**Lecture 10:** 1-D Maps, Lorenz Map, Logistic Map, Sine Map, Period Doubling Bifurcation, Tangent Bifurcation, Transient and Intermittent Chaos in Maps, Orbit Diagram (or Feigenbaum Diagram), Feigenbaum Constants, Universals of Unimodal Maps, Universal Route to Chaos

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Handout: Orbit diagram of Logistic map
1 Lorenz map

In this lecture we continue to study the possibility that the Lorenz attractor might be long-term periodic. As in previous lecture, we use one-dimensional Lorenz map in the form

\[ z_{n+1} = f(z_n) \]  \hfill (1)

to gain insight into the continuous time three-dimensional flow of Lorenz attractor.

1.1 Cobweb diagram

The cobweb diagram (also called the cobweb plot) introduced in previous lecture is a graphical way of thinking about and determining the iterates of the Lorenz map, see Fig. 1. In order to construct a cobweb plot move vertically to the function \( f \) and horizontally to the diagonal and repeat (see Lecture 9 for basic examples).

![Cobweb diagram](image)

**Figure 1:** (Left) Cobweb diagram of the Lorenz map. (Right) Lorenz map iterates corresponding to the cobweb diagram shown on the left.

In the previous lecture we showed, using the linearisation, that the fixed point \( z^* \) (period-1 point) satisfying

\[ f(z^*) = z^* \]  \hfill (2)

where \( f(z) \) is the function defining Lorenz map, is unstable. This conclusion can be graphically confirmed with the aid of a cobweb diagram. The following numerical file shows the Lorenz map and its iterates.

**Numerics:** cdf#1, nb#1

Cobweb diagram and iterates of the (generalised) Lorenz map. Lyapunov exponent \( \lambda(r) \) of Logistic map.

![Cobweb diagram and iterates](image)

**Figure 2:** (Left) Cobweb diagram of the normalised Lorenz map showing the first ten iterates for \( z_0 \approx z^* \) where the fixed point \( z^* \) is unstable, since \( |f(z^*)| > 1 \). The Lorenz map has property \( |f(z)| > 1, \forall z \) in the map basin. (Right) Lorenz map iterates corresponding to the cobweb diagram shown on the left.
Figure 3: (Top-left) Stable fixed point $x^*$ of a one-dimensional system given by $\dot{x} = f(x)$ where $f'(x^*) < 1$ is shown with the filled bullet. (Top-right) Family of time-series solutions shown for several initial conditions corresponding to the phase portrait shown on the left. (Bottom-left) Unstable fixed point shown with the empty bullet where $f'(x^*) > 1$. (Bottom-right) Family of time-series solutions corresponding to the phase portrait shown on the left.

1.2 Comparison of fixed point $x^*$ in 1-D continuous systems and 1-D maps

The fixed points of one-dimensional continuous systems (ODEs) and of discrete one-dimensional maps (with period-1 point) are clearly analogous. Figures 3 and 4 show that analogy.

Figure 4: (Top-left) Stable fixed point $x^*$ (period-1 point) of a one-dimensional map given by $x_{n+1} = f(x_n)$ where $|f'(x^*)| < 1$ is shown with the filled bullet. (Top-right) Three sets of map iterates $x_n$ shown for initial conditions $x_0$, $x^*$ and $x_02$ corresponding to the cobweb diagram shown on the left. (Bottom-left) Unstable fixed point (period-1 point) shown with the empty bullet where $|f'(x^*)| > 1$. (Bottom-right) Three sets of map iterates $x_n$ shown for initial conditions $x_0$, $x^*$ and $x_02$ corresponding to the cobweb diagram shown on the left.
1.3 Period-p orbit and stability of period-p points

We ended the previous lecture with two open ended questions: Can trajectories in the cobweb diagram of the Lorenz map close onto themselves? And, how does a closed trajectory in the cobweb plot translate into the three-dimensional continuous time Lorenz flow?

![Image of Lorenz attractor and cobweb diagram]

Figure 5: (Left) Period-4 orbit, shown with the red graph, where \( z_{n+4} = z_n \). (Right) Lorenz map iterates \( z_n \) corresponding to the cobweb diagram shown on the left. Map iterates are repeat every four iterates.

So, can trajectories in the cobweb diagram of the Lorenz map close onto themselves? We could imagine a trajectory of Lorenz map’s cobweb plot closing onto itself in a manner shown in Fig. 5. Visual inspection of the cobweb diagram shown in Fig. 5 reveals that the iterates \( z_n \) (local maxima of the three-dimensional Lorenz flow) repeat themselves such that \( z_0 = z_1 \) or more generally

\[
z_{n+4} = z_n, \quad \forall n,
\]

if this dynamics can be shown to be possible, then it would strongly suggest that a limit-cycle might be possible in the three-dimensional Lorenz attractor. This conclusion can be generalised further:

\[
z_{n+p} = z_n, \quad \forall n,
\]

where \( p \in \mathbb{Z}^+ \) is the period of the limit-cycle. This type of fixed point is called the period-p point and it represents the period-p orbit of the map. The period-p points are a new type of fixed points. The period-1 point coincides with our old friend—fixed point \( z^* \). The period-p points where \( p > 1 \) don’t have corresponding analogies in one-dimensional continuous systems in the manner as was discussed in Sec. 1.2. This is because the period-p points represent oscillatory limit-cycle solutions which are not possible in one-dimensional systems (see Lecture 2: Impossibility of oscillations in 1-D systems).

It is natural to assume that period-p points or orbits, very much like fixed points \( z^* \) (period-1 point), can be either stable (attracting trajectories) or unstable (repelling trajectories). If we can show that stable period-p orbits are possible in Lorenz map, then that would strongly suggest a possibility of periodic solutions in the continuous time Lorenz attractor.

Let’s find the analytical expression for the relationship between \( z_{n+p} \) and \( z_n \) in (4). The \( n + 1 \) iterate of \( z_0 \) is

\[
z_{n+1} = f(f(\ldots f(z_0)\ldots))) \equiv f^n(z_0).
\]

The subsequent iterates of the closed period-p orbits, similar to the one shown in Fig. 5 with the red trajectory, expressed analytically are

\[
\begin{align*}
( & z_1 = f(z_0)) \\
( & z_2 = f(z_1) = f^1(z_1) \\
( & z_3 = f(z_2) = f(f(z_1)) \equiv f^2(z_1) \\
( & z_4 = f(z_3) = f(f(f(z_1))) \equiv f^3(z_1) \\
& \vdots \\
( & z_{n+p} = f^p(z_n),
\end{align*}
\]

\[[Image of period-p orbit and iterative formula for Lorenz map]]
where \( z \) is the period-\( p \) point in the period-\( p \) orbit and \( f^p \) is the \( p \)th iterate map.  

**Definition:** \( z \) is a **period-\( p \) point** if equation

\[
f^p(z) = z.
\]  

(7)

where \( p \) is minimal, is satisfied.

The stability of the period-\( p \) point is determined via linearisation. For simplicity we consider stability of period-2 point

\[
f^2(z) \equiv f(f(z)) = z.
\]  

(8)

Note that \( z \) is the period-2 point for map \( f \) but the fixed point (period-1) for map \( f^2 \). In the previous lecture we showed that the stability of fixed point \( z^\ast \) depends on the slope of the map at that point. Small perturbations \( |\eta| \ll 1 \) evolve according to

\[
\eta_{n+1} \approx |f^\prime(z^\ast)|\eta_n.
\]  

(9)

We find

\[
|(f^2(z))'| = \frac{d}{dz} |f(f(z))| = \left[ \begin{array}{c}
\text{chain rule} \\
\text{orbit points}
\end{array} \right]_{z_{n+2} = z_n, \text{we select } z_1} = |f^\prime(f(z_1)) \cdot f^\prime(z_1)| = |f^\prime(z_2)| \cdot |f^\prime(z_1)| > 1,
\]  

(10)

because for the Lorenz map \( |f^\prime(z)| > 1, \forall z \) in the map basin. The period-2 point is thus **unstable**. The evolution of perturbations defined by (9) generalised to all period-\( p \) points in any closed period-\( p \) orbit are the following:

\[
\eta_{n+p} \approx \left| \prod_{k=0}^{p-1} f^\prime(z_{n+k}) \right| \eta_n,
\]  

(11)

here, again, by the Lorenz map property

\[
\prod_{k=0}^{p-1} f^\prime(z_{n+k}) > 1.
\]  

(12)

Thus, all period-\( p \) points are unstable. The above analysis of the Lorenz map has strongly demonstrated (not proven) that periodic solutions of Lorenz system are not possible and that the flow is indeed **long-term aperiodic** for \( t \to \infty \).

### 2 1-D maps, a proper introduction

This section deals with a new class of dynamical systems (introduced in Lecture 9) in which time is **discrete**, rather than continuous. These systems are known variously as difference equations, recursion relations, iterated maps, or simply maps. The Lorenz map is such a system. When we say “map,” do we mean the function \( f \) or the difference equation

\[
x_{n+1} = f(x_n)?
\]  

(13)

Following common usage, we’ll call both of them maps. If you’re disturbed by this, you must be a pure mathematician... or should consider becoming one! Fixed point \( x^\ast \) of one-dimensional map (13) satisfies Eq. (2) and period-\( p \) point \( x \) satisfies Eq. (7) for minimal \( p \).

Maps arise in various ways:

1. **As tools for analysing differential equations.** We have already encountered maps in this role. For instance, Lorenz map provided strong evidence that Lorenz attractor is truly strange, and is not just a long-period limit-cycle. In future lectures Poincaré maps will allow us to prove the existence of a periodic solutions, and to analyse the stability of periodic solutions in general. Maps will prove to be superb tools for studying and analysing chaotic systems.
2. **As models of natural phenomena.** In some scientific contexts it is natural to regard time as discrete. This is the case in digital electronics, in parts of economics and finance theory, in impulsively driven mechanical systems, and in the study of certain animal populations where successive generations do not overlap.

3. **As simple examples of chaos.** Maps are interesting to study in their own right, as mathematical laboratories for chaos. Indeed, maps are capable of much wilder behaviour than differential equations because the points $x_n$ *hop* along their orbits/trjectories/iterates rather than flow continuously. Continuity is very much a restriction on possible dynamics (see Lecture 6: Poincaré-Bendixson theorem).

## 3 Logistic map

![Logistic map](image.png)

Figure 6: Logistic map and the diagonal.

In a fascinating and influential review article (linked below), Robert May (1976) emphasised that even simple nonlinear maps could have very complicated dynamics. May illustrated his point with the **Logistic map** given by

$$x_{n+1} = rx_n(1 - x_n),$$

(14)

a discrete-time analog of the **Logistic equation** for population growth, where $x_n$ is the dimensionless measure of the population in the $n$th generation and $r$ is the intrinsic growth rate. As shown in Fig. 6, the graph of map (14) is a parabola with a maximum value of $r/4$ at $x = 1/2$. We restrict the control parameter $r$ to the range ($-2$ or) $0 \leq r \leq 4$ so that Eq. (14) maps the interval $0 \leq x \leq 1$ into itself.

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**Logistic map**

Logistic map\(^1\) has the form

$$x_{n+1} = rx_n(1 - x_n), \quad x_0 \in [0, 1], \quad r \in [0, 4], \quad n \in \mathbb{Z}^+,$$

(1)

where $r$ is the control parameter.


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**Reading suggestion**

<table>
<thead>
<tr>
<th>Link</th>
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3.1 Lyapunov exponent

The calculation of Lyapunov exponents of differential equations is not a trivial task. In the case of maps it is much easier. We remind that the positive Lyapunov exponent $\lambda$ is a sign of chaos.

Lyapunov exponent of Logistic map

Chaos is characterised by sensitive dependence on initial conditions. If we take two close-by initial conditions, say $x_0$ and $y_0 = x_0 + \eta$ with $\eta \ll 1$, and iterate them under the map, then the difference between the two time series $y_n = y_n - x_n$ should grow exponentially

$$|y_n| \sim |\eta| e^{\lambda n},$$

where $\lambda$ is the Lyapunov exponent. For maps, this definition leads to a very simple way of measuring Lyapunov exponents. Solving (2) for $\lambda$ gives

$$\lambda = \frac{1}{n} \ln \frac{|y_n|}{|\eta|}.$$  

By definition $y_n = f^n(x_0 + \eta) - f^n(x_0)$. Thus

$$\lambda = \frac{1}{n} \ln \left| \frac{f^n(x_0 + \eta) - f^n(x_0)}{\eta} \right|.$$  

Lyapunov exponent of Logistic map

For small values of $\eta$, the quantity inside the absolute value signs is just the derivative of $f^n$ with respect to $x$ evaluated at $x = x_0$:

$$\lambda = \frac{1}{n} \ln \left| \frac{df^n}{dx} \right|_{x = x_0}.$$  

(5)

Since $f^n(x) = f(f(\ldots f(x))\ldots)$, by the chain rule

$$\left| \frac{df^n}{dx} \right|_{x = x_0} = \left| f'(f^{n-1}(x_0)) \cdot f'(f^{n-2}(x_0)) \cdots f'(x_0) \right| = \prod_{i=0}^{n-1} |f'(x_i)|.$$  

(6)

Our expression for the Lyapunov exponent becomes

$$\lambda = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$  

(7)

The obtained algorithm is used in the following numerical file to calculate the Lyapunov exponent of the Logistic map as a function of the growth rate $r$.

Lyapunov exponent of Logistic map

$$\lambda = \frac{1}{n} \ln \left| \prod_{i=0}^{n-1} f'(x_i) \right| = \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$  

Lyapunov exponent is the large iterate $n$ limit of this expression, and so we have,

$$\lambda = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i)|.$$  

(8)

This formula can be used to study Lyapunov exponent as a function of control parameter $r$

$$\lambda(r) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln |f'(x_i, r)|.$$  

(9)

2See Mathematica .nb file uploaded to course webpage.

NUMERIC: CDF#1, NB#1

Cobweb diagram and iterates of the (generalised) Lorenz map. Lyapunov exponent $\lambda(r)$ of the Logistic map.

Figure 7: Lyapunov exponent of the Logistic map as a function of the growth rate $r$. Calculation uses 5000 iterations for every $r$ value plotted.
3.2 Bifurcation analysis and period doubling bifurcation

Next, let’s consider only the stable fixed point and stable period-p points as we incrementally increase the value of control parameter \( r \geq 0 \). We do that in order to simplify our analysis and to save some lecture time. The stable fixed point (period-1 point) of Logistic map given by (14) and satisfying condition (2) is

\[
f(x^*) = x^* \Rightarrow rx^*(1-x^*) = x^* \quad \Leftrightarrow \quad x^*,
\]

\[
r(1-x^*) = 1,
\]

\[
r - rx^* = 1, \quad |x^*| < r,
\]

\[
1 - x^* = \frac{1}{r},
\]

\[
x^* = 1 - \frac{1}{r}.
\]

Additionally, there are the trivial solutions \( x^* = 0 \) and \( x^* = 1 \) (for initial condition \( x_0 = x^* = 1 \), and for \( n > 1, x_n \to x^* = 0 \)). Fixed point (19) is stable for \( |f'(x^*)| < 1 \). Using map definition (14) we write

\[
|f'(x^*)| = \left| \left[ r(1-x) \right]' \right|_{x=x^*} = |r - 2rx^*| < 1.
\]

Using condition (20) and for the trivial fixed point \( x^* = 0 \) we get

\[
|r - 0| < 1,
\]

\[
|r| < 1.
\]

The other trivial fixed point \( x^* = 1 \) gives the same result

\[
|r - 2r| < 1,
\]

\[
|r - 2r| < 1.
\]

Thus, the fixed points \( x^* = 0 \) and \( x^* = 1 \) are stable for \( |\pm r| < 1 \). For the non-trivial fixed point \( x^* = 1 - 1/r \) and for condition (20) we find

\[
|r - 2r \left( 1 - \frac{1}{r} \right)| < 1,
\]

\[
|r - 2r + 2| < 1,
\]

\[
|2 - r| < 1,
\]

\[
1 < |r| < 3.
\]

The fixed points \( x^* = 1 - 1/r \) exist and is stable for \( 1 < |r| < 3 \).

It seems that the found intervals (22), (24) and (28) excluded \( r = 1 \) (obviously it does not satisfy \( |f'(x^*)| < 1 \)). Let’s find the value of the map slope \( |f'(x^*)| \) for \( r = 1 \) using (20) in the case of the trivial solutions \( x^* = 0 \)

\[
|f'(x^*)| = |1 - 0| = 1,
\]

\[
x^* = 1
\]

\[
|f'(x^*)| = |1 - 2| = 1,
\]

and in the non-trivial case for \( x^* = 1 - 1/r = 1 - 1 = 0 \). Which should obviously generate the same result

\[
|f'(x^*)| = \left| 1 - 2 \left( 1 - \frac{1}{r} \right) \right| = |1 - 0| = 1.
\]

Below, it will also be beneficial to know what happens for \( r = 3 \), the \( r \) value just after the interval (28). We consider the non-trivial fixed point \( x^* = 1 - 1/r \) and find the slope

\[
|f'(x^*)| = \left| 3 - 2 \cdot 3 \left( 1 - \frac{1}{r} \right) \right| = |3 - 4| = |1 - 1| = 1.
\]

Usually, slope \( |f'(x^*)| = 1 \) corresponds to the (period doubling or flip) bifurcation point. Values \( r = 1 \) and \( r = 3 \) are the bifurcation points. Figure 8 shows the map, positions of the fixed points, and map slopes \( |f'(x^*)| \) evaluated at the non-trivial fixed point \( x^* = 1 - 1/r \) for \( r = 1 \) and \( r = 3 \).
The following interactive numerical file show the dynamics of Logistic map for $0 \leq |r| < 1$ and $1 < |r| < 3$.

**Numerics: cdf#2, nb#2**


![Cobweb Diagram](image1)

**Figure 9:** (Top-left) Cobweb diagram of the Logistic map shown for $r = 0.98 < 1$. The fixed point $x^* = 0$. (Top-right) Map iterates corresponding to the cobweb diagram shown on the left. (Bottom-left) Cobweb diagram of the Logistic map shown for interval $1 < |r| < 3$ with $r = 2.82$. The fixed point $x^* = 1 - 1/r$. (Bottom-right) Map iterates corresponding to the cobweb diagram shown on the left.

What happens for $r \geq 3$? What happens after the **period doubling** or **flip bifurcation** at $r = 3$? The name “flip” refers to the fact that the map trajectories start to flip between two values — period-2 points in period-2 orbit. Let’s check the dynamics using a computer.

**Numerics: cdf#2, nb#2**


After the **initial transient behaviour** has decayed the dynamics of the map settles to a **stable period-2 orbit**. It can be showed that period-2 orbits exist for $3 \leq r < 1 + \sqrt{6}$. 

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9/17  
As of March 7, 2020
Let’s try to think about this outcome some more...

If you want to study the dynamics of the attractors of second iterate map $f^2$ analytically consider one of the equations in (11, slide numbering). The fourth order polynomial defined by Eq. (7)

$$r[x(1-x)][1-(rx(1-x))]=x,$$

$$-r^3x^4+2r^3x^3-r^2(1+r)x^2+r^2x=x.$$

Intersections between the graph of second iterate map $f^2$ and the diagonal correspond to the solutions of $f^2(x)=x$ (34).

**Stability of f.p.s of $f^2$ map in period-2 orbit**

We need to know the slopes of period-2 points

$$\begin{align*}
(f(p))' &= \frac{df(p)}{dx} = f'(p) = f(f(x))' = f^2(x), \\
(f(q))' &= \frac{df(q)}{dx} = f'(q) = f(f(x))' = f^2(x).
\end{align*}$$

According to the chain rule it holds that

$$f^2(x) = f(f(x)),$$

In our case

$$\begin{align*}
(f^2(p))' &= f'(f(p)) \cdot f'(p) = f'(q) \cdot f'(p), \\
(f^2(q))' &= f'(f(q)) \cdot f'(q) = f'(p) \cdot f'(q).
\end{align*}$$

Above follows from the commutative property of multiplication.
The slopes of map \( f^2 \) at its fixed points \( p \) and \( q \) are equal and they are the products of map \( f \) slopes at its respective period-2 points.

What happens for \( r \geq 1 + \sqrt{6} \)? Once again we use a computer...

**Numerics: cdf#2, nb#2**


A **period-4 orbit** has emerged. It can be shown that period-4 orbits exist for \( 1 + \sqrt{6} \leq r < 3.54409 \).

![Cobweb diagram](image1)

![Orbit diagram](image2)

Figure 11: (Left) Cobweb diagram of the Logistic map shown for \( r = 3.535 \geq 1 + \sqrt{6} \) featuring the stable period-4 point. The blue graph shows fourth iterate map \( f^4 \). (Right) Map iterates corresponding to the cobweb diagram of map \( f \) shown on the left.

Figure 12 counts the period-4 points/iterates in the period-4 orbit of the Logistic map. Compare Fig. 12 to Fig. 5.

![Period-4 orbit](image3)

Figure 12: Period-4 orbit in the Logistic map.

What have we seen so far? As we have incrementally increased the value of control parameter \( r \) the periods of the fixed points have increased as well — from period-1 to period-2 and finally to period-4. The period is **clearly doubling**. This type of bifurcation is called the **period doubling bifurcation**. Further period-doublings to orbits of period-8, -16, -32, etc., occur as \( r \) increases. Specifically, let \( r_n \) denote the value of \( r \) where a stable \( 2^n \)-orbit first appears. Then computer experiments reveal that:
Logistic map, period doubling

Even number periods.
\[ r_n - \text{bifurcation point, onset of stable period-}2^n \text{ orbit.} \]

<table>
<thead>
<tr>
<th>( r_n )</th>
<th>period</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.0</td>
<td>2</td>
</tr>
<tr>
<td>1 + \sqrt{6} \approx 3.44949</td>
<td>4</td>
</tr>
<tr>
<td>3.54409</td>
<td>8</td>
</tr>
<tr>
<td>3.56441</td>
<td>16</td>
</tr>
<tr>
<td>3.56875</td>
<td>32</td>
</tr>
<tr>
<td>3.56969</td>
<td>64</td>
</tr>
</tbody>
</table>

\[ r_\infty - \text{onset of chaos (the accumulation point).} \]

\[ \delta = \lim_{n \to \infty} \frac{r_{n-1} - r_{n-2}}{r_{n} - r_{n-1}} \approx 4.669201609... \quad (15) \]

Note that the successive bifurcations come faster and faster. Ultimately the \( r_n \) converge to a limiting value \( r_\infty \). The convergence is (essentially) geometric (it’s geometric near \( r_\infty \)): in the limit of large \( n \), the distance between successive transitions shrinks by a constant factor of \( \delta \). This ratio is called the Feigenbaum constant. After the accumulation point \( r > r_\infty \) the dynamics becomes chaotic as we’ll see below.

3.3 Orbit diagram

The orbit diagram is also called the fig tree diagram (Feigenbaum in German means “fig tree”) or (incorrectly) the Feigenbaum diagram. You might guess that the system would become more and more chaotic for \( r > r_\infty \) as \( r \) increases, but in fact the dynamics are more subtle than that. To see the long-term behaviour for all values of \( r \) at once, we plot the orbit diagram a special kind of bifurcation diagram. Orbit diagram plots the system’s attractor (stable fixed points and stable period-\( p \) points) as a function of control parameter \( r \).

To generate the orbit diagram for yourself, you’ll need to write a computer program with two loops. First, choose a value of \( r \). Then generate an orbit starting from some random initial condition \( x_0 \). Iterate for 300 cycles or so, to allow the system to settle down to its eventual stable behaviour. Once the transients have decayed, plot many points, say \( x_{301}, \ldots, x_{900} \) above that \( r \). Then move to an adjacent value of \( r \) and repeat, eventually sweeping across the whole diagram.

The following interactive numerical file shows the orbit diagram for the Logistic map.


The connection between the cobweb diagram, orbit diagram and the Lyapunov exponent is shown in the interactive numerical file linked below.

Logistic map: cobweb diagram, orbit diagram and map iterates, Lyapunov exponent.
The period doubling is driven by the subsequent flip bifurcations or supercritical pitchfork bifurcations (if we use the nomenclature introduced in Lecture 2). The unstable fixed point and period-p points are omitted from the orbit diagram. Feigenbaum diagram shows both stable and unstable fixed points and period-p points or simply diagram branches (not shown here).

The orbit diagram of the Logistic map features self-similarity. In mathematics, a self-similar object is (exactly or) approximately similar to a part of itself, i.e., the whole has the same shape or quality as one or more of its sub-parts.

3.4 Tangent bifurcation and odd number period-p points

Can an odd number period appear in the Logistic map — e.g., period-3 point? One of the most intriguing features of the orbit diagram is the occurrence of periodic windows for \( r > r_\infty \). The period-3 window that occurs for \( 1 + \sqrt{8} \leq r \leq 3.8415... \) is the most conspicuous. Suddenly, against a backdrop of chaos, a stable period-3 orbit appears out of the blue.

Let’s see if we can find the period-3 window of Logistic map using a computer.

The tangent bifurcation is a new type of bifurcation for us, that occurs at \( r = 1 + \sqrt{8} \).

Figure 13: (Left) Period-3 orbit of map \( f \) can be found in the period-3 window. Here, showing the trajectory for \( r = 3.83 \). (Right) Map iterates corresponding to the cobweb diagram of map \( f \) shown on the left.

The mechanism responsible for the occurrence of the odd number periods-p points is shown in Fig. 14. The intersections between the graph of \( f^3 \), shown in red, and the diagonal correspond to solutions of...
Figure 14: Period-3 orbit can be found in period-3 window of the orbit diagram.

$f^3(x) = x$. There are eight solutions for $r \geq 1 + \sqrt{8}$, six of interest to us are marked with the blue dots, and two imposters that are not genuine period-3; they are actually fixed points, or period-1 points for which $f(x*) = x*$. The blue filled dots in Fig. 14 correspond to a stable period-3 cycle; note that the slope of $f^3(x)$ is shallow at these points, consistent with the stability of the cycle. In contrast, the slope exceeds 1 at the cycle marked by the empty blue dots; this period-3 orbit is therefore unstable.

Now suppose we decrease $r$ toward the chaotic regime $r < 1 + \sqrt{8}$. Then the red graph in Fig. 14 changes shape — the hill moves down and the valleys rise up. The curve therefore moves towards the diagonal. Figure 14 shows that when $r = 1 + \sqrt{8}$, the six blue intersections have merged to form three black filled period-3 points by becoming tangent to the diagonal. At this critical value of $r$, the stable and unstable period-3 cycles coalesce and annihilate in a tangent bifurcation. This transition defines the beginning of the periodic period-3 window.

All odd number periods will also undergo the period doubling. This means that all number periods $[1, \infty)$ are eventually represented in the orbit diagram.

Figure 14 also explains the self-similarity present in the orbit diagram as shown on Slide 12 for different scales of magnification. The peaks and valley of map $f^3$ are smaller distorted copies of the original map $f$. Thus, the local dynamics of the cobweb trajectories for map $f^3$ must be similar to the dynamics of the original map $f$.

Intermittency$^3$ and period-3 window

Transitive chaos and intermittency in dynamical systems.
Tangent bifurcation occurs at $r = 1 + \sqrt{8} \approx 3.8284$ (period-3 orbit).

Iterates of Logistic map shown for $r = 3.8282$ and $x_0 = 0.15$.

$^3$See Mathematica.nb file uploaded to course webpage.

The dynamics closely related to the tangent bifurcation is intermittency. Logistic map exhibits intermittent chaos for $r$ values just before the period-3 window.
4 Sine map and universality of period doubling

It can be shown that in all unimodal maps the same dynamics of period doubling occurs. For example we consider the sine map.

The graph of the sine map has the same basic shape as the graph of the Logistic map. Both curves are smooth, concave down, and have a single maximum. Such maps are called unimodal.

Sine is also a transcendental function opposed to an algebraic function as is the polynomial that describes the Logistic map. A transcendental function is an analytic function that does not satisfy a polynomial equation. In other words, a transcendental function “transcends” algebra in that it cannot be expressed in terms of a finite sequence of the algebraic operations of addition, multiplication, and root extraction.

The transcendental nature of the sine map must makes the underlying higher-dimensional physics represented by the sine map fundamentally different from, e.g., the three-dimensional Lorenz flow sampled by Lorenz map.

Let’s study the dynamics of the sine map using a computer.

The sine map iterates, cobweb and orbit diagrams ($x_n \in [0, 1]$).

The dynamics is surprisingly similar to the dynamics of the Logistic map. Mitchell J. Feigenbaum was first to discover the quantitative laws that are independent of unimodal map functions $f$. By that we mean that the algebraic form of $f(x)$ is irrelevant, only its overall shape matters.
Here \( x_m = \max f(x) \) is the maximum of the map graph. The **Feigenbaum constants** are valid up to the onset of chaos at \( r=\infty \) and inside each periodic window for \( r > r_{\infty} \).

In addition to scaling law in control parameter \( r \) direction, shown earlier, Feigenbaum also found scaling law for the vertical \( x \)-direction of the orbit diagram. The **Feigenbaum constants** are universal the same convergence rate appears no matter what unimodal map is iterated! They are new mathematical constants, as basic to period doubling as \( \pi \) is to circles.

### Reading suggestion

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### 5 Universal route to chaos

We showed that the qualitative dynamics of the Logistic and sine maps are identical. They both undergo period doubling routes to chaos, followed by periodic windows interwoven with chaotic bands. Even more remarkably, the periodic windows occur in the same order, and with the same relative sizes. For instance, the period-3 window is the largest in both cases, and the next largest windows preceding it are period-5 and period-6. But there are quantitative differences. For instance, the period doubling bifurcations occur later (for greater parameter \( r \) value) in the Logistic map, and the periodic windows are thinner.

Turns out that the onset of chaos via period doubling is predominant in nature and in artificial chaotic systems. Feigenbaum constants have real predictive power in various scientific applications. The period doubling bifurcation is the “route” taken by nonlinear systems to reach chaotic solution. In term of the bifurcations introduced in Lecture 2, the period doubling bifurcation can also be seen as a series of subsequent/succeeding supercritical pitchfork bifurcations, see Slide 11.

### Revision questions

1. What is cobweb diagram?
2. What is recurrence map or recurrence relation?
3. What is 1-D map?
4. How to find fixed points of 1-D maps?
5. What is Lorenz map?
6. What is Logistic map?
7. What is sine map?
8. What is period doubling?
9. What is period doubling bifurcation?
10. What is tangent bifurcation.
11. Do odd number periods (period-p orbits) exist in chaotic systems?
12. Do even number periods (period-p orbits) exist in chaotic systems?
13. Can maps produce transient chaos?
14. Can maps produce intermittency?
15. Can maps produce intermittent chaos?
16. What is orbit diagram (or Feigenbaum diagram)?
17. What are Feigenbaum constants?