Lecture notes #11
Nonlinear Dynamics
YFX1520

Lecture 11: Feigenbaum’s analysis of period doubling, superstability of fixed points and period-p points, renormalisation, universal limiting function, discrete time dynamics analysis methods, Poincaré section, Poincaré map, Lorenz section, attractor reconstruction

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1 Feigenbaum’s analysis of period doubling

In this lecture we are continuing our study of one-dimensional maps as simplified models of chaos and as tools for analysing higher order differential equations. We have already encountered maps in this role. The Lorenz map provided strong evidence that the Lorenz attractor is truly strange, and is not just a long-period limit-cycle, see Lectures 9 and 10.

1.1 Feigenbaum constants

The Feigenbaum constants were presented during Lecture 10.

\[ \delta = \lim_{n \to \infty} \frac{\Delta_{n-1}}{\Delta_n} = \lim_{n \to \infty} \frac{r_{n-1} - r_{n-2}}{r_n - r_{n-1}} \approx 4.669201609... \]  

\[ \alpha = \lim_{n \to \infty} \frac{d_{n-1}}{d_n} \approx -2.502907875... \]

In the above orbit diagram \( x_m = \max f(x) \) is the maximum of the unimodal map graph. The Feigenbaum constants are valid up to the onset of chaos at the accumulation point \( r = r_\infty \) and inside each periodic window for \( r > r_\infty \). The Feigenbaum constants are universal the same convergence rate appears no matter what unimodal map is iterated! They are new mathematical constants, as basic to period doubling as \( \pi \) is to circles.

Showing the values of Feigenbaum constants to 30 decimal places:

\[ \delta = 4.669201609102990671853203821578, \]  

\[ \alpha = -2.5029078750958928282283902873218. \]

In this lecture we’ll take a closer look at the Feigenbaum \( \alpha \) constant.

1.2 Superstable fixed points and period-p points

The fixed point defined by

\[ f(x^*) = x^*, \]

Figure 1: Superstable fixed point where \( f'(x^*) = 0 \) and \( x^* = x_m \) is the maximum of given map function.
is said to be superstable if its slope
\[ f'(x^*) = 0. \] (4)
This holds true when the maximum (or minimum) of map function \( f \) is the fixed point
\[ x^* = x_m = \max f(x). \] (5)
Figures 1, 2 and Slide 4 show the superstable fixed point of the unimodal Logistic map.

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In Lecture 9 we showed that the stability of fixed point \( x^* \) depends on the absolute slope of \( f \) at \( x^* \), i.e., \( |f'(x^*)| \). If \( |f'(x^*)| < 1 \), then the fixed point is stable and for \( |f'(x^*)| > 1 \) it is unstable.

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Linearisation of a map is derived by studying the slightly perturbed iterate
\[ x_n = x^* + \eta_n, \] (6)
where \( |\eta| \ll 1 \). We study the next iterate dynamics through linearisation of map \( f \)
\[ x_{n+1} = f(x_n) = f(x^* + \eta_n) = \left[ \text{Taylor series about } x_n = x^* \right] = f(x^*) + \frac{f'(x^*)}{1!}(x^* + \eta_n - x^*) + O(\eta_n^2) \approx x^* + f'(x^*)\eta_n, \] (7)
\[ x_{n+1} = x^* + f'(x^*)\eta_n + O(\eta_n^2), \] (8)
\[ x_{n+1} - x^* = f'(x^*)\eta_n + O(\eta_n^2). \] (9)
The left-hand side of the result follows from definition (6). The linearised form of map \( f \) is obtained by neglecting the higher order terms \( O(\eta_n^2) \)
\[ \eta_{n+1} = |f'(x^*)|\eta_n. \] (10)

---

We know that iterates \( x_n \) converge towards stable fixed point \( x^* \) where \( |f'(x^*)| < 1 \) according to the leading term of the Taylor expansion. But what happens for superstable fixed point \( x^* \)? How does the solution converge for \( f'(x^*) = 0 \)? Obviously, the linearisation can’t be used, since the leading term will be zero. We need to include an additional term in the Taylor series expansion
\[ \eta_{n+1} = |f'(x^*)|\eta_n + \frac{|f''(x^*)|}{2!}\eta_n^2 + O(\eta_n^3) = \frac{|f''(x^*)|}{2!}\eta_n^2 + O(\eta_n^3), \] (11)
\[ \eta_{n+1} = \frac{|f''(x^*)|}{2!} \eta_n^2 + O(\eta_n^3). \]  

(12)

This result means that iterates converge **faster than exponentially** when they approach a superstable fixed point. We say that iterates converge quadratically. A very beneficial property to keep in mind when analysing maps numerically (less calculations required to reach results).

The following interactive numerical file demonstrates the superstable convergence of iterate \( x_n \) in the case of Logistic map where \( r = 2 \).

Comparison of superstable and stable fixed points in the Logistic map.

Comparison of map iterate convergence to superstable fixed point and a stable fixed point.

![Comparison of superstable and stable fixed points in the Logistic map.](image1)

Figure 2: (Top) Cobweb diagram and map iterates of Logistic map featuring superstable fixed point. Initial condition \( x_0 = 0.1 \) and parameter \( r = 2 \). (Bottom) Cobweb diagram and map iterates of Logistic map featuring stable fixed point. Initial condition \( x_0 = 0.1 \) and parameter \( r = 2.85 \).

Can there be **superstable period-p points**?

Superstable period-p point (Logistic map)

For example in the case of period-2 point

\[ f^2(x^*, r) = x^*, \quad (x^* = x_m) \Rightarrow r = 1 + \sqrt{5} \]  

(5)

\[ (f^2(x^*, r))' = \frac{d}{dx^*}[f(f(x^*, r))] = f'(f(x^*, r)) \cdot f'(x^*, r) = 0. \]  

(6)
Superstable period-p point: fixed point \( x^* \) that is also the local minimum or maximum of map \( f^p \). Algebraically we write
\[
f^p(x^*) = x^*,
\]
where \( x^* = x_m = \max f(x) \). The superstable point of map \( f \) where \( f(x^*) = x^* \) is also one of the points in the superstable period-p orbit.

The superstable points of the Logistic map are shown below on the orbit diagram where \( r_n \) is the onset of the period-\( 2^n \) orbit (period doubling or flip bifurcation points) and \( r = R_n \) corresponds to superstable period-\( 2^n \) orbits. The \( R_n \) occurs always between \( r_n \) and \( r_{n+1} \). Unsurprisingly, the ratios of succeeding \( R_n \) also converges to \( \delta \) for \( n \to \infty \).

**Period doubling bifurcation points**

| \( r_0 \) – onset of chaos (accumulation point) | \( R_0 = 2.0 \) | t.p. |
| \( r_1 \) | \( R_1 = 1 + \sqrt{5} \approx 3.6180 \) | period-2 |
| \( r_2 \approx 3.5441 \) | \( R_2 \approx 3.8284 \) | period-4 |
| \( r_3 \approx 3.56441 \) | \( R_3 \approx 3.8346 \) | period-8 |
| \( r_4 \approx 3.56875 \) | \( R_4 \approx 3.83667 \) | period-16 |
| \( \vdots \) | \( \vdots \) | \( \vdots \) |
| \( r_\infty \approx 3.5699457 \) | \( R_\infty \approx 3.56994567 \) | period-\( 2^\infty \) |

\[
limit_{n \to \infty} \frac{R_{n-1}}{R_n} = \delta
\]

**1.3 Renormalisation and the Feigenbaum constants**

Let’s try to tap into the observation, presented in Lecture 10, that the orbit diagram of unimodal maps is **self-similar** under magnification. How do the scaling constant \( \alpha \) and \( \delta \) govern the period doubling bifurcations in the Logistic and other unimodal maps?

![Orbit diagram, superstable period-2^n points](image)

Next, we consider the local dynamics of the orbit diagram near \( r = R_0 \) and then compare it to the situation near \( r = R_1 \), see Fig. 4. We **renormalise** one region into another, Region 1 into Region 2 as they appear in Fig. 3. Following slides show graphically what we mean by **renormalisation procedure**.

Figure 3: Regions of interest: the proximity to \( r = R_0 \) and \( r = R_1 \). Orbit diagram is showing bifurcation points \( r_n \) corresponding to the onsets of period-\( 2^n \) orbits, and superstable fixed points \( R_n \) corresponding to superstable period-\( 2^n \) orbits, cf. Slide 6.
Let’s express the renormalisation procedure algebraically:

\[
 f(x, R_0) \approx \alpha f^2\left(\frac{x}{\alpha}, R_1\right),
\]

notice that we have scaled the vertical and horizontal directions simultaneously. In summary, map \( f \) has been renormalised by taking its second iterate, rescaling \( x \to x/\alpha \), shifting \( r \) to the next superstable value, and multiplying everything by \( \alpha \).

The higher order renormalisations of \( f(x, R_0) \) are given by

\[
 f(x, R_0) \approx \alpha^2 f^4\left(\frac{x}{\alpha^2}, R_2\right),
 \approx \alpha^3 f^8\left(\frac{x}{\alpha^3}, R_3\right),
 \approx \alpha^4 f^{16}\left(\frac{x}{\alpha^4}, R_4\right),
 \vdots
 \approx \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right).
\]

What happens when we reach the accumulation point \( r = r_\infty \) — reach the chaotic regime?

\[
 \lim_{n \to \infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_n\right) = g_0(x).
\]

Feigenbaum found numerically that this limit converges only for \( \alpha = -2.502907875\ldots \) by producing the resulting limiting function \( g_0 \). The function \( g_0 \) is called the universal limiting function. This function is similar in shape to map \( f \) and it retains the superstability of its fixed point.

The descriptor “universal” in this context means that function \( g_0 \) is almost independent of the unimodal function \( f \). How can \( g_0 \) be independent of \( f \)? Function \( g_0 \) depends on \( f \) only through its behaviour near \( x = 0 \), since that’s all that survives in the argument \( x/\alpha^n \) as \( n \to \infty \). With each renormalisation, we are blowing up smaller and smaller neighbourhoods of the maximum of \( f \) until practically all of the information (global shape further away from \( x = 0 \)) of \( f \) becomes meaningless. The only remaining information is the quadratic nature of the maximum. In the case of unimodal maps the Taylor expansion at \( x^* = x_m \) has to have the second (quadratic) \( x^2 \) term (superstability). The fixed point has to be what is called the second degree maximum point or the quadratic maximum.

You can find the universal limiting functions related to any superstable period-\(2^i\) orbit. To obtain other universal limiting functions \( g_i(x) \), start with \( f(x, R_i) \) instead of \( f(x, R_0) \):

\[
 g_i(x) = \lim_{n \to \infty} \alpha^n f^{2^n}\left(\frac{x}{\alpha^n}, R_{n+i}\right).
\]
What happens at the onset of chaos \( i = \infty \)? Let’s find the universal limiting function at the onset of chaos where \( R_i = R_\infty \). We need to renormalise \( f(x, R_\infty) \) one time

\[
g_\infty(x) \approx \alpha f^2\left(\frac{x}{\alpha}, R_\infty + 1\right) = \alpha f^2\left(\frac{x}{\alpha}, R_\infty\right) = g(x). \tag{18}
\]

For once, we don’t have to shift \( r \) when we renormalise (we became independent of \( i \))! The limiting function \( g_\infty(x) \), usually called simply \( g(x) \), satisfies

\[
g(x) = \alpha g^2\left(\frac{x}{\alpha}\right). \tag{19}
\]

This is a functional equation for \( g(x) \) and the universal scale factor \( \alpha \). It is self-referential: primary unknown \( g(x) \) is defined in terms of itself.

Let’s try to solve Eq. (19). The functional equation is not complete until we specify boundary conditions on \( g(x) \). After the shift of origin, all our unimodal \( f \)'s have a maximum at \( x = 0 \), so we require

\[
g'(0) = 0. \tag{20}
\]

Also, we can set

\[
g(0) = \left[ \text{From Def. (18)} \right] = g(0, R_\infty) = \text{const.}, \tag{21}
\]

to

\[
g(0) = 1, \tag{22}
\]

without loss of generality. This just defines the normalised scale for \( x \); if \( g(x) \) is a solution of Eq. (19), so is \( \mu g(x/\mu) \) where \( \mu \in \mathbb{R} \), with the same \( \alpha \). Naturally, this selection agrees with boundary condition (20).

Now we can solve Eq. (19) for function \( g(x) \) and \( \alpha \) at the boundary \( x = 0 \)

\[
g(0) = \alpha g^2\left(\frac{0}{\alpha}\right) = \alpha g^2(0) = \alpha g(g(0)), \tag{23}
\]

since \( g(0) = 1 \) we have

\[
1 = \alpha g(1), \tag{24}
\]

\[
\alpha = \frac{1}{g(1)}. \tag{25}
\]

The universal scaling constant \( \alpha \) depends on the limiting function \( g \). As of now no closed form solution exist for function \( g \). We can resort to numerical method to find the shape of the limiting function \( g \) and the value of \( \alpha \). Next, we solve Eq. (19) with boundary condition (25) using power series solution.

### 1.4 Numerical determination of the Feigenbaum \( \alpha \) constant

#### Limiting function \( g(x) \) and Feigenbaum constant \( \alpha \)

Let’s consider a functional equation in the following form:

\[
g(x) = a g\left(\frac{x}{\alpha}\right) = a g^2\left(\frac{x}{\alpha}\right), \tag{9}
\]

where \( \alpha \) acts as scaling coefficient and

\[
\alpha = \frac{1}{g(1)}. \tag{10}
\]

The power series solution is obtained by assuming power expansion

\[
g(x) = 1 + ax^2 + bx^4 + cx^6 + dx^8 + ex^{10} + \ldots, \tag{11}
\]

which assumes that the map maximum is quadratic.

#### Limiting function \( g(x) \) and Feigenbaum constant \( \alpha \)

**Numeric solution**\(^1\): the one term approximation where

\[
g(x) = 1 + ax^2 + O(x^4), \tag{12}
\]

results in

\[
a = -\frac{1}{2}(1 + \sqrt{3}), \quad \alpha = \frac{1}{g(1)} \approx -2.73206, \quad \text{(9.2\% error).} \tag{13}
\]

\(^1\)See Mathematica nb file uploaded to course webpage.
The power series expansion uses Maclaurin expansion and assumes a quadratic maximum. The quadratic maximum follows from the superstability property. The above process can generate lots of extraneous roots you will need to shift through them to find the correct solution.

The solution shown above is taken from the following numerical file. The file is commented for the benefit of students who are interested in numerical methods in general.

Calculation of universal limiting function \( g \) and value of the Feigenbaum constant \( \alpha \) in the case of the Logistic map using power series expansion (one and four term approximations).

In order to find a better match more terms in the power series expansion are needed.

The solution shown above is taken from the following numerical file.

Calculation of universal limiting function \( g \) and value of the Feigenbaum constant \( \alpha \) in the case of the Logistic map using power series expansion (one and four term approximations).

The renormalisation theory also explains the value of \( \delta \). Unfortunately, the derivations and analysis of the Feigenbaum \( \delta \) constant requires advanced functional analysis know-how (operators in function space, Frechet derivatives, etc.) and for this reason is omitted from this course.

Conclusion: Something qualitative (unimodality) gave us something quantitative (The Feigenbaum’s constants \( \alpha \) and \( \delta \)). In science it is usually the other way around!

Reading suggestion

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2 Discrete time analysis methods

We have already used mapping of a continuous three-dimensional flow into a discrete one-dimensional map — Lorenz attractor \( \rightarrow \) Lorenz map. Our goal in this section is to generalise this approach.
2.1 Poincaré section and Poincaré map

The Poincaré section is a set of intersection points between an $n-1$ dimensional hyperspace, hypersurface, or hyperplane and a trajectory of $n$-dimensional continuous-time flow. The Poincaré map or return map is the (algebraic) relationship between these preceding and succeeding intersection points, i.e., point $\bar{x}_k$ and $\bar{x}_{k+1}$, where $k$ is the intersection point, found on a Poincaré section, see Slides 13 and 14.

Poincaré maps are useful for studying swirling flows, such as the flow near a periodic orbit or as we’ll see later, the flow in some chaotic systems. Consider an $n$-dimensional system

$$\dot{x} = \vec{f}(\vec{x}).$$  \hspace{1cm} (26)

Let $S$ be an $n-1$ dimensional surface or section, see Slide 13. $S$ is required to be transverse to the flow, i.e., all trajectories starting on $S$ flow through it, not parallel to it. The Poincaré map $\vec{P}$ is a mapping from $S$ to itself, obtained by following trajectories from one intersection with $S$ to the next. If $\bar{x}_k \in S$ denotes the $k$-th intersection, then the Poincaré map is defined by

$$\bar{x}_{k+1} = \vec{P}(\bar{x}_k).$$ \hspace{1cm} (27)

Suppose that $\bar{x}^*$ is a fixed point of $\vec{P}$, i.e., $\vec{P}(\bar{x}^*) = \bar{x}^*$. Then a trajectory starting at $\bar{x}^*$ returns to $\bar{x}^*$ after some time $t$, and is therefore a closed orbit for the original system [26]. Moreover, by looking at the behaviour of $\vec{P}$ near this fixed point, we can determine the stability of the closed orbit.

Thus the Poincaré map converts problems about closed orbits (which are difficult) into problems about fixed points of a mapping (which are easier in principle, though not always in practice). The snag is that it’s typically impossible to find a formula for $\vec{P}$.

2.1.1 Example of finding a Poincaré map formula

For the sake of illustration, we present an examples for which the Poincaré map $\vec{P}$ can be computed explicitly. Consider the vector field given in polar coordinates by

$$\dot{r} = r(1 - r^2), \quad \dot{\theta} = 1.$$ \hspace{1cm} (28)

This decoupled system is familiar to us. We analysed it in Lecture 6. Angular velocity $\dot{\theta}$ is constant and positive. The behaviour of a trajectory in the radial direction is described by the first equation. 1-D phase portrait of the first equation is shown in Fig.4. We can sketch the phase portrait corresponding to Sys. (28) by combining the above observations. Figure 5 shows the resulting phase portrait and the stable limit-cycle associated with the fixed point $r^* = 1$ of the first equation present in Sys. [25].
Figure 4: Phase portrait of 1-D equation featured in (28) where the stable fixed point $r^* = 1$.

Figure 5: Phase portrait shown in polar coordinates corresponding to Sys. (28). Stable limit-cycle is shown with the red closed trajectory.

Figure 6: (Left) Cobweb diagram of the found Poincaré map where initial condition $r_0 \lesssim r^* = 1$. (Right) Cobweb diagram where initial condition $r_0 \gtrsim r^* = 1$.

Let $S$ be the positive $x$-axis (coincides with the $r$-axis in polar coordinate representation). Compute the Poincaré map. Show that the system has a unique and stable periodic orbit defined by the stable limit-cycle.

**Solution:** Let $r_0$ be an initial condition on $S$. Since $\dot{\theta} = 1$, the first return to $S$ occurs after a time of flight $t = 2\pi$. Then $r_1 = P(r_0)$, where $r_1$ satisfies

$$\frac{dr}{dt} = r(1 - r^2),$$

$$dt = \frac{dr}{r(1 - r^2)},$$

$$\int_{r_0}^{r_1} \frac{dr}{r(1 - r^2)} = \int_0^{2\pi} dt = 2\pi,$$  \hspace{1cm} (31)

$$\int_{r_0}^{r_1} \frac{dr}{r(1 - r^2)} = 2\pi.$$  \hspace{1cm} (32)

Evaluation of the integral and solving for $r_1$ yields

$$r_1 = [1 + e^{-4\pi(r_0^2 - 1)}]^{-1/2}.$$  \hspace{1cm} (33)
Hence the sought map formula

\[ P(r) = \left[ 1 + e^{-4\pi(r^2 - 1)} \right]^{-1/2}, \]  

(34)

and the Poincaré map takes the following form:

\[ r_{n+1} = P(r_n) = \left[ 1 + e^{-4\pi(r_n^2 - 1)} \right]^{-1/2}. \]  

(35)

The graph of \( P \) and the cobweb diagram for \( r_0 \) is plotted in Fig.6. The cobweb diagram shows that the fixed point \( r^* = 1 \) is stable and unique. No surprise, since we knew from above that this system has a stable limit-cycle at \( r = 1 \).

Slides 15 and 16 show two additional ways of visualising the Poincaré sections. These ideas are expanded below.

**Note (from Lecture 9):** The Poincare mapping is not the same as the Lorenz mapping discussed in Lecture 9. In both cases we’re trying to simplify the analysis of a differential equation by reducing it to an iterated map of some kind. But there’s an important distinction: To construct a Poincare map for a three-dimensional flow, we compute a trajectory’s successive intersections with a two-dimensional surface. The Poincare map takes a point on that surface, specified by two coordinates, and then tells us how those two coordinates change after the first return to the surface. The Lorenz map is different because it characterises the trajectory by only one number, not two. This simpler approach works only if the attractor is very “flat,” i.e., close to two-dimensional, as the Lorenz attractor is.

### 2.2 Poincaré analysis of the periodically driven damped Duffing oscillator

The Duffing equation (or the Duffing oscillator), named after Georg Duffing (1861–1944), is a nonlinear second-order differential equation used to model certain damped and driven oscillators.

The first term on the left-hand side of Eq. (16, slide numbering) is the inertial term, the third term is the nonlinear stiffness, and the fourth term is the damping. The non-homogeneous right-hand side of the equation describes the external \( 2\pi \)-periodic forcing with amplitude \( F \) and frequency \( \omega = 1 \). The solution for \( F > 0 \) may be a strange attractor. The external forcing of two-dimensional systems is often the source of chaos. But wait, how can two-dimensional system feature chaos? Isn’t chaos supposed to be reserved only for three or higher-dimensional systems. The external forcing term allows for this second order system to be represented as a three-dimensional system.
The non-autonomous equation of motion has the following form:
\[
\ddot{x} - x + x^3 + \delta \dot{x} = F \cos \omega t,
\] (17)
where \(\delta\) is the damping coefficient, \(F\) and \(\omega\) are the forcing strength
and frequency, respectively.

If you remove the damping and the external forcing terms the system becomes equivalent to the
problem of a particle in a double-well potential given by potential function
\[
V(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2,
\] (36)
that was presented/studied in Lecture 5.

Figure 7 shows three ways you can think graphically about the solution trajectories of the Duffing oscillator
and its Poincaré section cf. NB#3, Slides 15 and 16. Below we assume that in the external forcing term
\(\omega = 1\).

The following numerical file shows the solution to Duffing oscillator, and the quantitatively accurate
renderings of the three different ways to plot trajectories of the Duffing oscillator shows in Fig. 7.
Poincaré section of the Duffing oscillator (static).

The following slides show how the Poincaré section of the Duffing oscillator for \( t \gg 1 \) changes as we continuously move it along the \( 2\pi \)-periodic time or \( z \)-axis as shown in Fig. 8.

Figure 8: Poincaré section dynamics and periodic \( z \)-axis. A moving Poincaré section featuring a single intersection point shown with the blue dot.

Notice that the Poincaré section is depicting something that can be described as continuous stretching, folding and re-injection of the initial shape of points into itself. Careful visual inspection of the intersection points will reveal that they are being progressively mixed, within the region defined by the attractor, by this stretching–folding–re-injection process. This type of Poincaré section dynamics is common for strange attractors.

The dynamics of the Poincaré section of the Duffing oscillator is explored in the following numerical file.

2.3 Poincaré analysis of the Rössler attractor

Originally studied by Otto Rössler (1940–). The system is a product of pure imagination and it exhibits chaotic dynamics by design.
Rössler attractor

Rössler attractor is given by (as mentioned in Lecture 9):

\[
\begin{align*}
\dot{x} &= -y - x, \\
\dot{y} &= x + ay, \\
\dot{z} &= b + z(x - c).
\end{align*}
\]  

(21)

Chaotic solution exists for \(a = 0.1, b = 0.1, c = 14\).

The system has only one nonlinearity the \(zx\) term in the third equation.

Rössler attractor, Poincaré sections

The following numerical file was used to calculate the attractor and the Poincaré sections shown above.

The orbit diagram shown on Slide 25 is calculated using the Rössler map, shown on Slide 24. It clearly features period doubling bifurcations and chaotic region with periodic window bands in it, just like the Logistic and Sine maps do. Slide 26 shown the period doubling happening in time-integrated solution.
Numerical solution of the Rössler attractor. The Poincaré section of the Rössler attractor.

The following interactive numerical file shows the period doubling bifurcation happening in the $xy$-projection of Rössler attractor, cf. Slide 25 and 26.

Periodic orbits in the Rössler attractor for varied $c$ value.

The following slide shows the $2\pi$-periodic dynamics of the Poincaré section of the Rössler attractor.

Once again, notice the folding, stretching and re-injection, and consequent mixing of the Poincaré section, cf. Slides 19 and 20. This type of Poincaré section dynamics will be explored further in the next week’s lecture.

3 Lorenz section of 3-D attractor

The slide above shows a two-dimensional Poincaré section of a strange attractor where we sliced the attractor with a perpendicular plane, thereby exposing its cross section. If we take a further one-dimensional slice or the Lorenz section through the Poincaré section, we find an infinite set of points separated by gaps of various sizes. In the next week’s lecture we will look into this discovery and into
4 Attractor reconstruction

In applications it is often the case that you are able to measure or are restricted to only measuring one time-series related to a higher order system — one aspect of a problem. Let’s say you need to analyse the attractor that governs your measured one-dimensional time-series signal $s(t)$, e.g., show that it is strange. At first it might seem that there is not enough information. But, there exist a surprising data analysis technique known as the attractor reconstruction.

For systems governed by an attractor, the dynamics in the full higher dimensional phase space can be reconstructed from measurements of just a single time series $s(t)$! A single variable carries sufficient information about all the others. The method is based on time delays $\tau$. For instance, define a two-dimensional vector

$$\vec{x}(t) = (s(t), s(t + \tau))^T$$

for some time delay $\tau > 0$ and signal $s(t)$ where $\vec{x} = (x, y)^T$. Signal $s(t)$ can therefore represent either the variable $x(t)$ or $y(t)$. The time-series $s(t)$ generates a trajectory $\vec{x}(t)$ in a two-dimensional phase space that represents its attractor’s trajectory. You can also considered the attractor in three dimensions, by defining the three-dimensional vector

$$\vec{x}(t) = (s(t), s(t + \tau), s(t + 2\tau))^T,$$

for time delay $\tau > 0$. For four-dimensional attractor use

$$\vec{x}(t) = (s(t), s(t + \tau), s(t + 2\tau), s(t + 3\tau))^T,$$

etc.

The following interactive numerical file demonstrates attractor reconstruction using three examples. First, a sine wave is used to reconstruct its two-dimensional attractor — circle. Second, a quasi-periodic signal is used to reconstruct its three-dimensional attractor — torus. Third, time-series $y(t)$ of the Lorenz system is used to reconstruct the three-dimensional Lorenz attractor.

Higher-dimensional attractor reconstruction from 1-D time-domain signals. Examples: sine wave, quasi-periodic signal, $y(t)$ of the Lorenz attractor.

Revision questions

1. What are the values of the Feigenbaum constants?
2. What are the Feigenbaum constants (more in-depth answer)?
3. Define superstable fixed point of a map.
4. Define superstable period-p point (or period-p orbit) of a map.
5. What are universals of unimodal maps?
6. What is the universal route to chaos?
7. Idea behind renormalisation?
8. What are universal limiting functions in the context of maps?
9. Name discrete time dynamics analysis methods.
10. What is the Poincaré section?
11. What is the Poincaré map (return map)?
12. What is the Lorenz section?