

Turning lax monoidal categories into strict ones

Miloslav Štěpán

Masaryk University
miloslav.stepan@mail.muni.cz

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Plan of the presentation

- 1 Colax monoidal categories & lax monoidal functors
- 2 Strictification process
- 3 Bonus features, generalizations

Colax monoidal categories

Definition

A *colax monoidal category* $(\mathcal{A}, \otimes, \gamma, \iota)$ is a category \mathcal{A} together with:

- a functor $\otimes_n : \mathcal{A}^n \rightarrow \mathcal{A}$ for all $n \geq 0$, denote $a_1 \otimes \cdots \otimes a_n := \otimes_n(a_1, \dots, a_n)$, $[a] := \otimes_1(a)$, $I := \otimes_0(*)$
- for every list of lists $(\vec{a}_1, \dots, \vec{a}_k)$ of objects of \mathcal{A} an *associator* morphism. For instance:

$$\gamma((a_1, a_2), (a_3, a_4)) : a_1 \otimes a_2 \otimes a_3 \otimes a_4 \rightarrow (a_1 \otimes a_2) \otimes (a_3 \otimes a_4),$$

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- for every object $a \in \mathcal{A}$ a *unit* morphism:

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Remark - variants

- γ, ι identities \Rightarrow a *strict monoidal category*,
- γ, ι isomorphisms \Rightarrow an *unbiased monoidal category*. (are equivalent to ordinary monoidal categories [Leinster, 2004, I.3]).
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Example, [BW, 2011]

Let (t, δ, ϵ) be a comonad on a category \mathcal{A} with coproducts. Define a colax monoidal category structure on \mathcal{A} by putting:

$$a_1 \otimes \cdots \otimes a_n := \coprod_{i=1}^n ta_i.$$

The unitor is given by the comonad counit: $\iota_a := \epsilon_a : [a] = ta \rightarrow a$.
The associator is given by:

$$\begin{aligned}
 a_1 \otimes a_2 \otimes b &= ta_1 + ta_2 + tb \xrightarrow{\delta_{a_1} + \delta_{a_2} + \delta_{a_3}} t^2 a_1 + t^2 a_2 + t^2 b \\
 &\cong \downarrow \\
 (a_1 \otimes a_2) \otimes [b] \otimes I &= t(ta_1 + ta_2) + t^2 b + t\emptyset \leftarrow t^2 a_1 + t^2 a_2 + t^2 b + \emptyset
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- a set of objects $\text{ob } \mathcal{M}$,
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subject to associativity and unit laws.

Example

Vect_k . A morphism $f : V_1, \dots, V_n \rightarrow W$ is a k -multilinear map $V_1 \times \dots \times V_n \rightarrow W$.

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Multicategories vs monoidal categories

Proposition

There is a functor $I : \text{ColaxMonCat}_l \rightarrow \text{Mult}$ sending a colax monoidal category $(\mathcal{A}, \otimes, \gamma, \iota)$ to its *underlying multicategory*. Its objects are the objects of \mathcal{A} , its hom set is given by:

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A multicategory lies in the essential image of I if and only if it is *weakly representable*.

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Multicategories vs monoidal categories

Proposition

There is a functor $I : \text{ColaxMonCat}_l \rightarrow \text{Mult}$ sending a colax monoidal category $(\mathcal{A}, \otimes, \gamma, \iota)$ to its *underlying multicategory*. Its objects are the objects of \mathcal{A} , its hom set is given by:

$$I\mathcal{A}(a_1, \dots, a_n; b) := \mathcal{A}(a_1 \otimes \dots \otimes a_n, b).$$

A multicategory lies in the essential image of I if and only if it is *weakly representable*.

Lax monoidal functors

Definition

A lax monoidal functor $(F, \bar{F}) : (\mathcal{A}, \otimes, \gamma, \iota) \rightarrow (\mathcal{B}, \odot, \gamma', \iota')$ between colax monoidal categories consists of:

- a functor $F : \mathcal{A} \rightarrow \mathcal{B}$,
- for every (a_1, \dots, a_n) a morphism in \mathcal{B} :

$$\bar{F}_{a_1, \dots, a_n} : Fa_1 \odot \dots \odot Fa_n \rightarrow F(a_1 \otimes \dots \otimes a_n),$$

subject to associativity and unit axioms.

Example

Take two sup-semilattices A, B with a lowest element. Regard them as monoidal categories $(A, \vee), (B, \vee)$.

Any order-preserving map $f : A \rightarrow B$ automatically lax monoidal because we have:

$$f(a) \vee f(b) \leq f(a \vee b).$$

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Example

A lax monoidal functor $* \rightarrow (\mathcal{A}, \otimes, \gamma, \iota)$ is precisely a **monoid** in the monoidal category \mathcal{A} .

Example

The forgetful functor $U : \text{Vect}_k \rightarrow \text{Set}$ becomes lax monoidal if we define:

$$\begin{aligned} \bar{U}_{V_1, \dots, V_n} : UV_1 \times \dots \times UV_n &\rightarrow U(V_1 \otimes_k \dots \otimes_k V_n), \\ (v_1, \dots, v_n) &\mapsto v_1 \otimes \dots \otimes v_n. \end{aligned}$$

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Turning colax monoidal categories into strict ones

Question: How to turn a colax monoidal category into a strict one in a best possible way?

Answer: By applying the left 2-adjoint:

$$\text{StrMonCat}_s \perp \text{ColaxMonCat}_l$$

StrMonCat_s is 2-cocomplete, so by [Lack, 2002], this 2-adjoint exists. **What does it look like?**

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Ingredient: partial evaluations

Definition [PF2020]

Let $(\mathcal{A}, \otimes, \gamma, \iota)$ be a colax monoidal category. A *partial evaluation* $(\vec{a}_1, \dots, \vec{a}_k)$ is a list of lists of objects of \mathcal{A} .

Its *source* and *target* are defined to be $\text{conc}(\vec{a}_1, \dots, \vec{a}_k)$ and $(\otimes_{n_1}(\vec{a}_1), \dots, \otimes_{n_k}(\vec{a}_k))$.

Example

For $(\mathbb{N}, +)$, an example is:

$$\begin{aligned} ((1, 2), (4, 1, 0), (7)) &: (1, 2, 4, 1, 0, 7) \rightarrow (3, 5, 7), \\ ((), (), (1)) &: (1) \rightarrow (0, 0, 1). \end{aligned}$$

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Example

For a terminal monoidal category $*$, a partial evaluation is precisely an *order-preserving functions* between finite ordinals. For instance:

$$((\bullet, \bullet), ()), (\bullet), ()) : (\bullet, \bullet, \bullet) \rightarrow (\bullet, \bullet, \bullet, \bullet)$$

corresponds to the unique order-preserving function $f : \{0 \rightarrow 1 \rightarrow 2\} \rightarrow \{0 \rightarrow 1 \rightarrow 2 \rightarrow 3\}$ whose fibres over 0, 1, 2, 3 have sizes 2, 0, 1, 0.

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Category of corners

Construction: Let $(\mathcal{A}, \otimes, \gamma, \iota)$ be a colax monoidal category. Define its *category of corners* $\text{Cnr}(\mathcal{A})$ as follows.

- the objects (a_1, \dots, a_n) are lists of objects of \mathcal{A} ,
- a morphism has two components: partial evaluation, list of morphisms of \mathcal{A} . For example:

$$\begin{array}{ccc}
 (a_1, a_2, a_3) & \xrightarrow{((\cdot), (a_1, a_2), (\cdot), (a_3))} & (I, a_1 \otimes a_2, I, [a_3]) \\
 & & \downarrow (f_1, f_2, f_3, f_4) \\
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- the identity morphism is defined as the following corner:

$$\begin{array}{ccc} (\mathbf{a}_1, \dots, \mathbf{a}_n) & \longrightarrow & ([\mathbf{a}_1], \dots, [\mathbf{a}_n]) \\ & & \downarrow (\iota_{\mathbf{a}_1}, \dots, \iota_{\mathbf{a}_n}) \\ & & (\mathbf{a}_1, \dots, \mathbf{a}_n) \end{array}$$

Category of corners - composition (1 of 6)

An example of composition of two corners:

$$\begin{array}{ccc}
 (a_1, a_2, a_3) & \xrightarrow{((a_1, a_2), (), (a_3))} & (a_1 \otimes a_2, I, [a_3]) \\
 & & \downarrow (f_1, f_2, f_3) \\
 & & (b_1, b_2, b_3) \xrightarrow{((b_1), (b_2, b_3))} ([b_1], b_2 \otimes b_3) \\
 & & \downarrow (g_1, g_2) \\
 & & (c_1, c_2)
 \end{array}$$

Category of corners - composition (2 of 6)

$$\begin{array}{ccc}
 (a_1, a_2, a_3) \xrightarrow{((a_1, a_2), ()), (a_3))} (a_1 \otimes a_2, I, [a_3]) & \xrightarrow{((a_1 \otimes a_2), (I, [a_3]))} & ([a_1 \otimes a_2], I \otimes [a_3]) \\
 \downarrow (f_1, f_2, f_3) & & \downarrow ([f_1], f_2 \otimes f_3) \\
 (b_1, b_2, b_3) \xrightarrow{((b_1), (b_2, b_3))} & ([b_1], b_2 \otimes b_3) & \\
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Category of corners - composition (3 of 6)

$$\begin{array}{ccc}
 (a_1, a_2, a_3) & \xrightarrow{((a_1, a_2), (), (a_3))} & (a_1 \otimes a_2, I, [a_3]) & \xrightarrow{((a_1 \otimes a_2), (I, [a_3]))} & ([a_1 \otimes a_2], I \otimes [a_3]) \\
 & & & & \downarrow (g_1 \circ [f_1], g_2 \circ f_2 \otimes f_3) \\
 & & & & (c_1, c_2)
 \end{array}$$

Category of corners - composition (4 of 6)

$$\begin{array}{ccc}
 (a_1, a_2, a_3) & \xrightarrow{((a_1, a_2), (a_3))} & (a_1 \otimes a_2, [a_3]) \\
 \\
 (a_1, a_2, a_3) & \xrightarrow{((a_1, a_2), (), (a_3))} (a_1 \otimes a_2, l, [a_3]) & \xrightarrow{((a_1 \otimes a_2), (l, [a_3]))} ([a_1 \otimes a_2], l \otimes [a_3]) \\
 & & \downarrow (g_1 \circ [f_1], g_2 \circ f_2 \otimes f_3) \\
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Category of corners - composition (5 of 6)

$$\begin{array}{ccc}
 (a_1, a_2, a_3) & \xrightarrow{((a_1, a_2), (a_3))} & (a_1 \otimes a_2, [a_3]) \\
 & & \downarrow (\gamma, \gamma) \\
 (a_1, a_2, a_3) & \xrightarrow{((a_1, a_2), (a_3))} & (a_1 \otimes a_2, l, [a_3]) \xrightarrow{((a_1 \otimes a_2), (l, [a_3]))} ([a_1 \otimes a_2], l \otimes [a_3]) \\
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Category of corners - composition (6 of 6)

$$\begin{array}{ccc}
 (a_1, a_2, a_3) & \xrightarrow{((a_1, a_2), (a_3))} & (a_1 \otimes a_2, [a_3]) \\
 & & \downarrow \\
 & & (c_1, c_2)
 \end{array}$$

$(g_1 \circ [f_1] \circ \gamma, g_2 \circ f_2 \otimes f_3 \circ \gamma)$

Category of corners

The tensor product \boxplus in $\text{Cnr}(\mathcal{A})$ is given by concatenation:

$$(a_1, \dots, a_n) \boxplus (b_1, \dots, b_m) := (a_1, \dots, a_n, b_1, \dots, b_m),$$

The unit element is given by the empty list $I := ()$.

There is a lax monoidal functor $(P, \bar{P}) : (\mathcal{A}, \otimes, \gamma, \iota) \rightarrow (\text{Cnr}(\mathcal{A}), \boxplus, I)$:

$$\begin{array}{ccc}
 \begin{array}{c} a \\ \downarrow f \\ b \end{array} & \mapsto & \begin{array}{ccc} (a) & \longrightarrow & ([a]) \\ & & \downarrow (\iota_a) \\ & & (a) \\ & & \downarrow (f) \\ & & (b) \end{array}
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The lax monoidal structure, a collection of 1-cells in $\text{Cnr}(\mathcal{A})$ like this:

$$\bar{P}_{a_1, \dots, a_n} : Pa_1 \boxplus \cdots \boxplus Pa_n \rightarrow P(a_1 \otimes \cdots \otimes a_n),$$

is given by the corner:

$$\begin{array}{ccc} (a_1, \dots, a_n) & \longrightarrow & (a_1 \otimes \cdots \otimes a_n) \\ & & \parallel \\ & & (a_1 \otimes \cdots \otimes a_n) \end{array}$$

The lax monoidal structure, a collection of 1-cells in $\text{Cnr}(\mathcal{A})$ like this:

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Theorem

The category of corners construction is left adjoint to the inclusion:

$$\begin{array}{ccc}
 & \text{Cnr}(-) & \\
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 \text{StrMonCat}_s & \perp & \text{ColaxMonCat}_l \\
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 \end{array}$$

In other words:

$$\begin{array}{ccc}
 & \forall \text{ lax monoidal} & \nearrow (\mathcal{B}, \odot) \\
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Sketch of a proof.

For a lax monoidal functor $(F, \bar{F}) : (\mathcal{A}, \otimes, \gamma, \iota) \rightarrow (\mathcal{B}, \odot)$, the unique strict monoidal functor $F' : (\text{Cnr}(\mathcal{A}), \boxplus) \rightarrow (\mathcal{B}, \odot)$ sends $(a_1, \dots, a_n) \mapsto Fa_1 \odot \dots \odot Fa_n$,

and on morphisms it is defined like this:

$$\begin{array}{ccc}
 (a_1, a_2, a_3) \xrightarrow{((a_1, a_2), ()), (a_3))} (a_1 \otimes a_2, I, [a_3]) & & Fa_1 \odot Fa_2 \odot Fa_3 \\
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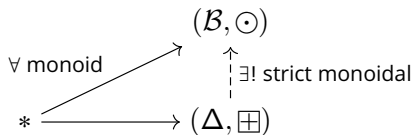
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Example

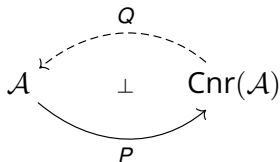
In particular, $\text{Cnr}(\ast) = \Delta$, the category of finite ordinals and order-preserving maps. It enjoys the universal property that:



Lax coherence theorem

Theorem

The underlying functor $P : \mathcal{A} \rightarrow \text{Cnr}(\mathcal{A})$ of the unit of the above adjunction admits a left adjoint whose counit is given by the unitor of $(\mathcal{A}, \otimes, \gamma, \iota)$:

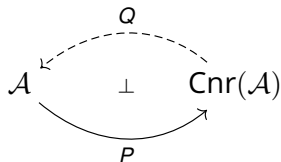


In particular, every **normal** colax monoidal category can be reflectively embedded in a strict monoidal category.

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The left adjoint $Q : \text{Cnr}(\mathcal{A}) \rightarrow \mathcal{A}$ sends $(a_1, \dots, a_n) \mapsto a_1 \otimes \cdots \otimes a_n$, and on morphisms is defined like this:

$$\begin{array}{ccc}
 (a_1, a_2, a_3) \xrightarrow{((a_1, a_2), (a_3))} & (a_1 \otimes a_2, [a_3]) & \\
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 & (c_1, c_2) & \\
 & & \mapsto \\
 & & \begin{array}{c}
 a_1 \otimes a_2 \otimes a_3 \\
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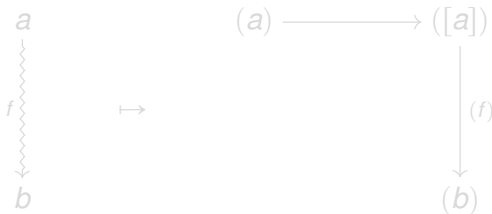
Bonus features

Relationship to the Kleisli category (1 of 2)

Remark

Every colax monoidal category $(\mathcal{A}, \otimes, \gamma, \iota)$ has an underlying comonad $(\mathcal{A}, [-] : \mathcal{A} \rightarrow \mathcal{A}, \gamma_{((-)} , \iota)$, with the comultiplication $[a] \rightarrow [[a]]$ being given by the associator γ evaluated at $((a))$.

There is an obvious functor $\text{Kl}(\mathcal{A}) \rightarrow \text{Cnr}(\mathcal{A})$:



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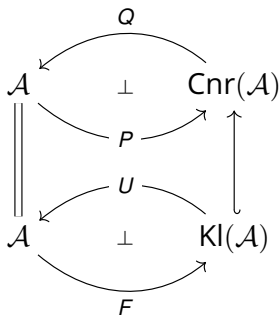
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Relationship to the Kleisli category (2 of 2)

Proposition

The adjunction from the above theorem restricts to the Kleisli adjunction:



Relationship to multicategories

Recall the adjunction between strict monoidal categories and multicategories:

$$\text{StrMonCat}_s \xrightarrow{J} \text{ColaxMonCat}_l \xrightarrow{I} \text{Multicat}$$

$\overset{F}{\curvearrowright} \perp \curvearrowleft$

Since I is fully faithful, this induces an adjunction:

$$\text{StrMonCat}_s \quad \overset{FI}{\curvearrowright} \perp \curvearrowleft \quad \text{ColaxMonCat}_l$$

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 & \searrow & \swarrow \\
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 \end{array}$$

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Some bonus observations

- Given $(\mathcal{A}, \otimes, \gamma, \iota)$ with γ, ι invertible, a strict monoidal category \mathcal{A}' **equivalent** to \mathcal{A} can be obtained from $\text{Cnr}(\mathcal{A})$ by formally inverting partial evaluations,
- The strict monoidal category $(\text{Cnr}(\mathcal{A}), \boxplus)$ is *flexible*: any monoidal functor out of $\text{Cnr}(\mathcal{A})$ is naturally isomorphic to a strict one
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Thank you.