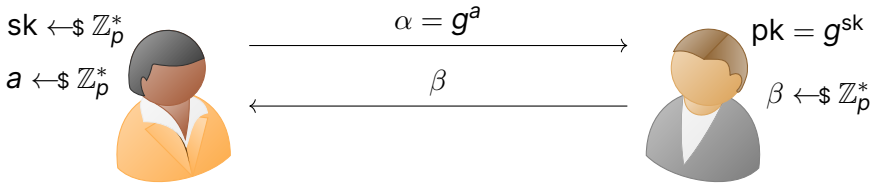

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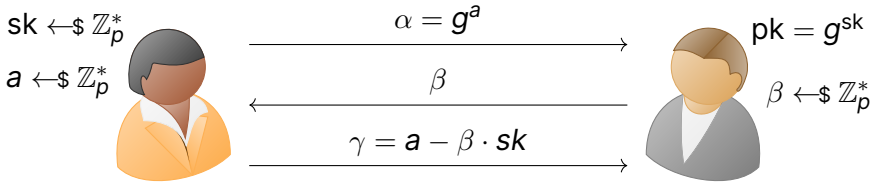

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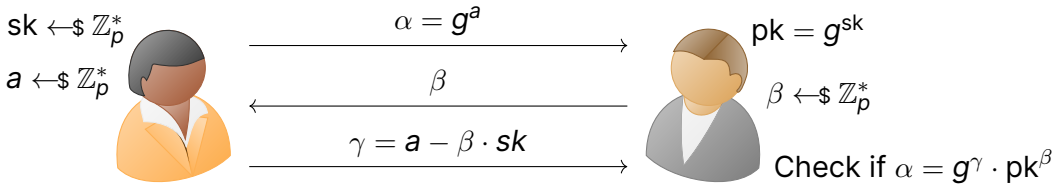
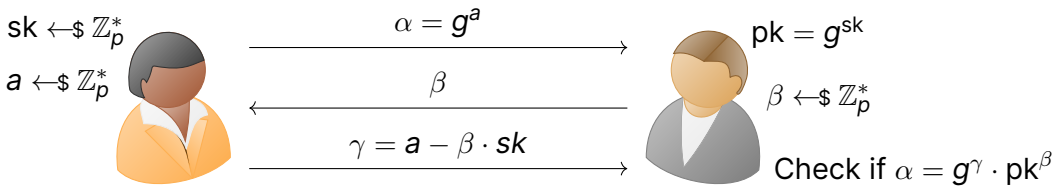
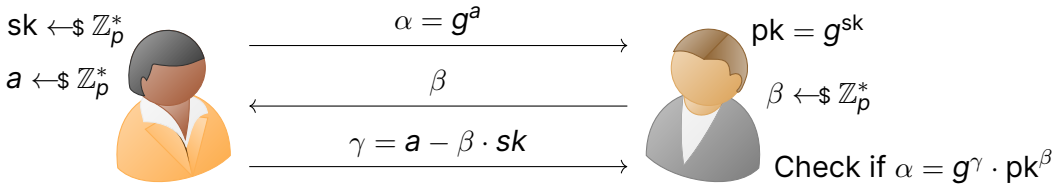


Proof of knowledge of exponent



- **Soundness:** is convinced: ability to respond to several different β-s (for the same α) implies knowledge of *sk*
  - $sk = (\gamma_1 - \gamma_2)/(\beta_2 - \beta_1)$

Proof of knowledge of exponent



- **Soundness:** is convinced: ability to respond to several different β-s (for the same α) implies knowledge of *sk*
  - sk $= (\gamma_1 - \gamma_2)/(\beta_2 - \beta_1)$
- **ZK:** learns nothing: can sample triples (α, β, γ) without
  - β, γ ←$ $\mathbb{Z}_p^*$. Compute $\alpha \leftarrow g^\gamma \cdot pk^\beta$

Fiat-Shamir heuristic

- β is random. A (one-way) hash function H has “random” outputs
- What if 🤔 herself computes $\beta = H(\alpha, \text{ctxt})$?
  - This is called the *Fiat-Shamir heuristic*
- The proof of knowledge is then a single message (α, γ) expressing that
  - 🤔 knows sk
  - 🤔 wanted to present this proof in the context ctxt
- 🧐 verifies by recomputing β, checking the equation

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- The proof of knowledge is then a single message (α, γ) expressing that
  -  knows sk
  -  wanted to present this proof in the context ctxt
-  verifies by recomputing β, checking the equation
- If β is shorter than α, and α can be computed from $\beta, \gamma, \text{pk}, \text{ctxt}$, then the proof message may be (β, γ)

ElGamal with proof-of-knowledge of exponent

- $\mathbb{G}$ — a large cyclic group. $\mathbb{G} = \langle g \rangle$. $|\mathbb{G}| = p$
- KeyGen works by: $\mathsf{sk} \leftarrow\!\!\$\ \mathbb{Z}_p$, $\mathsf{pk} \leftarrow g^{\mathsf{sk}}$
- Message space: **Msg** $= \mathbb{G}$

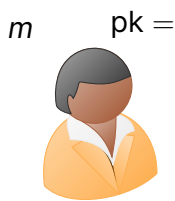
Enc(pk, m)

- $r, r' \leftarrow\!\!\$\ \mathbb{Z}_p$
- $c_1 \leftarrow g^r$, $c_2 \leftarrow m \cdot \mathsf{pk}^r$, $\alpha \leftarrow g^{r'}$
- $\beta \leftarrow H(c_1, c_2, \alpha)$
- $\gamma \leftarrow r' + \beta \cdot r$
- output $(c_1, c_2, \alpha, \gamma)$

Dec(sk, $(c_1, c_2, \alpha, \gamma)$)

- $\beta \leftarrow H(c_1, c_2, \alpha)$
- $g^\gamma \stackrel{?}{=} \alpha \cdot c_1^\beta$
- output c_2 / c_1^{sk}

Random oracle model



Random oracle model

m

$\text{pk} = g^{\text{sk}}$

sk

$r, r' \leftarrow\$ \mathbb{Z}_p$

$c_1 \leftarrow g^r$

$c_2 \leftarrow m \cdot \text{pk}^r$

$\alpha \leftarrow g^{r'}$

Random oracle model

m

$r, r' \leftarrow\$ \mathbb{Z}_p$

$c_1 \leftarrow g^r$

$c_2 \leftarrow m \cdot \mathrm{pk}^r$

$\alpha \leftarrow g^{r'}$

$\mathrm{pk} = g^{\mathrm{sk}}$

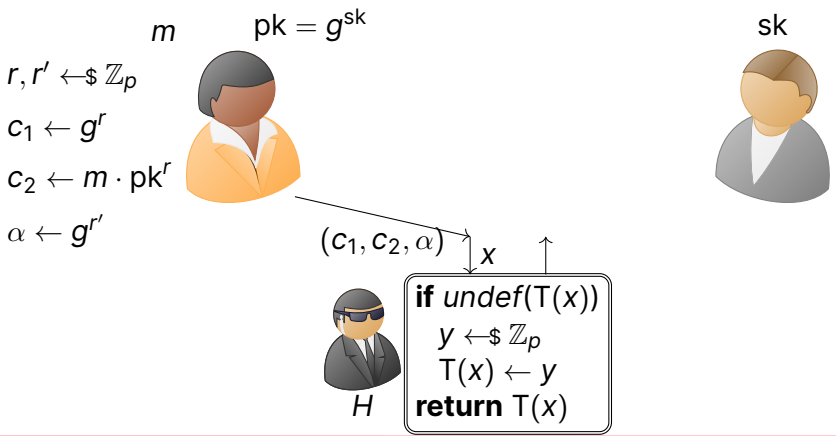
sk

x

H

if *undef*$(\mathsf{T}(x))$
  $y \leftarrow\$ \mathbb{Z}_p$
  $\mathsf{T}(x) \leftarrow y$
return $\mathsf{T}(x)$

Random oracle model



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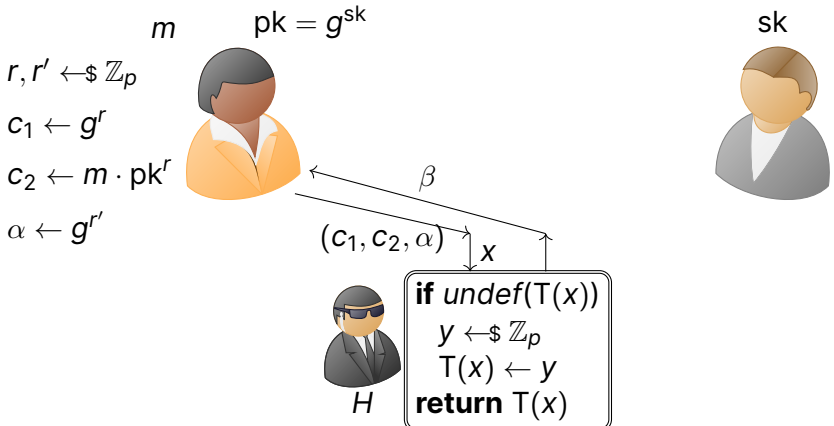

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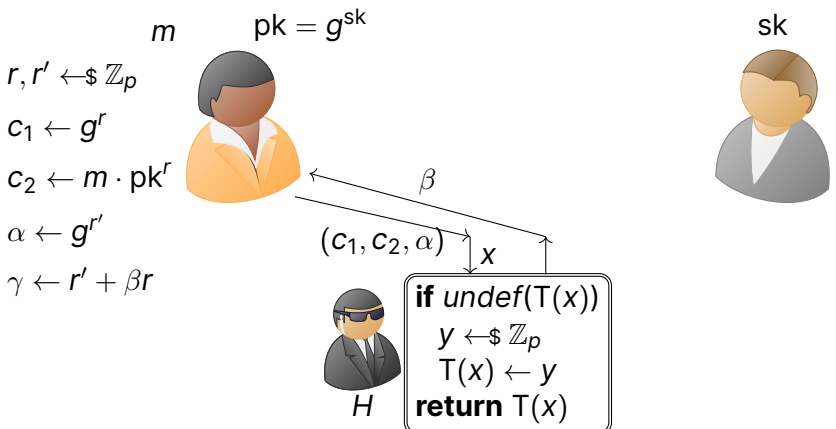

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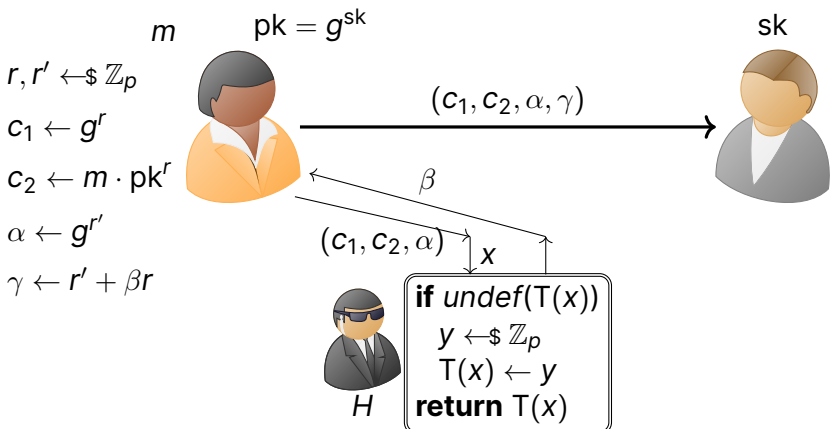
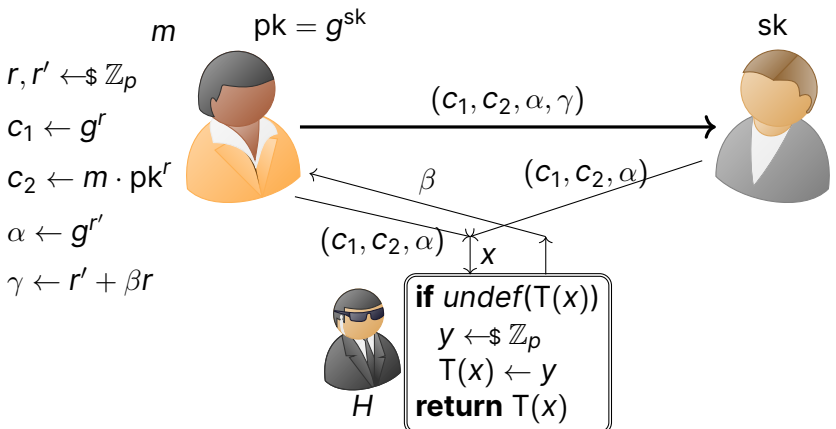


Random oracle model



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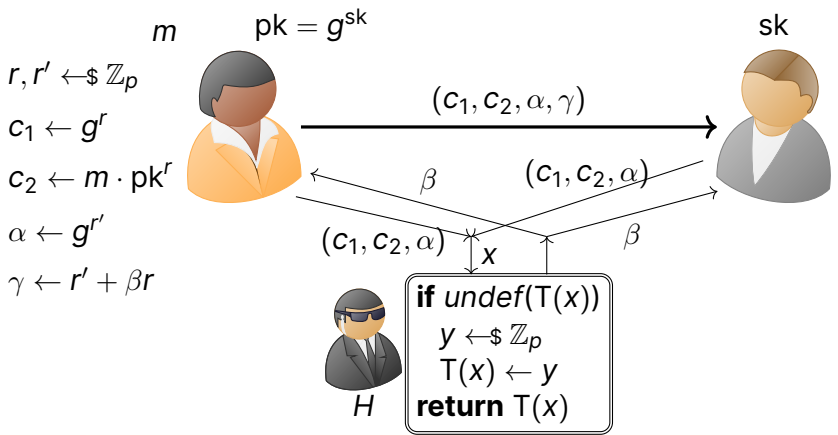

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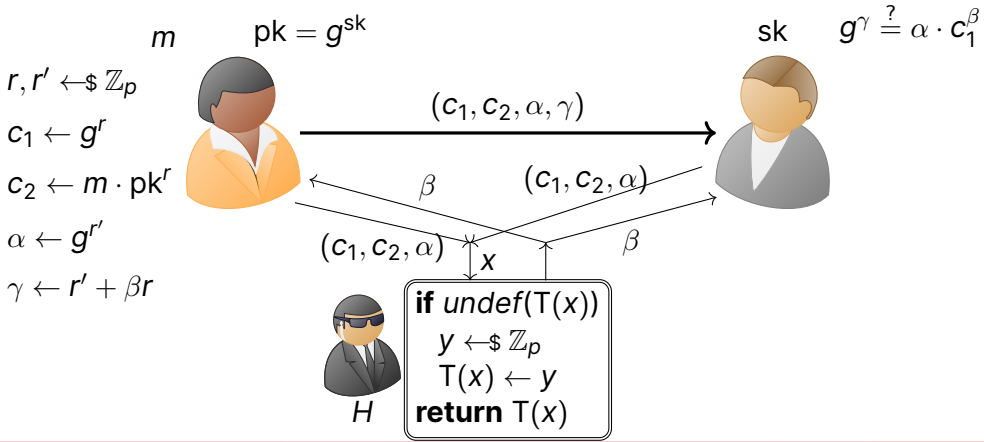

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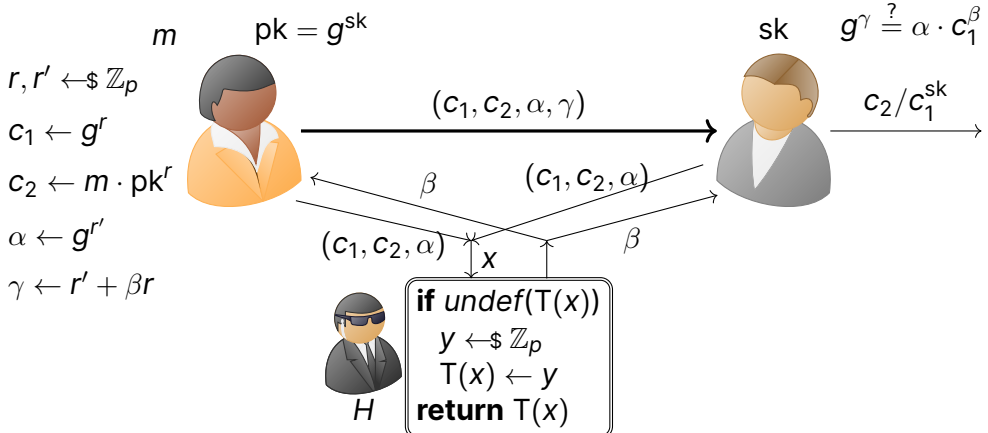
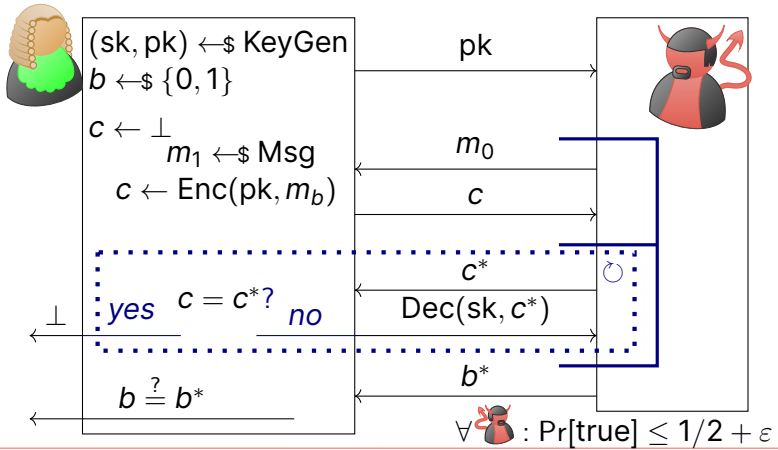
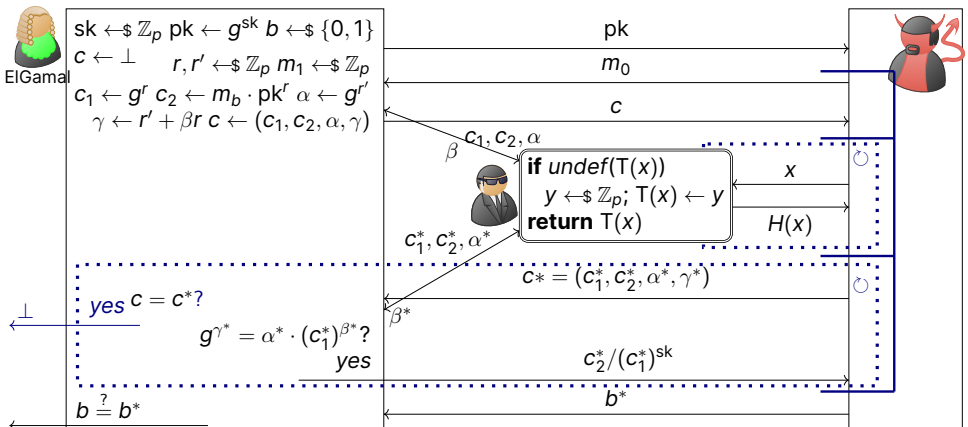


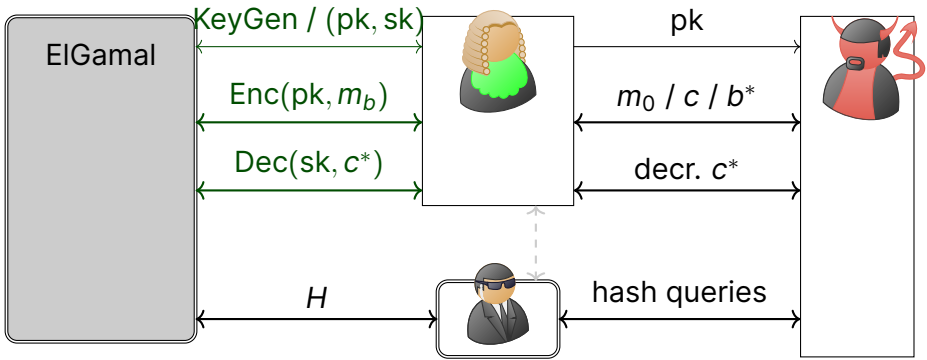
IND-CCA security



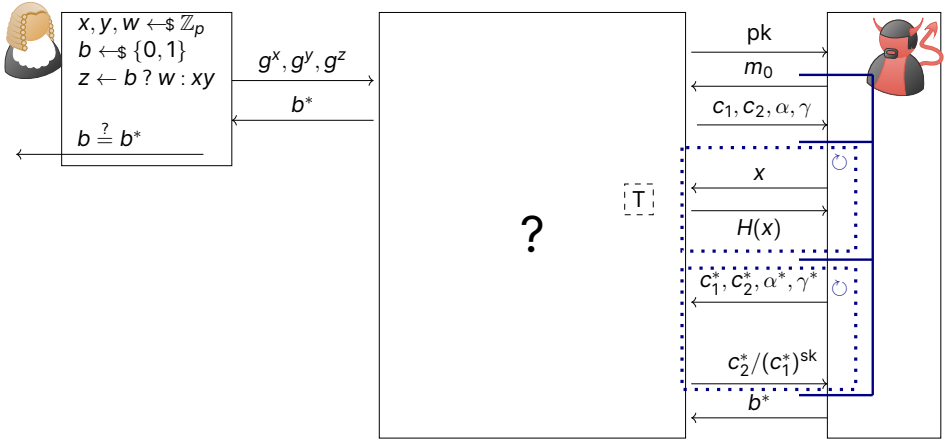
ElGamal with proof of kn. of. exp. & IND-CCA



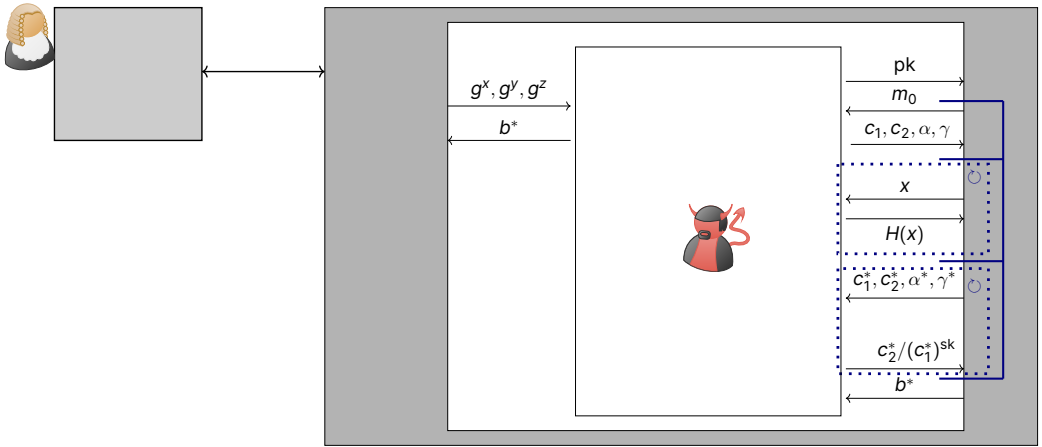
The same, with more boxes



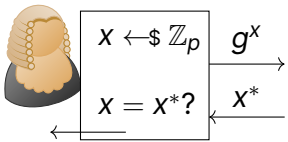
The reduction?



Meta-reductions



Rewinding / rewindability



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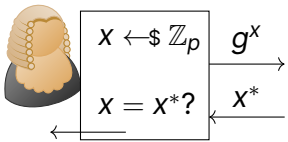

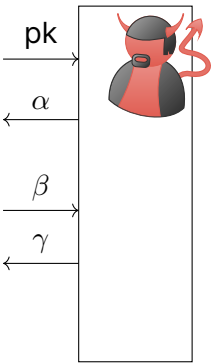

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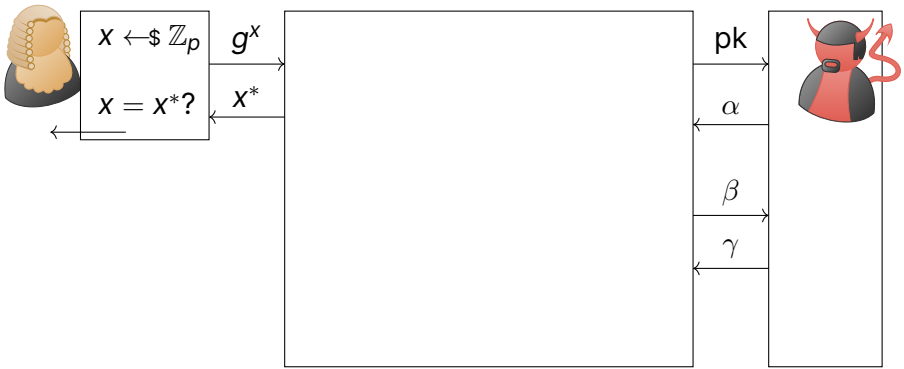
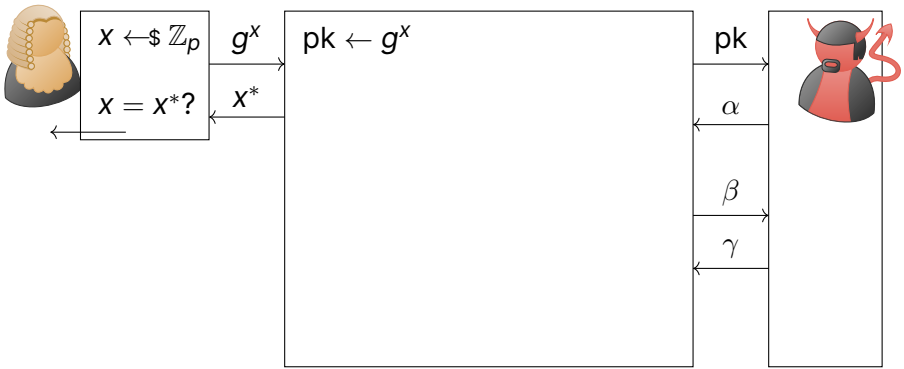


Rewinding / rewindability



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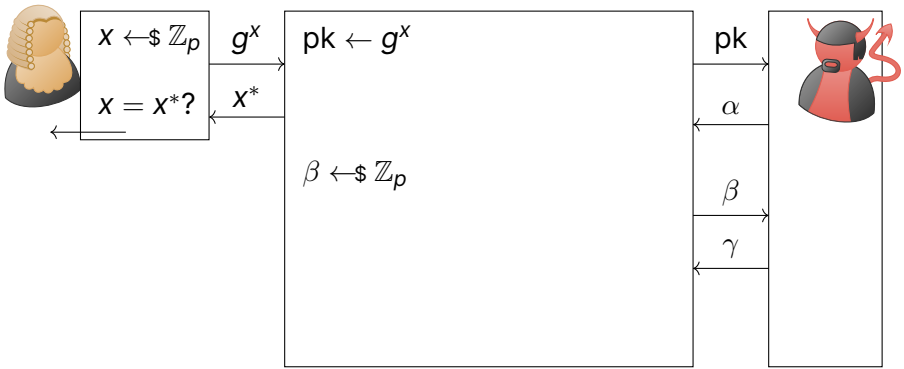
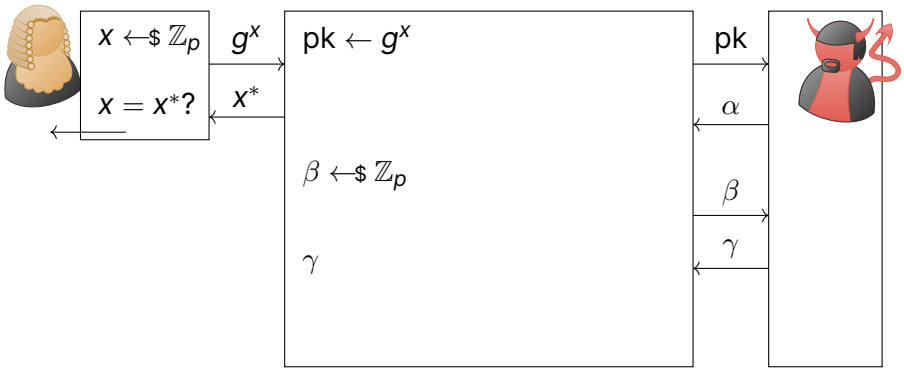


Rewinding / rewindability



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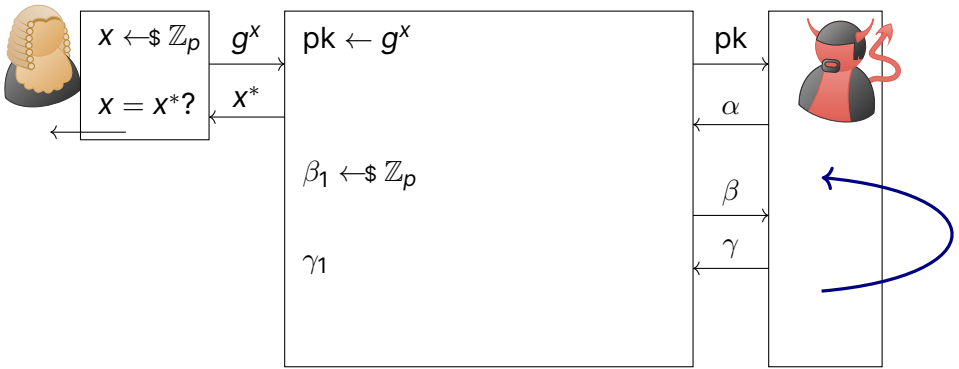

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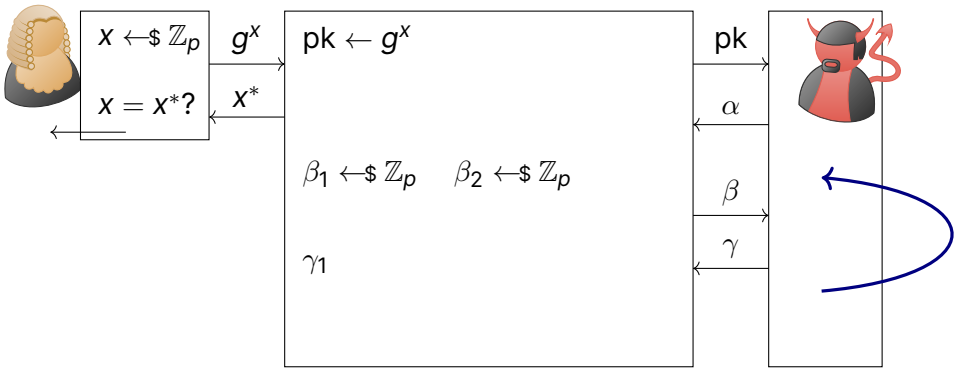

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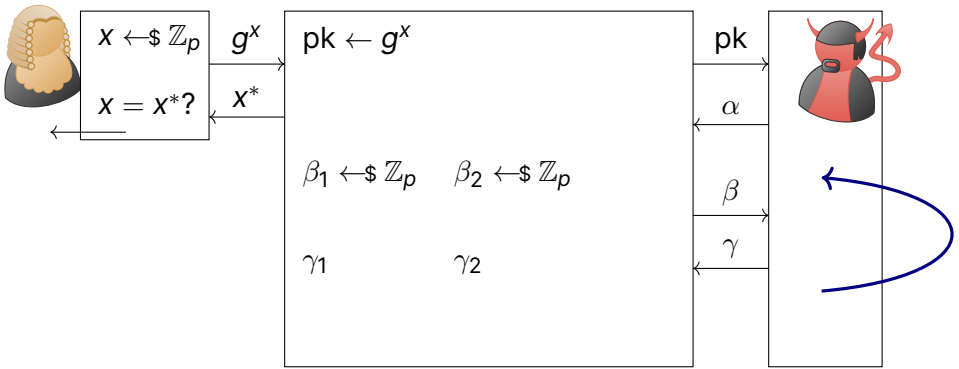

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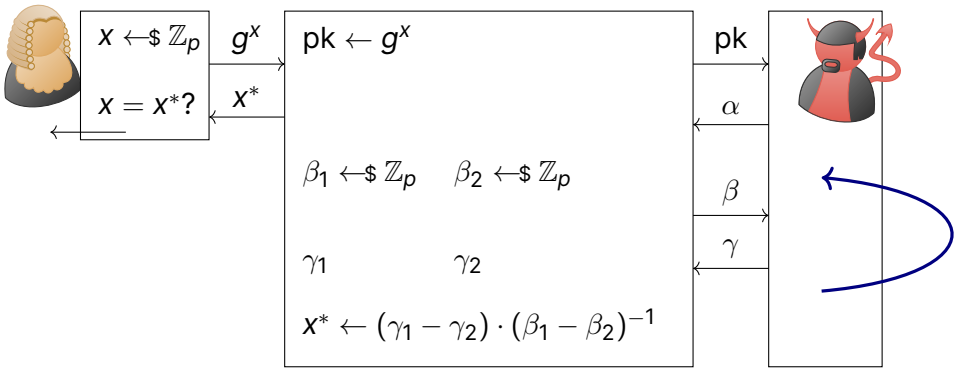

Example: bulletproofs

- Public elements $g_1, \ldots, g_n, h_1, \ldots, h_n, \mathrm{P}, u \in \mathbb{G}$
- wants to convince  that she knows $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{Z}_p$, such that

$$u^{\sum_{i=1}^{n} a_i b_i} \cdot \prod_{i=1}^{n} g_i^{a_i} h_i^{b_i} = \mathrm{P}$$

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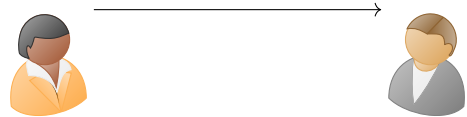
- Privacy is not important
  - Could just send $a_1, \ldots, a_n, b_1, \ldots, b_n$
  - Can we be more efficient than this?

The protocol

- Let $m = n/2$.



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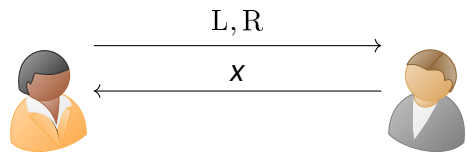

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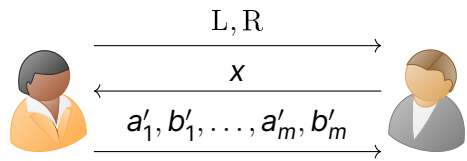

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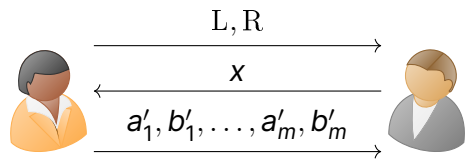

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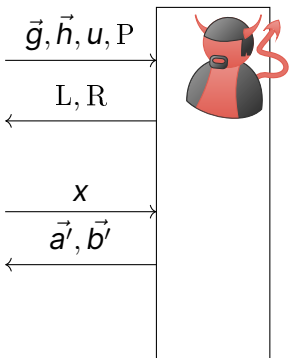
Reduction



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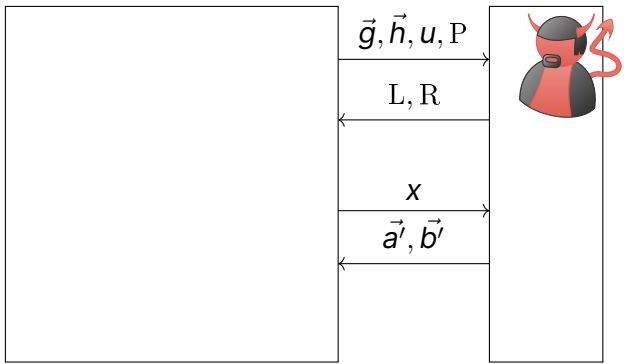

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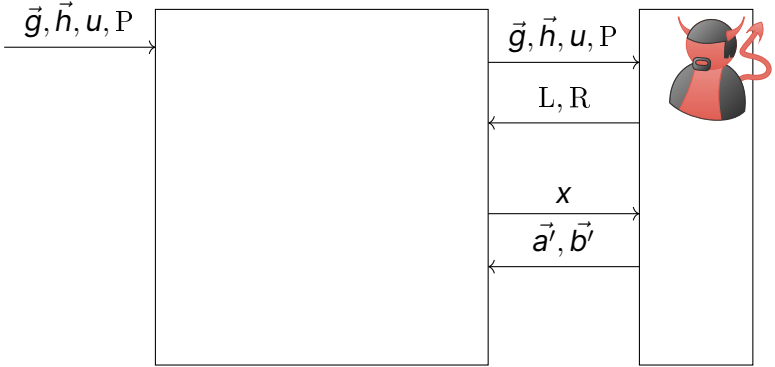
$\vec{g}, \vec{h}, u, \mathrm{P}$
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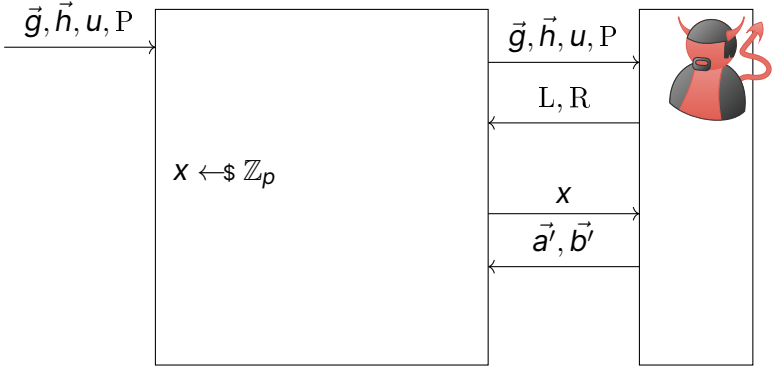
L, R
x
$\vec{a}', \vec{b}'$

Reduction


$\vec{g}, \vec{h}, u, \mathrm{P}$
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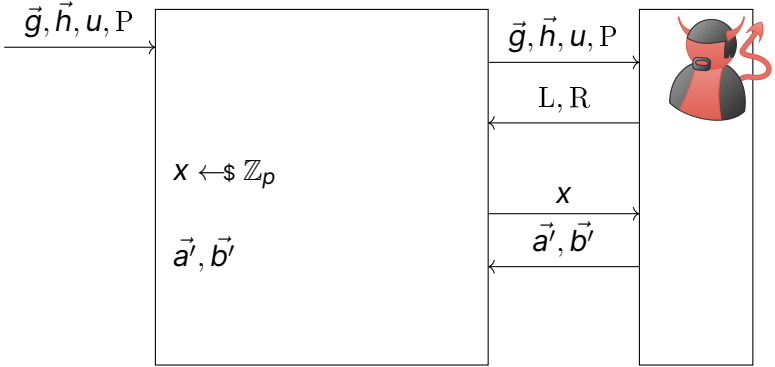
$\vec{g}, \vec{h}, u, \mathrm{P}$
$x \leftarrow\$ \mathbb{Z}_p$
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$\vec{g}, \vec{h}, u, \mathrm{P}$
L, R
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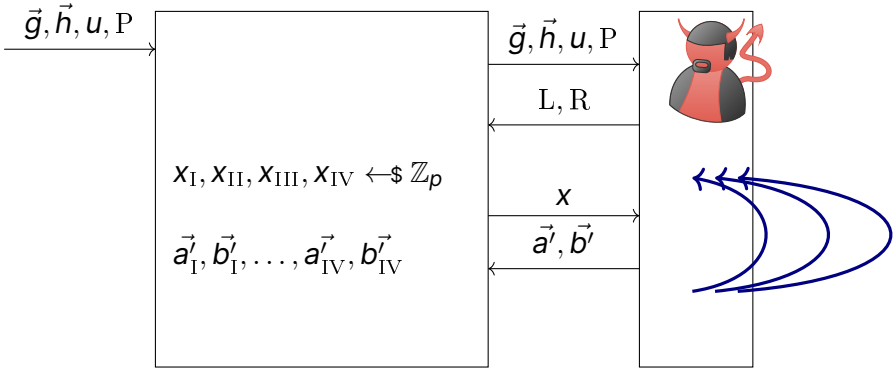
Reduction


$\vec{g}, \vec{h}, u, \mathrm{P}$
$x_{\mathrm{I}}, x_{\mathrm{II}}, x_{\mathrm{III}}, x_{\mathrm{IV}} \leftarrow\$ \mathbb{Z}_p$
$\vec{a'_{\mathrm{I}}}, \vec{b'_{\mathrm{I}}}, \ldots, \vec{a'_{\mathrm{IV}}}, \vec{b'_{\mathrm{IV}}}$
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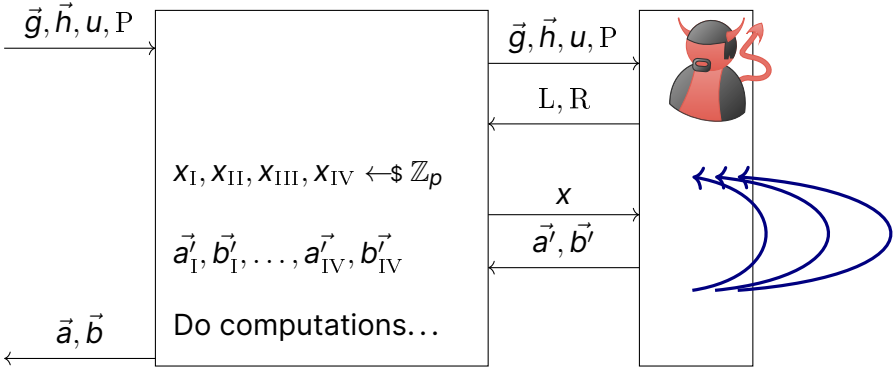
$\vec{g}, \vec{h}, u, \mathrm{P}$
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Do computations...
$\vec{a}, \vec{b}$
$\vec{g}, \vec{h}, u, \mathrm{P}$
L, R
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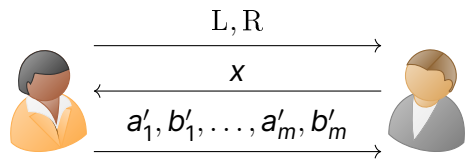

Recursion

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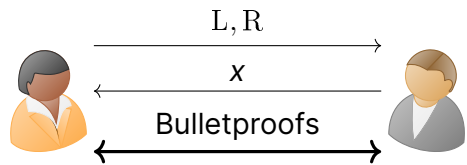

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-  sends random challenge $x \xleftarrow{\$} \mathbb{Z}_p$
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Game-based vs. ideal functionality based definitions

Think of functionalities, talking to multiple parties

Game-based

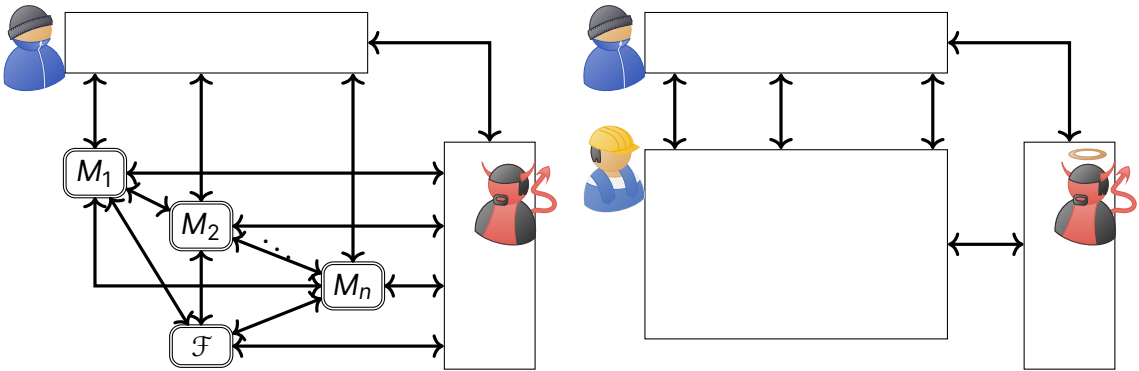
- Implementation must satisfy a number of properties
- Each property is given by a game b/w  and 
  -  runs some parts of the implementation, and does some checks
-  may not “win”

Id. fun. based

- There’s an ideal machine 
  - Talks to all parties
  - Also talks to 
- Implementation must be *at least as secure* as 

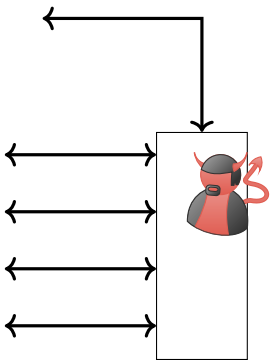
Benefit of this style of definition: we are less likely to mis-specify the security requirements

Universal composability



“at least as secure as”: For each , there must be a , such that the views of are indistinguishable in these two scenarios

Simulator, or: where do we get 



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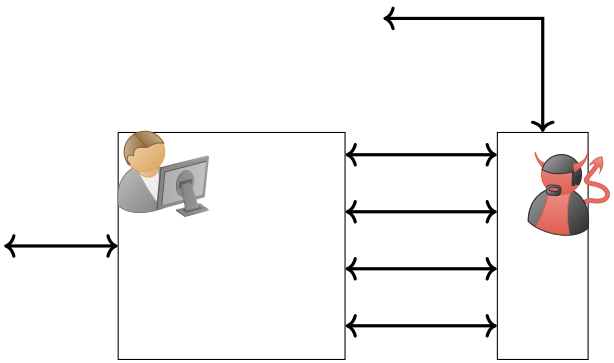

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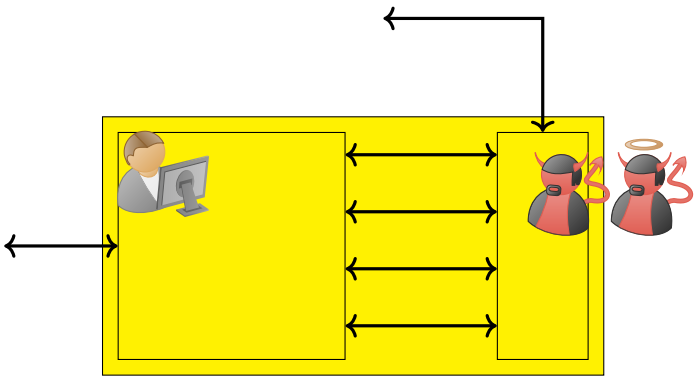

To do (in proofs):

- Propose 
- Show that the real implementation is indistinguishable from the composition of  and 

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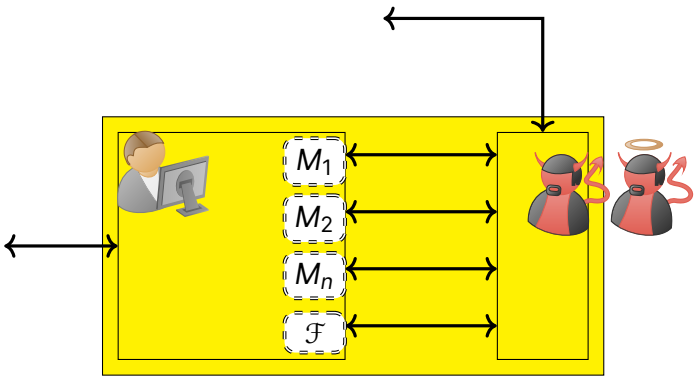

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Example. 👷 for signatures. Key generation

- P_i (in 🥷) asks 👷 for pk_i

- 👷 asks 😇 for some bit-string pk_i
- 👷 receives pk_i from 😇
- 👷 stores (i, pk_i) in the table of keys
- 👷 sends pk_i to P_i

Example.  for signatures. Key generation

- P_i (in ) asks  for pk_i
-  ignores the request if $(i, \ldots)$ is in the table of keys. Otherwise:
-  asks  for some bit-string pk_i
-  receives pk_i from 
-  stores (i, pk_i) in the table of keys
-  sends pk_i to P_i

Example. for signatures. Signing

- P_i (in) asks to sign message M
- ignores the request if $(i, \ldots)$ is not in the table of keys. Otherwise:
- sends M to and asks for the value σ of a signature
- receives σ from
- stores $(\mathrm{pk}_i, M, \sigma, 1)$ in the table of signatures
- sends σ to P_i

Example. for signatures. Verification

- P_i (in) asks to verify $(\mathrm{pk}_j, M, \sigma)$
- If $(\mathrm{pk}_j, M, \sigma, b)$ is in the table of signatures, then answers "b" to P_i
- If $(\ldots, \mathrm{pk}_j)$ is in the table of keys, but there is no $(\mathrm{pk}_j, M, \ldots)$ in the table of signatures, then answers "0" to P_i
- Otherwise, sends $(\mathrm{pk}_j, M, \sigma)$ to , getting back b^*
- stores $(\mathrm{pk}_j, M, \sigma, b^*)$ in the table of signatures, and answers "b^*" to P_i

Example. Signatures. Implementation

- Let **Sig** $=$ (**KeyGen**, **Sign**, **Verify**) be an existentially unforgeable (under chosen-message attacks) signature scheme
- Machine M_i is totally intuitive:
  - uses **KeyGen** to generate a key; stores sk
  - uses **Sign**$(\mathrm{sk}, \cdot)$ to sign
  - uses **Verify** to verify signatures

Resources

- There is a well-defined interface b/w  and  , and b/w  and 
-  makes queries (sends messages) to  , and vice versa
- (In general:)  makes queries (sends messages) to  and receives replies
  - May correspond to  playing a *corrupted* party
  - May correspond to  simulating  controlling the network
- (As we saw:)  makes queries to  and receives replies
  - Assumption:  responds

Alternative to responsive 

- At the start,  gives **KeyGen**, **Sig**, **Verify** to 
  - These algorithms are given as executable code
- Instead of consulting  ,  runs these algorithms
  - But maintains and uses the tables, too

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- At the start,  gives **KeyGen**, **Sig**, **Verify** to 
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  - But maintains and uses the tables, too
- In general…this is less flexible
  - Recall $\mathcal{F}$ as part of 

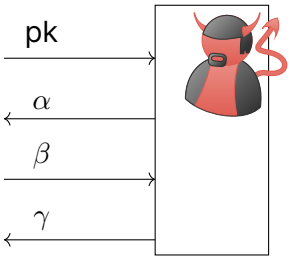
Information flows

- Making queries to  gives information to 
- Can we model that  do not learn anything while  responds to the query?

Information flows

- Making queries to  gives information to 
- Can we model that  do not learn anything while  responds to the query?
- Is the ability to update descriptions of algorithms sufficient?

Knowledge extraction



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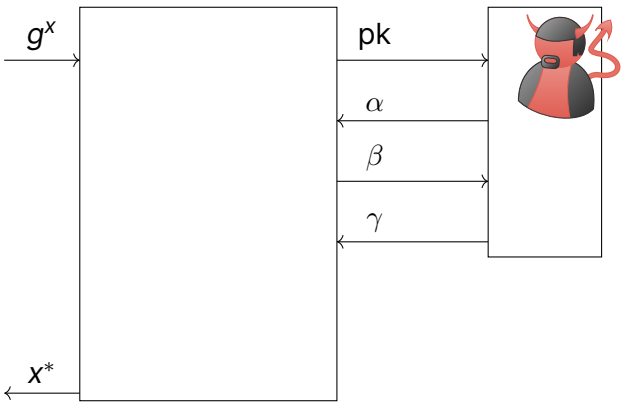

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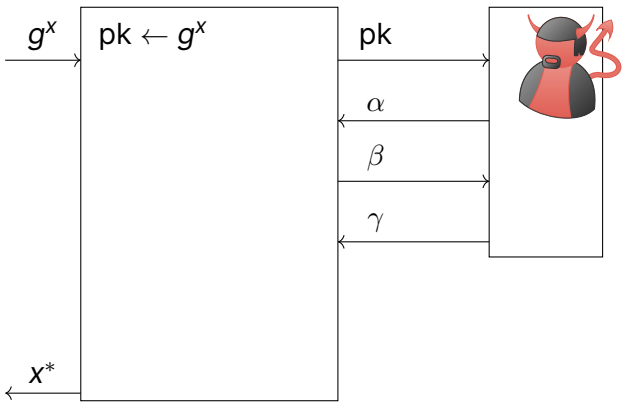

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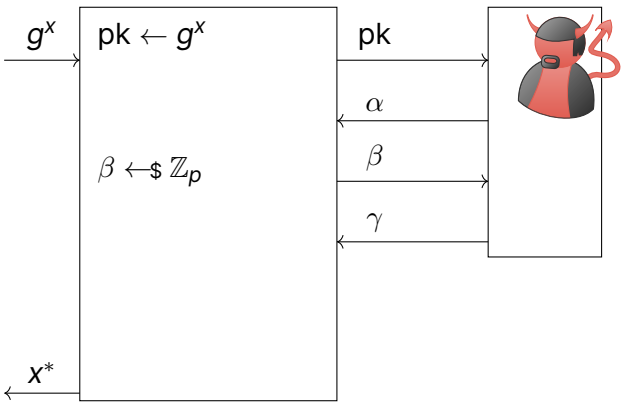

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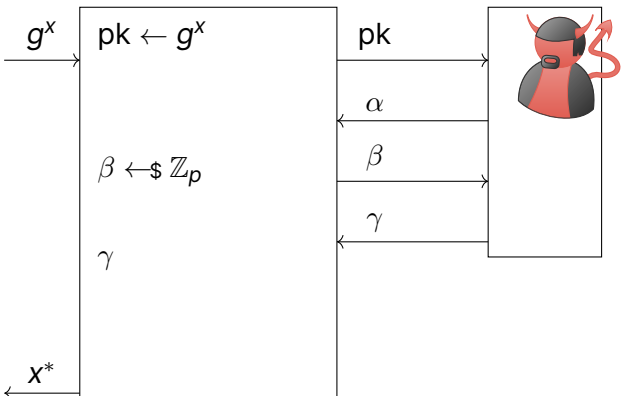

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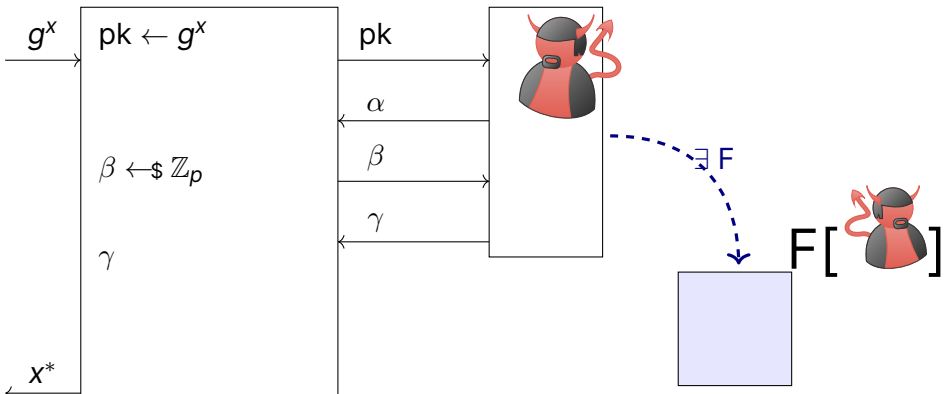

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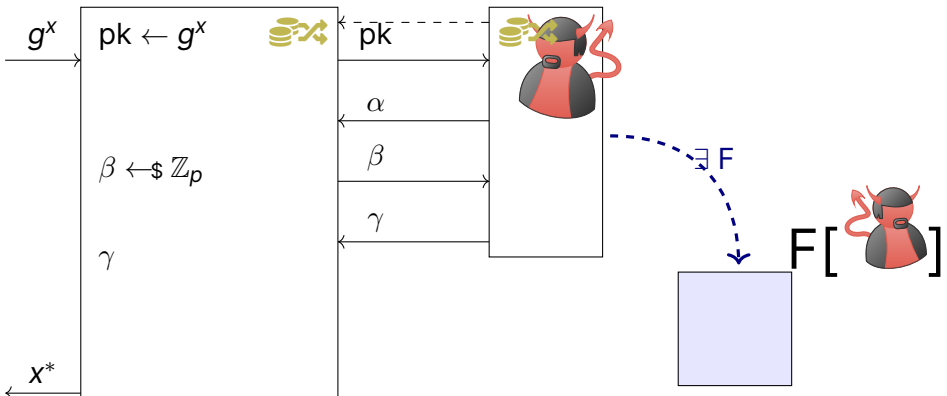

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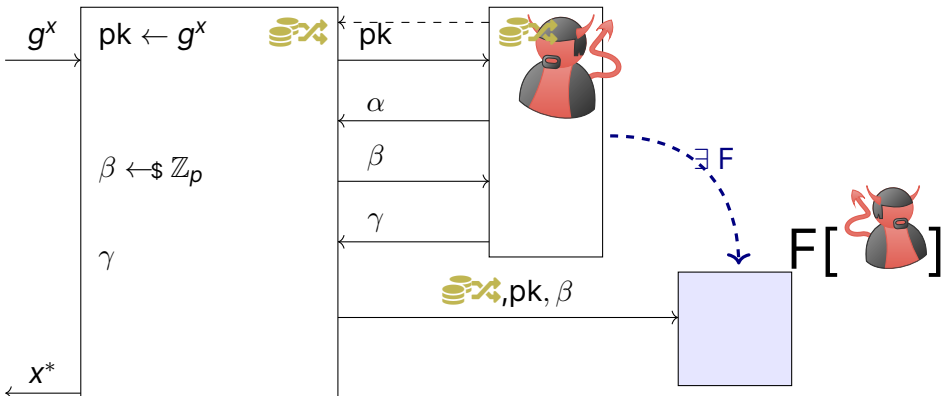

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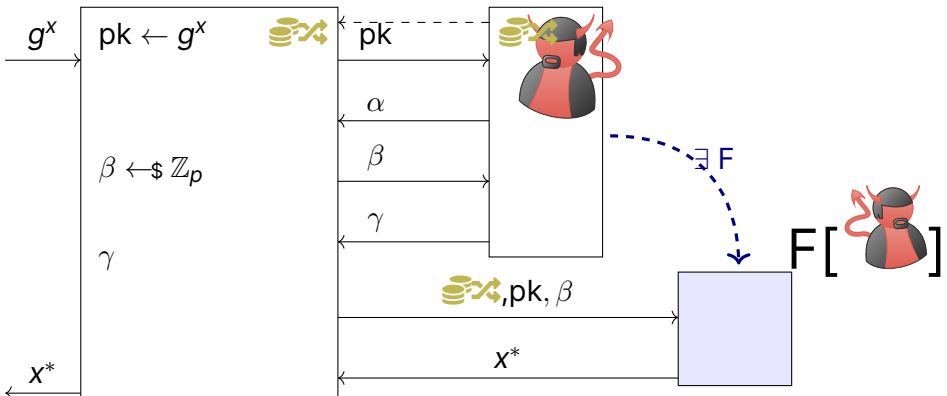
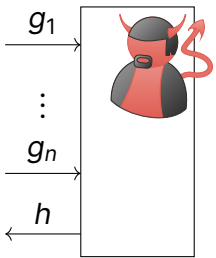
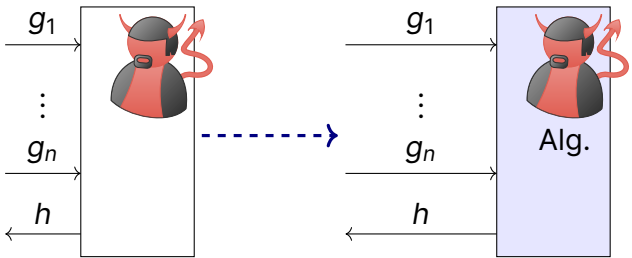


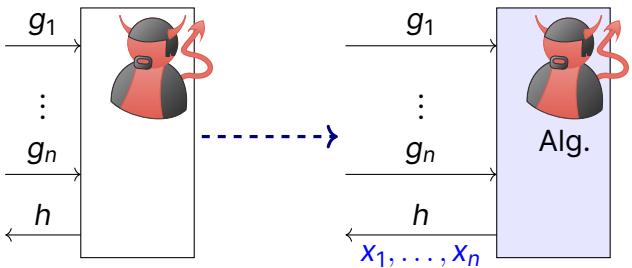
Algebraic Group Model



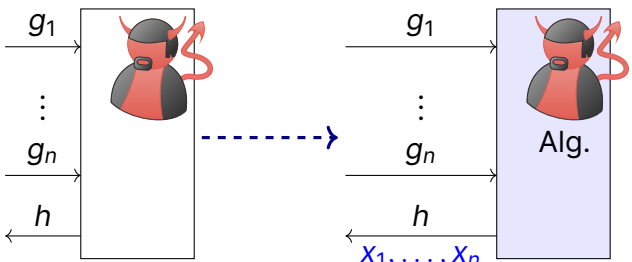
Algebraic Group Model



Algebraic Group Model



Algebraic Group Model



Such that $h = g_1^{x_1} \cdots g_n^{x_n}$